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# LAGRANGE'S FOUR SQUARE THEOREM WITH ALMOST-PRIME NUMBERS HAVING A SPECIAL FORM*

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In this paper we consider the Lagrange's equation with almost-prime numbers satisfying a diophantine inequality.

**1. Introduction and statement of the result.** In 1770 Lagrange proved that for any positive integer $N$ the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = N \tag{1}$$

has a solution in integer numbers $x_1, \dots, x_4$. Later Jacobi found an exact formula for the number of the solutions [7, Ch. 20]. A lot of researchers studied the equation (1) for solvability in integers satisfying additional conditions. There is a hypothesis stating that if $N$ is sufficiently large and $N \equiv 4 \pmod{24}$ then (1) has a solution in primes. This hypothesis has not been proved so far, but several approximations to it have been established.

In 1994 J. Brüdern and E. Fouvry [1] proved that for any large $N \equiv 4(\mathrm{mod}24)$, the equation (1) has a solution in $x_1, \dots, x_4 \in \mathcal{P}_{34}$. (We say that an integer $n$ is almost-prime of order $r$ if $n$ has at most $r$ prime factors, counted with their multiplicities, and denote by $\mathcal{P}_r$ the set of all almost-primes of order $r$.) This result is improved by D. R. Heath-Brown and D. I. Tolev [8]. They showed that for the same restrictions for $N$, the equation (1) has a solution in prime $x_1$ and almost-prime $x_2, x_3, x_4 \in \mathcal{P}_{101}$. In their paper they also proved that the equation has a solution in $x_1, \dots, x_4 \in \mathcal{P}_{25}$. In 2010 Tak Wing Ching [2] improved this result with three of them being $P_3$-numbers and the other – a $P_4$-number.

On the other hand, let us consider a subset of the set of integers having the form

$$\mathcal{A} = \{n | a < \{\eta n\} < b\},$$

where $\eta$ is a fixed quadratic irrational number, and $a, b \in [0, 1]$.

Let $I(N)$ be the number of solutions of (1) in arbitrary integers and $J(N)$ be the number of solutions of (1) in integers of the set $\mathcal{A}$.

In 2011, S. A. Gritsenko and N. N. Motkina [5] proved that for any positive small $\varepsilon$, the following formula holds

$$J(N) = (b-a)^4 I(N) + O\left(N^{0,9+3\varepsilon}\right).$$

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S. A. Gritsenko and N. N. Motkina consider many others additive problem in witch variables are in special set of numbers similar to $\mathcal{A}$ [3, 4, 6]. In 2013 A. V. Shutov [11] considered solvability of diophantine equation in integer numbers from $\mathcal{A}$. Further research in this area was made by A. V. Shutov and A. A. Zhukova [12].

**2. Main results.** Our result is

**Theorem 1.** *Let $\eta_1$, $\eta_2$, $\eta_3$, $\eta_4 \in \mathbb{R}\backslash\mathbb{Q}$ and at least one of them be a quadratic irrational number, $0 < \lambda < \frac{1}{10}$ and $k = \left[\frac{54}{1-10\lambda}\right]$. Then for every sufficiently large integer $N$, the equation* (1) *has a solution in almost-prime numbers $x_1, \ldots, x_4 \in \mathcal{P}_k$, such that $||\eta_1 x_1 + \eta_2 x_2 + \eta_3 x_3 + \eta_4 x_4|| < N^{-\lambda}$.*

The present paper is an extension of the work of Zh. H. Petrov and T. Todorova [14].

**3. Notations.** In the present paper we use the following notations.

We denote by $N$ a sufficiently large odd integer. The letters $a$, $b$, $k$, $l$, $m$, $n$, $q$, $p$ denote always integers. By $(n_1, \ldots, n_k)$ we denote the greatest common divisor of $n_1, \ldots, n_k$. We denote by $\vec{n}$ four dimensional vectors and let

$$|\vec{n}| = \max(|n_1|, \ldots, |n_4|). \tag{2}$$

As usual $\mu(q)$ is the Möbius function and $\tau(q)$ is the number of positive divisors of $q$. Sometimes we write $a \equiv b\ (q)$ as an abbreviation of $a \equiv b \pmod q$. We write $\sum\limits_{x\ (q)}$ for a sum over a complete system of residues modulo $q$ and respectively $\sum\limits_{x\ (q)}^{*}$ is a sum over a reduced system of residues modulo $q$. Let $e(t) = e^{2\pi i t}$.

We use Vinogradov's notation $A \ll B$, which is equivalent to $A = O(B)$. By $\varepsilon$ we denote an arbitrarily small positive number, which is not the same in different formulas. The constants in the $O$-terms and $\ll$-symbols are absolute or depend on $\varepsilon$.

