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SOME NOTES ON THE SOLUTIONS OF TRANSPORTATION PROBLEMS*

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In the present paper we correct some inaccuracies in the solutions of three transportation problems of the text-book [1]. In these problems the author indicates only two solutions for every problem. The parametrization of the optimization model of the transportation problem provides more opportunities for companies to take into account additional factors.

The transportation problem is a special application of Linear Programming. The problem deals with the transport of a single product from different sources to different destinations, so that the total transportation cost is minimal [2]. Many companies use the transportation problem even in a three-dimensional version. This motivated our idea to propose a parameterization of the problem. We will start with some necessary information and notations as given in the textbook [1, p. 92] in my translation from Bulgarian: "Let us denote by a_1, a_2, $\dots a_m$ the quantities of homogeneous goods supplied by the producer at the destinations supplied by the producer A_1, A_2, $\dots$, A_m, and by b_1, b_2, $\dots b_n$ the quantities of goods demanded by consumers B_1, B_2, $\dots B_n$. The values of c_{ij} denoting the transportation cost of a unit of cargo from the i-th supplier to the j-th consumer are also known". "Let us denote by x_{ij} the planned amount of cargo from the i-th supplier to the j-th consumer; then we call the matrix $X = (x_{ij})$, $i = \overline{1..m}$, $j = \overline{1..n}$, the *transportation matrix.*

"The mathematical model of the transportation problem is in the form: Minimize the function $Z = c_{11}x_{11} + c_{12}x_{12} + ... + c_{mn}x_{mn} \to \min$, subject to some additional constraints" ([1], pp. 92–93).

The condition of the transportation problem is recorded in a table of the form:

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Table 1 [1, Table 8.1]

user / supplier	B_1	B_2	...	B_n	Supply a_i
A_1	c_{11} x_{11}	c_{12} x_{12}	...	c_{1n} x_{1n}	a_1
A_2	c_{21} x_{21}	c_{22} x_{22}	...	c_{2n} x_{2n}	a_2
...	...	...	...	...	...
A_m	c_{m1} x_{m1}	c_{m2} x_{m2}	...	c_{mn} x_{mn}	a_m
Demand b_j	b_1	b_2	...	b_n	$\sum a_i$ $\sum bj$

Let us consider the following task as formulated in [1, p. 86]:

"Example 8.1. The furniture company owns 4 factories B_1, B_2, B_3, B_4 and for the forthcoming period it must supply them with 30, 70, 70 and 110 m^3 of timber, respectively. In three of its warehouses A_1, A_2, A_3, the company has 30, 150 and 100 m^3 of timber, respectively". Find an optimal plan of the transportation costs for the transportation of timber from the warehouses to the factories.

There seems to be a gap in [1] in the solution of the problem. In our opinion, the solution should be look as follows:

Table 2 [1, Table 8.6]

Factories / Warehouses	B_1 $v_1 = 3$	B_2 $v_2 = 1$	B_3 $v_3 = 9$	B_4 $v_4 = 5$	Supply a_i
A_1 $u_1 = 0$	3 t	8 –	10 –	5 $30 - t$	30
A_2 $u_2 = -3$	1 –	4 –	6 $70 - t$	2 $80 + t$	150
A_3 $u_3 = 0$	3 $30 - t$	1 70	9 T	7 –	100
Demand	30	70	70	110	

In [1] only two solutions of the task are found, i.e. only two optimal matrices for the transportation costs:

$$X'_{opt} = \begin{pmatrix} 0 & 0 & 0 & 30 \\ 0 & 0 & 70 & 80 \\ 30 & 70 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X''_{opt} = \begin{pmatrix} 30 & 0 & 0 & 0 \\ 0 & 0 & 40 & 110 \\ 0 & 70 & 30 & 0 \end{pmatrix}.$$

The second matrix X''_{opt} is called an *alternative solution.*

In addition, the optimal value Z_2 of the shipments is obtained in [1], namely $Z_2 = 30.5 + 70.6 + 80.2 + 30.3 + 70 = 890$ BGN.

