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BRIDGING THE GAP BETWEEN SCHOOL AND COMPETITIVE MATHEMATICS: AN EXAMPLE

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The paper discusses the gap between the mathematics taught and learnt in school and the mathematics needed for competitions. A problem from a mathematics competition is used as an example of how this gap could be bridged.

Introduction. There are many ways of measuring the performance of students in mathematics, that range from large-scale international assessments, such as PISA [1] and TIMSS [2] to various mathematical olympiads and competitions, the most prestigious one being the International Mathematical Olympiad (IMO) [3]. A low PISA or TIMSS result generally causes concerns and provokes multiple initiatives to improve students' mathematical abilities country-wise. At the same time, good performance or a medal from IMO (or another similar event) is a reason for pride. And somehow, the students in-between, the "general", the average, in fact, the majority of the students, remain somewhat unnoticed. While intense training camps with the best trainers are available for the best dozen or two of students, and few hundreds can benefit from regular after-school or weekend sessions organized by their schools, there seems to be a lack of resources that bridge the gap between the mathematical knowledge and skills taught at school on one hand, and the ones needed for competitions, on the other. This is explored in more detail in the following sections.

A Specific Example. The rest of this paper will use Problem 1 for Grade 8 of the Bulgarian Fall Math Tournament 2019. The original text of the problem and the official solution (both in Bulgarian) can be found in [4].

Find all integer values of $\boldsymbol{x}$ *such that the value of the expression*

$$M = \left|x^3 - 2x^2 - 10x + 8\right|$$

is a prime number.

Official Solution (literal translation) Factor $M = |x-4|\left|x^2+2x-2\right|$. M is a prime number if one of the factors is equal to 1 and the other one is a prime number.
Case 1: $|x-4| = 1 \implies x \in \{3,5\}$. If $x=3$, we obtain $M=13$, which is a prime number. If $x=5$, we obtain $M=33$, which is not a prime number.
Case 2: $\left|x^2+2x-2\right| = 1 \implies x^2+2x-2 = \pm 1$. From the first equation, we obtain $x=1$ or $x=-3$, and, respectively, $M=3$ or $M=7$, which are prime numbers. The second equation is equivalent to $(x+1)^2 = 2$, which has no integer solutions.
Finally, the solutions are $x=1$ and $x=\pm 3$.

The official solution in the competition booklet only shows the main steps and results. Details on methods used and intermediate computations are omitted[1]. This should be enough for students coming out of the competition after spending 4.5 hours thinking on this problem (and three others) to check their answer and/or predict their scores, although the results [5] suggest otherwise, as almost half of the students scored 0 points on this problem. This may also be enough for the aforementioned few hundred students, and their teachers, in preparation for upcoming competitions. However, some of the main questions to be asked here are if the type of solution proposed in [4] is sufficient for:

- a "regular" student who simply likes mathematics and enjoys solving problems that are more complex and perhaps more difficult than the textbook? Because there are problems, like the one in question, that do not require gaining additional knowledge, but rather using already acquired knowledge in different ways and/or contexts.
- students wishing to take part in competitions (and stand a chance to win) but do not have access to training at the proper level?
  and
- teachers who do not have prior experience with competitions or are not confident in their competition subject knowledge to be able to transfer these skills and knowledge to their students?

Prerequisites. Below is a list of the prerequisites required to solve the problem:

- absolute value – definition and basic properties; solving absolute value equations (linear and quadratic)
- prime numbers – definition, determining if a number is prime/knowing the first few prime numbers
- factor polynomials – this can be done by using any valid method, such as
  - factor by grouping
  - factor by using long division
  - factor by using synthetic division
- solve linear equations
- solve quadratic equations – using the quadratic formula and the discriminant will be enough to solve any quadratic equation, however, being able to solve quadratic equations by grouping or by completing the square may be easier in certain situations
- factor quadratic trinomials
- evaluate algebraic expressions.

It must be noted that students will most likely face all these topics[2] in school. However, the following points should be considered:

- These topics are not usually treated in the same grade level.
- The examples used in the textbook might or most likely will differ in their difficulty.
- Textbook examples do not usually use that many pieces of knowledge within the same example.

[1]While this paper focuses on one particular problem, many of the statements are valid in general. Methods considered simpler or easier and intermediate computations are usually omitted. Another example can be given in geometry, where quite often the competition booklet does not include a graph, let alone multiple graphs showing different parts of the big picture or steps of the problem.

[2]Or at least enough topics to be able to fully solve the problem. For example, students do not need to be able to factor polynomials by using all the methods listed above, as long as they are able to successfully use one of them.

A Very Detailed Solution. Below is a suggested solution that shows every step and every computation. A comparison between the original solution and this detailed solution is done afterwards.

Factor M (this can be done by any valid method, factoring by grouping is shown below):

$$\begin{aligned} M &= \left|x^3 - 2x^2 - 10x + 8\right| \\ &= \left|x^3 - 4x^2 + 2x^2 - 8x - 2x + 8\right| \\ &= \left|x^2(x-4) + 2x(x-4) - 2(x-4)\right| \\ &= \left|(x-4)\left(x^2 + 2x - 2\right)\right| \\ &= |x-4|\left|x^2 + 2x - 2\right| \end{aligned}$$

The value of M is a prime number if and only if one of the factors $|x-4|$ and $\left|x^2+2x-2\right|$ has a value of 1 and the other is a prime number.