**4. Auxiliary results.** Now we introduce some lemmas, which shall be used later.

**Lemma 2.** *Suppose that $D \in \mathbb{R}, D > 4$. There exist arithmetical functions $\lambda^{\pm}(d)$ (called Rosser's functions of level $D$) with the following properties:*

1. *For any positive integer $d$ we have*
$$|\lambda^{\pm}(d)| \le 1, \qquad \lambda^{\pm}(d) = 0 \quad \text{if} \quad d > D \quad \text{or} \quad \mu(d) = 0.$$

2. *If $n \in \mathbb{N}$ then*
$$\sum_{d|n} \lambda^{-}(d) \le \sum_{d|n} \mu(d) \le \sum_{d|n} \lambda^{+}(d).$$

3. *If $z \in \mathbb{R}$ is such that $z^2 \le D$ and if*
$$P(z) = \prod_{2<p<z} p, \ \ \mathcal{B} = \prod_{2<p<z} \left(1 - \frac{1}{p-1}\right), \ \ \mathcal{N}^{\pm} = \sum_{d|P(z)} \frac{\lambda^{\pm}(d)}{\varphi(d)}, \ \ s_0 = \frac{\log D}{\log z}, \tag{3}$$
*then we have*
$$\mathcal{B} \le \mathcal{N}^{+} \le \mathcal{B}\left(F(s_0) + O\left((\log D)^{-\frac{1}{3}}\right)\right), \tag{4}$$
$$\mathcal{B} \ge \mathcal{N}^{-} \ge \mathcal{B}\left(f(s_0) + O\left((\log D)^{-\frac{1}{3}}\right)\right), \tag{5}$$

*where $F(s)$ and $f(s)$ satisfy*

$$\begin{aligned} &F(s) = 2e^{\gamma}s^{-1}, \quad \text{if} \quad 2 \le s \le 3,\\ &f(s) = 2e^{\gamma}s^{-1}\log(s-1), \quad \text{if} \quad 2 \le s \le 3,\\ &(sF(s))' = f(s-1), \quad \text{if } s > 3,\\ &(sf(s))' = F(s-1), \quad \text{if } s > 2. \end{aligned}$$

*Here $\gamma$ is Euler's constant*[1].

**Proof.** See Greaves [8, Chapter 4]. □

**Lemma 3.** *Suppose that $\Lambda_i, \Lambda_i^{\pm}$ are real numbers satisfying $\Lambda_i = 0$ or $1$, $\Lambda_i^- \le \Lambda_i \le \Lambda_i^+$, $i = 1, 2, 3, 4$. Then*

$$\begin{aligned} \Lambda_1\Lambda_2\Lambda_3\Lambda_4 \ge\ & \Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^-\Lambda_4^+ \\ & + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^- - 3\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+. \end{aligned} \tag{6}$$

**Proof.** The proof is similar to the proof of Lemma 13 in [1]. □

Let

$$w_0(t) = \begin{cases} e\left(\dfrac{1}{t^2 - \frac{16}{25}}\right) & \text{if } t \in \left(-\dfrac{4}{5}, \dfrac{4}{5}\right), \\ 0 & \text{if } t \notin \left(-\dfrac{4}{5}, \dfrac{4}{5}\right) \end{cases}$$

and

$$w(x) = w_0\left(\frac{x}{P} - \frac{1}{2}\right). \tag{7}$$

**Lemma 4.** *Let $u, \beta \in \mathbb{R}$ and*

$$J(\beta, u) = \int_{-\infty}^{+\infty} w_0\left(x - \frac{1}{2}\right) e(\beta x^2 + ux)dx. \tag{8}$$

*Then:*

1. *For every $k \in \mathbb{N}$ and $u \ne 0$ we have*

$$J(\beta, u) \ll_k \frac{1 + |\beta|^k}{|u|^k}.$$

2. *The following inequality holds*

$$J(\beta, u) \ll \min\left(1, |\beta|^{-\frac{1}{2}}\right).$$

**Proof.** See Lemma 9 in [8]. □

**Lemma 5.** *Suppose that $\vec{u} \in \mathbb{Z}^4$, $|\vec{u}| = \max(|u_1|, |u_2|, |u_3|, |u_4|) > 0$ and*

$$J(\gamma, \vec{u}) = \prod_{i=1}^{4} J(\gamma, u_i).$$

*Then we have*

$$\int_{-\infty}^{+\infty} |J(\alpha, \vec{u})|\, d\gamma \ll |\vec{u}|^{-1+\varepsilon}.$$

[1] $\gamma \approx 0.577$, also known as Euler-Mascheroni constant, is defined as the limiting difference between harmonic series and the natural logarithm.