If we denote by Z^t the value of the shipments in Table 2, then $0 \leq t \leq 30$, and we get:

$$Z^t = 3t + 5(30-t) + 6(70-t) + 2(80+t) + 3(30-t) + 1.70 + 9t =$$
$$= 5.30 + 6.70 + 2.80 + 3.30 + 70 + 3t - 5t - 6t + 2t - 3t + 9t = 890 = Z_2,$$

i.e. Z^t is the optimal transportation cost for $\forall t$, $0 \le t \le 30$.

Thus, from Table 2 we obtain the following optimal transportation matrix X^t_{opt}:

$$X^t_{opt} = \begin{pmatrix} t & 0 & 0 & 30-t \\ 0 & 0 & 70-t & 80+t \\ 30-t & 70 & t & 0 \end{pmatrix}, \quad 0 \le t \le 30.$$

It is clear that X^t_{opt}, unlike [1], yields innumerable optimal solutions (not necessarily integers) for the transportation costs, since t changes in the interval [0; 30].

Especially for $t = 0$ and $t = 30$, we obtain the optimal matrix X'_{opt} and the alternative solution X''_{opt} of [1], respectively.

Next we consider the following task [1, p. 92]:

"Example 8.3. A company is fulfilling a contract for the export of wheat, and it must transport for a certain period to 4 ports B_1, B_2, B_3, B_4, respectively 190, 220, 200 and 190 tons of wheat. At the same time, the company has provided in three different silos A_1, A_2 and A_3, the quantities of 250, 240 and 280 tons, respectively. The quantities available must be exported at minimum total transportation costs" ([1], p. 92).

The final table is (ibid. p. 94):

Table 3 [1, Table 8.13]

Factories / Warehouses	B_1 $v_1 = 3$	B_2 $v_2 = 1$	B_3 $v_3 = 9$	B_4 $v_4 = 5$	Supply a_i
A_1 $u_1 = 3$	2 / $30+t$	3 / $220-t$	6	4	250
A_2 $u_2 = 2$	1 / $160-t$	5	2 / $80+t$	8	240
A_3 $u_3 = 3$	7	4	3 / 90	1 / 190	280
A_f $u_f = 0$	0	0 / t	0 / $30-t$	0	30
Demand	30	70	70	110	800 / 800

The task includes a dummy supplier A_f with availability $a_f = 30$ tons in an additional 4th row of Table 3 with zero transportation costs.

The textbook also contains only two solutions for the optimal transportation matrices, namely

$$X'_{opt} = \begin{pmatrix} 30 & 220 & 0 & 0 \\ 160 & 0 & 80 & 0 \\ 0 & 0 & 90 & 190 \\ \hline 0 & 0 & 30 & 0 \end{pmatrix}$$

and the alternative solution

$$X''_{opt} = \begin{pmatrix} 60 & 190 & 0 & 0 \\ 130 & 0 & 110 & 0 \\ 0 & 0 & 90 & 190 \\ \hline 0 & 30 & 0 & 0 \end{pmatrix}.$$

The hypothetical (dummy) supplier added to the matrix is separated below the line.

In [1] the optimal value $Z_{\min}$ of the transportations is also determined, namely $Z_{\min} = 1500$ BGN.

If we denote by Z^t the value of the transportation cost, which depend on t, $0 \le t \le 30$ (not necessarily integers), then from Table 3 we calculate $Z^t = Z_{\min} - 3t + 2t - t + 2t = Z_{\min} = 1500$, i.e. Z^t is the optimal value for $\forall t$, $0 \le t \le 30$. Thus we obtain, in contrast to [1], infinitely many optimal transportation matrices X^t_{opt}, namely

$$X^t_{opt} = \begin{pmatrix} 30+t & 220-t & 0 & 0 \\ 160-t & 0 & 80+t & 0 \\ 0 & 0 & 90 & 190 \\ \hline 0 & t & 30-t & 0 \end{pmatrix}, \quad 0 \le t \le 30.$$

Since t changes in the interval $[0; 30]$, then for $t = 0$ and $t = 30$, we obtain the above matrix X'_{opt} and the alternative solution X''_{opt}, respectively.

