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ON CERTAIN INEQUALITIES FOR REAL-ROOT POLYNOMIALS

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We review certain inequalities satisfied by real-root polynomials, which are a refinement of the Jensen inequalities for functions from the Laguerre–Polya class. Then we demonstrate how these inequalities yield bounds for the zeros of classical orthogonal polynomials.

1. The Laguerre-Pólya class and inequalities of Jensen. A real entire function φ belongs to the Laguerre-Pólya class $\mathcal{LP}$ if φ admits representation of the form

$$\varphi(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{k=1}^{\omega} \Big(1 + \frac{x}{x_k}\Big) e^{-x/x_k}, \qquad 0 \le \omega \le \infty,$$

where $c, \beta, x_k \in \mathbb{R}$, $\alpha \ge 0$, n is a non-negative integer and $\sum_{k=1}^{\omega} x_k^{-2} < \infty$. The importance of the Laguerre-Pólya class $\mathcal{LP}$ stems from the fact that functions in this class, and only these, are uniform limits on compact subsets of $\mathbb{C}$, of sequences of algebraic polynomials having only real zeros (see, e.g., [9, Chapter 8]). In particular, $\mathcal{LP}$ encompasses the set $\mathcal{RP}$ of all real–valued algebraic polynomials with only real zeros. Throughout, $\mathcal{RP}_n$ will stand for the subclass of $\mathcal{RP}$ of polynomials of degree not exceeding n.

The entire functions from the Laguerre-Pólya class play an important role in mathematics. The reader is referred to [17] for various properties and characterizations of the functions from $\mathcal{LP}$.

Jensen [7] proved that if $f \in \mathcal{LP}$, then

$$\mathcal{L}_m(f;x) := \sum_{j=0}^{2m} (-1)^{m+j} \binom{2m}{j} f^{(j)}(x) f^{(2m-j)}(x) \ge 0 \quad \text{for all} \quad x \in \mathbb{R}. \tag{1}$$

The quantities $\mathcal{L}_m(f;x)$ appear in the MacLaurin series for $\big|f(z)\big|^2$, where $z = x + iy$ and $x, y \in \mathbb{R}$:

$$\big|f(z)\big|^2 = \sum_{m=0}^{\infty} \mathcal{L}_m(f;x) \frac{y^{2m}}{(2m)!}. \tag{2}$$

Formula (1) readily follows from the Leibnitz rule applied to $f(x+iy)f(x-iy)$.

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The particular cases of (1) obtained with $m = 1$ and $m = 2$ are respectively the well-known inequality of Laguerre [8, p. 17]

$$\left[f'(x)\right]^2 - f(x)f''(x) \geq 0, \qquad f \in \mathcal{LP},\ x \in \mathbb{R}, \tag{3}$$

and the inequality

$$3\left[f''(x)\right]^2 - 4f'(x)f'''(x) + f(x)f^{(4)}(x) \geq 0, \qquad f \in \mathcal{LP},\ x \in \mathbb{R}. \tag{4}$$

In the early seventies of the last century inequalities (1) were rediscovered by Patrick [15, 16]. Csordas and Varga [3] proved that the validity of (1) for all $m \in \mathbb{N}_0$ implies that f has only real roots, provided $f(z) = e^{-\alpha z^2} f_1(z)$, $\alpha \geq 0$, and f_1 is a real entire function of genus 0 or 1 (see also [2, Theorem 2.2]). Thus, in these cases the inequalities (1) provide a characterization of the Laguerre-Pólya class. Moreover, Csordas and Varga used (1) to formulate some necessary and sufficient conditions for the validity of the Riemann hypothesis. The importance of (1) in the study of $\mathcal{LP}$ and its relation to the Turán type inequalities was investigated by Craven and Csordas [1].

Of course, Jensen inequalities hold true in the more narrow class $\mathcal{RP}$, in which case the infinite sum in (2) terminates. Moreover, if f is a polynomial of degree n, then the condition $\mathcal{L}_m(f;x) \geq 0$ for $m = 0, 1, \dots, n$, is necessary and sufficient for $f \in \mathcal{RP}_n$. If $f(x) = (x - x_1)(x - x_2)\dots(x - x_n) \in \mathcal{RP}_n$, then

$$|f(x+iy)|^2 = \prod_{k=1}^{n} \left((x - x_k)^2 + y^2\right),$$

and comparison of the coefficients of y^{2m} here and in (2) implies the positivity of $\mathcal{L}_m(f;x)$ through the following explicit formula:

$$\mathcal{L}_m(f;x) = (2m)!f^2(x) \sum \frac{1}{(x - x_{i_1})^2 \dots (x - x_{i_m})^2}, \tag{5}$$

where the sum is over all m-combinations $\{i_1, \dots, i_m\}$ of $\{1, 2, \dots, n\}$.