**Proof.** The proof can be found in [8, Lemma 10]. $\square$

**Lemma 6.** *There exists a function $\sigma(\upsilon, q, \alpha)$ defined for $-\frac{q}{2} < \upsilon \le \frac{q}{2}$, $q \le P$, $|\gamma| \le \frac{P}{q}$, integrable with respect to $\gamma$, satisfying*

$$|\sigma(\upsilon, q, \gamma)| \le \frac{1}{1+|\upsilon|}$$

*and also*

$$\sum_{-\frac{q}{2}<\upsilon\le\frac{q}{2}} e\left(\frac{\overline{a}\upsilon}{q}\right)\sigma(\upsilon, q, \alpha) = \begin{cases} 1 & \text{if } \alpha \in \mathcal{N}(a, q), \\ 0 & \text{otherwise,} \end{cases}$$

*where*

$$\mathcal{N}(a, q) = \left(\frac{P^2}{q(q+q')}, \frac{P^2}{q(q+q'')}\right]$$

*and*

$$P < q + q', q + q'' \le P + q, \qquad aq' \equiv 1(\mathrm{mod} q), \qquad aq'' \equiv -1(\mathrm{mod} q). \tag{9}$$

**Proof.** See Lemma 45 [15]. $\square$

The Gauss sum is defined by

$$G(q, m, n) = \sum_{x(q)} e\left(\frac{mx^2 + nx}{q}\right). \tag{10}$$

For $\vec{d} = \langle d_1, \ldots, d_4\rangle \in \mathbb{Z}^4$, $\vec{n} = \langle n_1, \ldots, n_4\rangle \in \mathbb{Z}^4$ we denote

$$G(q, a\vec{d}^2, \vec{n}) = \prod_{i=1}^{4} G(q, ad_i^2, n_i).$$

We need to estimate an exponential sum of the form

$$V_q = V_q(N, \vec{d}, \upsilon, \vec{n}) = {\sum_{a(q)}}^{*} e\left(\frac{\overline{a}\upsilon - Na}{q}\right) G(q, a\vec{d}^2, \vec{n}). \tag{11}$$

To estimate $V_q$ we use the properties of the Gauss sum and the Kloosterman sum.

**Lemma 7.** *Suppose that $N, q \in \mathbb{N}, \upsilon \in \mathbb{Z}$ and $\vec{d}, \vec{n} \in \mathbb{Z}^4$. Then we have*

$$V_q(N, \vec{d}, \upsilon, \vec{n}) \ll q^{\frac{5}{2}}\tau(q)(q, N)^{\frac{1}{2}}(q, d_1)(q, d_2)(q, d_3)(q, d_4).$$

*Moreover, if some of the conditions*

$$(q, d_i) | n_i, \quad i = 1, \ldots, 4$$

*do not hold, then $V_q(N, \vec{d}, \upsilon, \vec{n}) = 0$.*

**Proof.** This result is analogous to the one in Lemma 1 [1]. $\square$

**Lemma 8** (Liouville)**.** *If $\eta$ is an irrational number which is the root of a polynomial $f$ of degree 2 with integer coefficients, then there exists a real number $A > 0$ such that, for all integers $p$, $q$, with $q > 0$,*

$$\left|\eta - \frac{p}{q}\right| \ge \frac{A}{q^2}.$$

**Proof.** See Theorem 1A [10]. $\square$

## 5. Proof of the theorem.

***5.1. Beginning of the proof.*** Let $N$ be a sufficiently large integer. We denote

$$z = N^{\alpha}, \qquad P(z) = \prod_{p<z} p, \qquad \delta = N^{-\lambda}.$$

We apply the well-known Vinogradov's "little cups" lemma [9, Chapter 1, Lemma A] with parameters

$$\alpha = -\frac{\delta}{2}, \qquad \beta = \frac{\delta}{2}, \qquad \Delta = \frac{\delta}{2}, \qquad r = [\log N]$$

and construct a function $\theta(t)$, which is periodic with period 1 and has the following properties:

$$\theta(t) = 1 \quad \text{for} \quad -\frac{\delta}{4} < t < \frac{\delta}{4};$$

$$0 < \theta(t) < 1 \quad \text{for} \quad -\frac{\delta}{2} < t < \frac{\delta}{4} \quad \text{or} \quad \frac{\delta}{4} < t < \frac{3\delta}{4}; \quad \theta(t) = 0 \quad \text{for} \quad \frac{3\delta}{4} < t < 1 - \frac{3\delta}{4}.$$