In the following problem in [1, p. 97] there are also only two solutions for the optimal matrices:

"Example 8.5. In a given region, the state reserve of wheat is located in three warehouses A_1, A_2, A_3 and due to its expiration date it must be replaced with new wheat. The old one must be transported to four mills B_1, B_2, B_3 and B_4, so that mill B_3 must be completely satisfied. The aim is to deliver the appropriate quantities of wheat to the mills at minimal transportation costs".

The final table given in [1, p. 100] is:

Table 4 [1, Table 8.21]

mills / warehouses	B_1 $v_1 = 2$	B_2 $v_2 = 3$	B_3 $v_3 = 5$	B_4 $v_4 = 4$	Supply a_i
A_1 $u_1 = -1$	3	2 / $130 - t$	4 / 30	3 / t	160
A_2 $u_2 = -4$	4	5	1 / 150	2	150
A_3 $u_3 = 0$	2 / 120	3 / $10 + t$	6	4 / $100 - t$	240
A_f $u_f = -4$	0	0 / t	60	0 / 50	50
Demand	120	140	180	160	600 / 600

The task includes a dummy warehouse A_f with supply $a_f = 50$ and an additional fourth row in Table 4 with zero transportation costs except for cell A_fB_3 with a transportation cost of 60.

Only optimal transportation matrices are defined in [1], namely the matrices

$$X'_{opt} = \begin{pmatrix} 0 & 130 & 30 & 0 \\ 0 & 0 & 150 & 0 \\ 120 & 10 & 0 & 110 \\ \hline 0 & 0 & 0 & 50 \end{pmatrix}$$

and ‘the alternative’ solution

$$X''_{opt} = \left(\begin{array}{cccc} 0 & 120 & 30 & 10 \\ 0 & 0 & 150 & 0 \\ 120 & 20 & 0 & 100 \\ \hline 0 & 0 & 0 & 50 \end{array} \right).$$

In addition, an optimal transportation value is obtained in [1], namely $Z_2 = 1240$.

If we denote by Z^t the value of the transportation cost, which depends on the parameter t, in Table 8.21, $0 \leq t \leq 110$ (not necessarily integers), then we obtain from it $Z^t = Z_2 + 3t - 4t + 3t - 2t = Z_2 = 1240$, i.e. Z^t is the optimal value of the transportation cost for $\forall t$, $0 \leq t \leq 110$. Then from Table 4 we get the following optimal matrix X^t_{opt} of the transportation,

$$X'_{opt} = \left(\begin{array}{cccc} 0 & 130 - t & 30 & t \\ 0 & 0 & 150 & 0 \\ 120 & 10 + t & 0 & 110 - t \\ \hline 0 & 0 & 0 & 50 \end{array} \right), \quad 0 \leq t \leq 110.$$

For $t = 0$ and $t = 10$, we obtain the optimal matrix X'_{opt} and the “alternative” solution X''_{opt} given in [1], respectively. Judging by Examples 8.1 and 8.3 of [1], the “alternative” solution should be obtained for $t = 110$ and not for $t = 10$.

It is clear that in the three problems considered the matrices X^t_{opt} yield infinitely many solutions for the values of the optimal transportation matrices, since t spans in a finite non-empty interval.

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НЯКОИ БЕЛЕЖКИ ВЪРХУ РЕШЕНИЯ НА ТРАНСПОРТНИ ЗАДАЧИ

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В настоящата статия разглеждаме някои неточности и непълноти при решаването на транспортни задачи. Авторите посочват само някои от решенията, а не всички възможни. Доказваме, че за всяка от посочените задачи съществуват безброй много решения, които зависят от параметър t, така че стойностите на t са в краен интервал. Параметризацията на оптимизационния модел на транспортната задача дава повече възможности на фирмите за съобразяване и с други фактори.