Case 1:
$|x-4| = 1$
Therefore,

$$\begin{array}{lll} x - 4 = 1 & \text{or} & x - 4 = -1 \\ x = 1 + 4 & \text{or} & x = -1 + 4 \\ x = 5 & \text{or} & x = 3 \end{array}$$

Case 1.1:
If $x = 5$, then

$$\begin{aligned} \left|x^2 + 2x - 2\right|_{x=5} &= \left|5^2 + 2 \cdot 5 - 2\right| \\ &= |25 + 10 - 2| \\ &= |33| \\ &= 33 \end{aligned}$$

Since 33 is **not** a prime number ($33 = 3 \cdot 11$), $x = 5$ is **not** a solution.

Case 1.2:
If $x = 3$, then

$$\begin{aligned} \left|x^2 + 2x - 2\right|_{x=3} &= \left|3^2 + 2 \cdot 3 - 2\right| \\ &= |9 + 6 - 2| \\ &= |13| \\ &= 13 \end{aligned}$$

Since 13 is a prime number, $x = 3$ is a solution.

Case 2:
$\left|x^2 + 2x - 2\right| = 1$
Therefore, $x^2 + 2x - 2 = 1$ or $x^2 + 2x - 2 = -1$.

Case 2.1:

$$x^2+2x-2=1$$
$$x^2+2x-2-1=0$$
$$x^2+2x-3=0$$
$$x^2-x+3x-3=0$$
$$x(x-1)+3(x-1)=0$$
$$(x-1)(x+3)=0$$

$$x-1=0 \quad \text{or} \quad x+3=0$$
$$x=1 \quad \text{or} \quad x=-3$$

Case 2.1.1:
If $x=1$, then

$$\begin{aligned} |x-4|_{x=1} &= |1-4| \\ &= |-3| \\ &= 3 \end{aligned}$$

Since 3 is a prime number, $x=1$ is a solution.

Case 2.1.2:
If $x=-3$, then

$$\begin{aligned} |x-4|_{x=-3} &= |-3-4| \\ &= |-7| \\ &= 7 \end{aligned}$$

Since 7 is a prime number, $x=-3$ is a solution.

Case 2.2:

$$x^2+2x-2=-1$$
$$x^2+2x-2+3=-1+3$$
$$x^2+2x+1=2$$
$$(x+1)^2=2$$

Since 2 is **not** a perfect square, the above equation does **not** have rational solutions, therefore it does **not** have integer solutions.

Hence, the values of x that satisfy the problem are $x=-3$, $x=1$, and $x=3$.

Table 1. Comparison of the Two Solutions

Step	Original Solution	Detailed Solution
factorized expression for M	directly given	all steps shown, method used specified
linear absolute value equation	answer directly given	solution with all intermediate steps shown
quadratic equations that follow from the quadratic absolute value equation	answers directly given	each equation solved, all intermediate steps shown
value of M for a specific value of x	answers directly given	all intermediate computations shown

Closing Notes on the Two Solutions. When comparing the two solutions, the following should be taken into account:

- Is the solution used in a printed book or as a part of an online resource? Omitting details and intermediate steps in a printed book could be due to limitations for the size of the printed book. In this case, a companion can be provided, online or in print.
- What is the target audience of the book/resource? If it is the few students participating in competitions, they would most likely not need very detailed explanations. However, if the book/resource is intended for a general audience, a detailed solution could be prepared, with the most important points (the ones from the official solution) highlighted.

Conclusion. Mathematical competitions, or at least the problems from such, can be made accessible to a larger population of students: the ones not interested in competing but interested in the mathematics behind these questions, and the ones who want and have the potential to compete but lack resources (mainly proper training). Using detailed solutions can help bridging the gap between school mathematics and mathematical competitions by:

- showing all intermediate steps and computations and avoiding "it is easy to show that ..." and "it is obvious that ..." statements, which could sometimes be tricky to show[3]
- specifying the methods used and therefore allowing the reader to find further information and practice problems on the topic
- providing a list of prerequisites or required knowledge for each problem which can help teachers who are new to competitions or ones who are not confident in their competition subject knowledge, to better prepare for after-school or weekend sessions. A list of prerequisites, or a few problems for each piece of required knowledge, can also be given to the students to prepare prior to such sessions.

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[4] Есенен математически турнир "Акад. Стефан Додунеков", София, 2019 г. (Fall Math Tournament "Acad. Stefan Dodunekov", Sofia, 2019)

[3] In the example shown, each of the omitted steps is worth a few lines. However, in other examples, it may take multiple pages to prove something "obvious".

[5] Тринадесети Есенен Математически Турнир “Акад. Стефан Додунеков”, 2019 г., Окончателен протокол, 8 клас, (Thirteenth Fall Math Tournament “Stefan Dodunekov”. 2019, final protocol, 8th grade) https://shorturl.at/efuwD, available through https://shorturl.at/alqFO
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ПРЕОДОЛЯВАНЕ НА ПРОПАСТТА МЕЖДУ УЧИЛИЩНАТА И СЪСТЕЗАТЕЛНАТА МАТЕМАТИКА: ПРИМЕР

Надежда Аплакова

В настоящата разработка се дискутира темата за разликата между математиката, преподавана и изучавана в училище, и математиката, необходима за математически състезания. Една задача от математическо състезание е използвана като пример как тази разлика може да бъде преодоляна.