Furthermore, the Fourier series of $\theta(t)$ is given by

(12)

$$\theta(t) = \delta + \sum_{\substack{0<|m|\leq H \\ m\neq 0}} c(m)\, e(mt) + O(P^{-A}), \text{ with } |c(m)| \leq \min\left(\delta, \frac{1}{|m|}\left(\frac{[\log N]}{\Delta\pi|m|}\right)^{[\log N]}\right),$$

where $A$ is an arbitrary large constant,

(13)
$$H = \frac{[\log N]^2}{\delta}$$

Let

$$\theta(\vec{\eta}\vec{x}) = \theta\big(\eta_1 x_1 + \eta_2 x_2 + \eta_3 x_3 + \eta_4 x_4\big)$$

and

$$w(\vec{x}) = w(x_1)w(x_2)w(x_3)w(x_4).$$

We consider the sum

$$\Gamma = \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=N \\ (x_i,P(z))=1,\ i=1,2,3,4}} \theta(\vec{\eta}\vec{x})w(\vec{x}).$$

From the condition $(x_i, P(z)) = 1$ it follows that any prime factor of $x_i$ is greater or equal to $z$. Suppose that $x_i$ has $l$ prime factors, counted with the multiplicity. Then we have

$$N^{\frac{1}{2}} \geq x_i \geq z^l = N^{\alpha l}$$

and hence $l \leq \dfrac{1}{2\alpha}$. This implies that if $\Gamma > 0$ then equation (1) has a solution in $x_1, \ldots, x_4$, which is almost-prime with at most $\left[\dfrac{1}{2\alpha}\right]$ prime factors, such that $||\eta_1 x_1 + \eta_2 x_2 + \eta_3 x_3 + \eta_4 x_4|| < N^{-\lambda}$.

For $i = 1, 2, 3, 4$ we define

$$\Lambda_i = \sum_{d|(x_i, P(z))} \mu(d) = \begin{cases} 1 & \text{if } (x_i, P(z)) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Then we find that

$$\Gamma = \sum_{x_1^2+x_2^2+x_3^2+x_4^2=N} \Lambda_1\Lambda_2\Lambda_3\Lambda_4\theta(\vec{\eta}\vec{x})w(\vec{x}).$$

We can write $\Gamma$ as

$$\Gamma = \sum_{x_i \in \mathbb{Z}} \Lambda_1\Lambda_2\Lambda_3\Lambda_4\theta(\vec{\eta}\vec{x})w(\vec{x}) \int_0^1 e(\alpha(x_1^2 + x_2^2 + x_3^2 + x_4^2 - N))d\alpha.$$

Suppose that $\lambda^{\pm}(d)$ are the Rosser functions of level $D$ (see Lemma 2). Let us also denote

$$\Lambda_i^{\pm} = \sum_{d|(x_i, P(z))} \lambda^{\pm}(d), \qquad i = 1, 2, 3, 4. \tag{15}$$

Then from Lemma 2, (14) and (15) we find that

$$\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+.$$

We use Lemma 3 and find that

$$\Gamma \geq \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 - 3\Gamma_5,$$

where $\Gamma_1, \ldots, \Gamma_4$ are the contributions coming from the consecutive terms of the right side of (6). We have $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4$ and

$$\Gamma_1 = \sum_{x_i \in \mathbb{Z}} \Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+\theta(\vec{\eta}\vec{x})w(\vec{x}) \int_0^1 e(\alpha(x_1^2 + x_2^2 + x_3^2 + x_4^2 - N))d\alpha,$$

$$\Gamma_5 = \sum_{x_i \in \mathbb{Z}} \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\theta(\vec{\eta}\vec{x})w(\vec{x}) \int_0^1 e(\alpha(x_1^2 + x_2^2 + x_3^2 + x_4^2 - N))d\alpha.$$

Hence, we get

$$\Gamma \geq 4\Gamma_1 - 3\Gamma_5. \tag{16}$$

***5.2. Asymptotic formula for $\Gamma_1$.*** We shall find an asymptotic formula for the integral $\Gamma_1$. We have

$$\Gamma_1 = \sum_{d_i|P(z)} \lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4) \sum_{x_i \equiv 0(d_i)} \theta(\vec{\eta}\vec{x})w(\vec{x}) \int_0^1 e(\alpha(x_1^2 + \cdots + x_4^2 - N))d\alpha.$$

Now using the Fourier series of $\theta(t)$, we find

$$\sum_{x_i\equiv 0(d_i)} \theta(\vec{\eta}\vec{x})w(\vec{x})e(\alpha(x_1^2+\cdots+x_4^2))$$
$$= \sum_{|m|\le H} c(m)\prod_{i=1}^{4}\sum_{x_i\equiv 0(d_i)} w(x_i)e(\alpha(x_i^2+m\eta_i x_i))$$
$$+ O\Bigg(P^{-A}\sum_{\substack{x_i\equiv 0(d_i)\\ x_1^2+\cdots+x_4^2=N}} w(x_1)w(x_2)w(x_3)w(x_4)\Bigg).$$

Let

$$S(\alpha,d_i,m) = \sum_{\substack{x\in\mathbb{Z}\\ x\equiv 0(d_i)}} w(x_i)e(\alpha x_i^2+m\eta_i x_i)\,, \tag{17}$$

$$S(\alpha,\vec{d},m) = S(\alpha,d_1,m)S(\alpha,d_2,m)S(\alpha,d_3,m)S(\alpha,d_4,m)\,. \tag{18}$$

and

$$\lambda(\vec{d}) = \lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4)\,. \tag{19}$$

Then

$$\Gamma_1 = \sum_{d_i|P(z)}\lambda(\vec{d})\sum_{|m|\le H} c(m)\int_0^1 S(\alpha,\vec{d},m)e(-N\alpha)d\alpha + O(1).$$

We split $\Gamma_1$ into two parts:

$$\Gamma_1 = \Gamma_1^0+\Gamma_1^*+O(1), \tag{20}$$

where

$$\Gamma_1^0 = c(0)\sum_{d_i|P(z)}\lambda(\vec{d})\sum_{\substack{x_i\equiv 0(d_i)\\ x_1^2+x_2^2+x_3^2+x_4^2=N}} w(\vec{x})$$

and

$$\Gamma_1^* = \sum_{d_i|P(z)}\lambda(\vec{d})\sum_{0<|m|\le H} c(m)\int_0^1 S(\alpha,\vec{d},m)e(-N\alpha)d\alpha\,, \tag{21}$$

So from (16) and (20) we get

$$\Gamma \ge 4\Gamma_1^0 - 3\Gamma_5^0 + O\big(\Gamma_1^*\big) + O\big(\Gamma_5^*\big) + O(1). \tag{22}$$

We will evaluate the sums $\Gamma_1^*$ and $\Gamma_5^*$.

***5.3. Estimation of $\Gamma_1^*$.*** In this subsection we find the upper bound for $\Gamma_1^*$ defined in (21). The function into integral in $\Gamma_1^*$ is periodic with period 1, so we can integrate over the interval $\mathcal{I}$ defined as

$$\mathcal{I} = \left(\frac{1}{1+[P]}, 1+\frac{1}{1+[P]}\right).$$

We apply the Kloosterman form of the Hardy-Littlewood circle method. We divide the interval into large arcs only. Using the properties of the Farey fractions we represent $\mathcal{I}$

as an union of disjoint intervals in the following way:

$$\mathcal{I} = \bigcup_{q\le P} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathcal{L}(q,a),$$

where

$$\mathcal{L}(a,q) = \left(\frac{a}{q} - \frac{1}{q(q+q')}, \frac{a}{q} + \frac{1}{q(q+q'')}\right]$$

and where the integers $q', q''$ are specified in (9). Then

$$\Gamma_1^* = \sum_{d_i|P(z)} \lambda(\vec{d}) \sum_{0<|m|\le H} c(m) \sum_{q\le P} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathcal{L}(a,q)} S(\alpha, \vec{d}, m) e(-N\alpha) d\alpha.$$

We change variable of integration $\alpha = \dfrac{a}{q} + \beta$, to get

$$\Gamma_1^* = \sum_{d_i|P(z)} \lambda(\vec{d}) \sum_{0<|m|\le H} c(m) \sum_{q\le P} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathcal{M}(a,q)} S\left(\frac{a}{q} + \beta, \vec{d}, m\right) e\left(-N\left(\frac{a}{q} + \beta\right)\right) d\beta,$$

where

$$\mathcal{M}(a,q) = \left(-\frac{1}{q(q+q')}, \frac{1}{q(q+q'')}\right].$$

From (9) we find that

$$\left[-\frac{1}{2qP}, \frac{1}{2qP}\right] \subset \mathcal{M}(a,q) \subset \left[-\frac{1}{qP}, \frac{1}{qP}\right]$$

and hence

(23) $$|\beta| \le \frac{1}{qP} \quad \text{for} \quad \beta \in \mathcal{M}(a,b).$$

Now we consider the sum $S(\alpha,\, d_i,\, m)$ defined in (17). Since $\eta_i$ is an irrational number $||s\eta_i|| \ne 0$ for all $s \in \mathbb{Z}$. Using that fact and working as in the proof of [8, Lemma 12], we find that for $\beta \in \mathfrak{M}(q,a)$ we have

(24)
$$S\left(\frac{a}{q} + \beta, d_i, m\right) = \frac{P}{d_i q} \sum_{|n - m d_i q \eta_i| < M_i} J\left(\beta P^2,\, (m\eta_i - \frac{n}{d_i q})P\right) G(q,\, a d_i^2,\, n) + O(P^{-B})$$

where $G(q,m,n)$ and $J(\gamma,u)$ are defined respectively by (10) and (8), $B$ is an arbitrarily large constant, $M_i = d_i P^{\varepsilon}$, $\varepsilon > 0$ is arbitrarily small and the constant in the $O$-term depends only on $B$ and $\varepsilon$. We leave the verification of the last formula to the reader.

Let

$$F(P,\, \vec{d}) = \sum_{0<|m|\le H} c(m) \sum_{q\le P} \sum_{a\,(q)}{}^{*} e\left(-\frac{aN}{q}\right) \int_{\mathcal{M}(a,q)} S\left(\frac{a}{q} + \beta, \vec{d}, m\right) e(-\beta N) d\beta.$$

It is obvious that

(25) $$\Gamma_1^* = \sum_{d_i|P(z)} \lambda(\vec{d}) F(P,\, \vec{d})\,.$$

Using (24) and Lemma 4 we get

$$F(P,\,\vec{d}) = F^*(P,\,\vec{d}) + O(1)\,, \tag{26}$$

where

$$F^*(P,\,\vec{d}) = \frac{P^4}{d_1d_2d_3d_4}\sum_{0<|m|\le H} c(m)\sum_{q\le P}\frac{1}{q^4}{\sum_{a\,(q)}}^{*} e\left(-\frac{aN}{q}\right)\times$$
$$\times\sum_{|n_i-md_iq\eta_i|<M_i} G(q,\,ad_i^2,\,\vec{n})\int_{\mathcal{M}(a,q)} J\left(\beta P^2,\,\left(m\vec{\eta}-\frac{\vec{n}}{\vec{d}q}\right)P\right)e(-\gamma)d\gamma$$

Working as in the proof of [13, Lemma 2] we find that

$$F^*(P,\,\vec{d}) = F^{'}(P,\,\vec{d}) + O\big(P^{3/2+\varepsilon}\big), \tag{27}$$

where

$$F^{'}(P,\,\vec{d}) = \frac{P^2}{d_1d_2d_3d_4}\sum_{0<|m|\le H} c(m)\sum_{q\le P}\frac{1}{q^4}\sum_{\substack{|n_i-md_iq\eta_i|<M_i\\ (q,\,d_i)|n_i,\,i=1,\ldots,4}} V_q(N,\,\vec{d},0,\,\vec{n})\times$$
$$\times\int_{|\gamma|\le\frac{P}{2q}} J\left(\gamma,\,\left(m\vec{\eta}-\frac{\vec{n}}{\vec{d}q}\right)P\right)e(-\gamma)d\gamma,$$

and $V_q(N,\,\vec{d},0,\,\vec{n})$ is defined by (11). We represent the sum $F^{'}(P,\,\vec{d})$ as

$$F^{'}(P,\,\vec{d}) = F_1 + F_2\,, \tag{28}$$

where $F_1$ is a contribution of these addends with $q \le Q$ and $F_2$ is the contribution for addends with $Q < q \le P$. Here $Q$ is parameter, which we shall choose later. Using Lemma 4 (2), Lemma 7 and (12) we get

(29)
$$F_2 \ll \frac{P^2\delta}{d_1d_2d_3d_4}\sum_{0<|m|\le H}\sum_{Q<q\le P}\frac{q^{5/2}\tau(q)(q,\,N)^{1/2}(q,\,d_1)\cdots(q,\,d_4)}{q^4}\sum_{\substack{|n_i-md_iq\eta_i|<M_i\\ (q,\,d_i)|n_i,\,i=1,\ldots,4}} 1$$

It is clear that the sum over $\vec{n}$ we have

$$\sum_{\substack{|n_i-md_iq\eta_i|<M_i\\ (q,\,d_i)|n_i,\,i=1,\ldots,4}} 1 \ll \prod_{1\le i\le 4}\sum_{\frac{-M_i+md_iq\eta_i}{(q,\,d_i)}<t_i<\frac{M_i+md_iq\eta_i}{(q,\,d_i)}} 1$$
$$\ll \frac{M_1M_2M_3M_4}{(q,d_1)(q,d_2)(q,d_3)(q,d_4)} \ll \frac{P^\varepsilon d_1d_2d_3d_4}{(q,d_1)(q,d_2)(q,d_3)(q,d_4)}$$

which, together with (5.3) and (13), gives

$$F_2 \ll P^{2+\varepsilon}\sum_{Q<q\le P}\frac{\tau(q)(q,\,N)^{1/2}}{q^{3/2}}$$

Now we apply Cauchy's inequality to get

$$F_2 \ll P^{2+\varepsilon}\left(\sum_{Q<q\le P}\frac{\tau^2(q)}{q}\right)^{\frac12}\left(\sum_{Q<q\le P}\frac{(q,N)}{q^2}\right)^{\frac12} \ll P^{2+\varepsilon}\left(\sum_{\substack{t|N\\ t\le P}} t \sum_{\frac{Q}{t}<q_1\le\frac{P}{t}}\frac{1}{t^2q_1^2}\right)^{\frac12}$$

$$\ll \frac{P^{2+\varepsilon}}{Q^{1/2}}. \tag{30}$$

To evaluate $F_1$ we first apply Lemma 5 to get

$$\int_{|\gamma|\le\frac{P}{2q}}\left|J\left(\gamma,\left(m\vec{\eta}-\frac{\vec{n}}{\vec{d}q}\right)P\right)\right|d\gamma \ll \left(\left|\left(m\vec{\eta}-\frac{\vec{n}}{\vec{d}q}\right)P\right|\right)^{-1+\varepsilon}$$

Then using Lemma 7 and (13) we receive

(31)

$$F_1 \ll \frac{P^2}{d_1d_2d_3d_4}\sum_{q\le Q}\frac{q^{5/2}\tau(q)(q,\,N)^{1/2}(q,\,d_1)\cdots(q,\,d_4)}{q^4}\sum_{\substack{|n_i-md_iq\eta_i|<M_i\\ (q,\,d_i)|n_i,\ i=1,\ldots,4}}\frac{1}{\left|(m\vec{\eta}-\frac{\vec{n}}{\vec{d}q})P\right|}$$

It is clear that if $n_i=(q,\,d_i)t_i$, $d_i=(q,\,d_i)d_i'$ and $\left|(m\eta_i-\dfrac{n_i}{d_iq})P\right|=\dfrac{P(q,\,d_i)}{qd_i}|t_i-md_i'\eta_iq|$ then for the sum over $(m\vec{\eta}-\dfrac{\vec{n}}{\vec{d}q})P$ we obtain

$$\sum_{\substack{|n_i-md_iq\eta_i|<M_i\\ (q,\,d_i)|n_i,\ i=1,\ldots,4}}\frac{1}{\left|(m\vec{\eta}-\frac{\vec{n}}{\vec{d}q})P\right|} \ll \frac{q}{P}\sum_{|t_i-md_i'q\eta_i|<\frac{M_i}{(q,\,d_i)}}\frac{1}{\max\limits_{1\le i\le4}(q,\,d_i)|t_i-md_i'\eta_iq|/d_i}. \tag{32}$$

Without loss of generality we can assume that $\eta_1$ is quadratic irrationality. Let $t_1^o$ is such that we can assume that $\eta_1$ is quadratic irrationality. Let $t_1^o$ is such that

$$|t_1^o-md_1'\eta_1q|=||-md_1'\eta_1q||=||md_1'\eta_1q||.$$

As $\eta_1$ is a quadratic irrational number then $||md_1'\eta_1q||\ne0$ and for $t_1\ne t_1^o$ we have $|t_1-md_1'\eta_1q|\ge1/2$. Hence

$$\max_{1\le i\le4}\frac{(q,\,d_i)|t_i-md_i'\eta_iq|}{d_1}\gg\frac{(q,\,d_1)}{d_1}$$

which, together with (32), gives

$$\frac{q}{P}\sum_{|t_i-md_i'q\eta_i|<\frac{M_i}{(q,\,d_i)}}\frac{1}{\max\limits_{1\le i\le4}(q,\,d_i)|t_i-md_i'\eta_iq|/d_i}$$

$$\ll\frac{q}{P}\left(\frac{d_1M_1M_2M_3M_4}{(q,\,d_1)^2(q,\,d_2)(q,\,d_3)(q,\,d_4)}+\frac{d_1M_2M_3M_4}{(q,\,d_1)(q,\,d_2)(q,\,d_3)(q,\,d_4)||md_1'\eta_1q||}\right)$$

$$\ll\frac{qP^{\varepsilon-1}Dd_1d_2d_3d_4}{(q,\,d_1)^2(q,\,d_2)(q,\,d_3)(q,\,d_4)}+\frac{qP^{\varepsilon-1}d_1d_2d_3d_4}{(q,\,d_1)(q,\,d_2)(q,\,d_3)(q,\,d_4)||md_1'\eta_1q||} \tag{33}$$

As $\eta_1$ is quadratic irrationality it has a periodic continued fraction and if $\dfrac{a_n}{b_n}$, $n\in\mathbb{N}$ is $n$-th convergent then $b_n\le c^n$ for some constant $c>0$. Using that $||md_1'q||\le\dfrac{HDQ}{(d_1,\,q)}$ and Liouville's inequality for quadratic numbers (see Lemma 8) we can find convergent

$\dfrac{a}{b}$ to $\eta$ with denominator such that

$$\frac{3HDQ}{(d_1,\,q)} < b \ll_c \frac{HDQ}{(d_1,\,q)}\,. \tag{34}$$

Since $(a,\,b)=1$ we have $md_1'q\dfrac{a}{b}\notin\mathbb{Z}$. As $\left|\eta_1-\dfrac{a}{b}\right|<\dfrac{1}{b^2}$ and (34) we get

$$\begin{aligned}
||md_1'q\eta_1|| &\geq \left|\left|md_1'q\frac{a}{b}\right|\right| - \left|\left|md_1'q\Big(\eta_1-\frac{a}{b}\Big)\right|\right| \geq \left|\left|md_1'q\frac{a}{b}\right|\right| - \frac{|m|d_1'q}{b^2}\\
&> \frac{1}{b}-\frac{|m|d_1'q(d_1,\,q)}{3bHDQ} \geq \frac{1}{b}-\frac{|m|d_1q}{3bHDQ}\\
&> \frac{1}{b}-\frac{|m|}{3bH} \geq \frac{1}{b}-\frac{1}{3b}=\frac{2}{3b}\\
&\gg \frac{(d_1,\,q)}{HDQ}.
\end{aligned}$$

From (33) and (32) it follows that

$$\sum_{\substack{|n_i-md_iq\eta_i|<M_i\\ (q,\,d_i)|n_i,\,i=1,\ldots,4}} \frac{1}{\big|(m\vec{\eta}-\frac{\vec{n}}{\vec{d}q})P\big|} \ll \frac{qP^{\varepsilon-1}d_1d_2d_3d_4HDQ}{(q,\,d_1)^2(q,\,d_2)(q,\,d_3)(q,\,d_4)}\,.$$

Then for $F_1$ (see (31))we receive

$$F_1 \ll \frac{P^{1+\varepsilon}DQ}{\delta}\sum_{q\leq Q}\frac{\tau(q)(q,\,N)^{1/2}}{q^{1/2}} \tag{35}$$

Applying Cauchy's inequality we get

$$\begin{aligned}
F_1 &\ll \frac{P^{1+\varepsilon}DQ}{\delta}\left(\sum_{q\leq Q}\tau^2(q)\right)^{\frac12}\left(\sum_{q\leq Q}\frac{(q,N)}{q}\right)^{\frac12}\\
&\ll \frac{P^{1+\varepsilon}DQ}{\delta}\cdot Q^{1/2}(\log Q)^{3/2}\left(\sum_{\substack{t|N\\ t\leq Q}}\sum_{q_1\leq\frac{Q}{t}}\frac{1}{q_1}\right)^{\frac12}\\
&\ll \frac{P^{1+\varepsilon}DQ^{3/2}}{\delta}
\end{aligned} \tag{36}$$

We choose $Q=\delta^{1/2}P^{1/2}D^{-1/2}$. Then

$$F_1,\,F_2 \ll P^{7/4+\varepsilon}\delta^{-1/4}D^{1/4}\,.$$

From the inequalities, (28), (27), (26), (37) it follows that

$$\Gamma_1^* \ll D^{17/4}P^{7/4+\varepsilon}\delta^{-1/4}\,. \tag{37}$$

***5.4. End of the proof of Theorem.*** From (22) we have

$$\Gamma \geq 4\Gamma_1^0-3\Gamma_5^0+O\big(\Gamma_1^*\big)+O\big(\Gamma_5^*\big)+O(1).$$

According to [8] and [1] for $D\leq P^{1/8-\varepsilon}$, $s=\dfrac{\log D}{\log z}=3.13$ we obtain the estimate

$$4\Gamma_1^0 - 3\Gamma_5^0 \gg \frac{C\delta N}{(\log N)^4} + O\big(\delta P^{3/2+\varepsilon} D^4\big) \tag{38}$$

with some constant $C$.

Since the sum $\Gamma_5^*$ is estimated in the same way as $\Gamma_1^*$, from (22), (37) and (38) we get

$$\Gamma \gg \frac{\delta N}{(\log N)^4} + D^{17/4} P^{7/4+\varepsilon} \delta^{-1/4}\,.$$

Then for a fixed small $\varepsilon > 0$, $\lambda < \dfrac{1-8\varepsilon}{10}$, $D < N^{\frac{1-10\lambda-8\varepsilon}{34}}$ and $z = D^{1/3,13}$ we get $\Gamma \gg \dfrac{\delta N}{(\log N)^4}$. Therefore, the equation (1) has solutions in almost-prime numbers $x_1, \dots, x_4 \in \mathcal{P}_k$, $k = \left[\dfrac{53,21}{1-10\lambda-8\varepsilon}\right]$ such that $||\eta_1 x_1 + \eta_2 x_2 + \eta_3 x_3 + \eta_4 x_4|| < N^{-\lambda}$.

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## ТЕОРЕМА НА ЛАГРАНЖ С ПОЧТИ ПРОСТИ ЧИСЛА ОТ СПЕЦИАЛЕН ТИП

**Татяна Л. Тодорова**

Разглеждаме проблем, свързан с теоремата на Лагранж за четирите квадрата с *почти прости числа* от подходящ ред, които удовлетворяват диофантово неравенство.