2. Refinements for real-root polynomials. As it may be expected, for real-root polynomials Jensen's inequalities can be sharpened. We begin with the following rather straightforward refinement of (1):

Theorem 2.1. *If $m \in \mathbb{N}$ and $f \in \mathcal{RP}_n$, where $n \geq m$, then*

$$\mathcal{L}_m(f;x) \geq \frac{\binom{2m}{m}}{\binom{n}{m}} \left[f^{(m)}(x)\right]^2 \quad \text{for every} \quad x \in \mathbb{R}. \tag{6}$$

Moreover, the equality in (6) *holds for every $x \in \mathbb{R}$ if f is either a polynomial of degree less than m or $f(x) = c(x - a)^n$ with $a, c \in \mathbb{R}$.*

Proof. The claim of Theorem 2.1 is clear, if $\deg f < m$. If $f(x) = (x - x_1)\dots(x - x_n)$, where $n \geq m$ and $\{x_i\}_{i=1}^n$ are real, then application of the quadratic mean – arithmetic mean inequality to (5) implies

$$\begin{aligned}
\mathcal{L}_m(f;x) &= (2m)!\, f^2(x) \sum \frac{1}{(x-x_{i_1})^2 \dots (x-x_{i_m})^2} \\
&= (2m)! \sum (x-x_{i_1})^2 \dots (x-x_{i_{n-m}})^2 \\
&\geq (2m)!\, \frac{\big(\sum (x-x_{i_1}) \dots (x-x_{i_{n-m}})\big)^2}{\binom{n}{m}} \\
&= (2m)!\, \frac{\big[f^{(m)}(x)\big]^2}{m!^2 \binom{n}{m}} = \frac{\binom{2m}{m}}{\binom{n}{m}} \big[f^{(m)}(x)\big]^2,
\end{aligned}$$

and the equality occurs if and only if $x_1 = x_2 = \ldots = x_n$. On the other hand, if $f \in \mathcal{RP}_n$ is of degree k, $m \leq k < n$, then by the above argument we get

$$\mathcal{L}_m(f;x) \geq \frac{\binom{2m}{m}}{\binom{k}{m}} \big[f^{(m)}(x)\big]^2 \geq \frac{\binom{2m}{m}}{\binom{n}{m}} \big[f^{(m)}(x)\big]^2$$

and the last inequality is strict if $f^{(m)}(x) \neq 0$. Theorem 1 is proved. $\square$

In the particular case $m = 1$, Theorem 2.1 provides the following improvement of the Laguerre inequality:

$$(n-1)\big[f'(x)\big]^2 - n f(x) f''(x) \geq 0, \quad x \in \mathbb{R}, \quad f \in \mathcal{RP}_n. \tag{7}$$

Inequality (7) has been proposed as a problem in [10]. It should be pointed out however that this inequality is much older. By a shift of the variable it is seen that (7) is equivalent to the same inequality for $x = 0$ only, thus, (7) provides an inequality for the coefficients of a real-root polynomial. Already Newton was aware (see, e.g., [6]) that, if $f(x) = \prod_{k=1}^{n}(x + x_k) = x^n + c_1 x^{n-1} + \cdots + c_n$, $\{x_k\}_1^n$ - real, then the coefficients of f must satisfy

$$c_k^2 \geq \frac{k+1}{k} \frac{n-k+1}{n-k} c_{k-1} c_{k+1} \quad \text{for} \quad k = 1, \ldots, n-1. \tag{8}$$

In fact, inequalities (8) are a consequence from (7), they follow by application of (7) to the derivatives of f (which are also real-root polynomials) with $x = 0$.

There is also an improvement of (4) for real-root polynomials, different from the one provided by Theorem 2.1 with $m = 2$, it reads as follows: for every $f \in \mathcal{RP}_n$ and $x \in \mathbb{R}$,

$$3(n-2)(n-3)\big[f''(x)\big]^2 - 4(n-1)(n-3) f'(x) f'''(x) + n(n-1) f(x) f^{(4)}(x) \geq 0. \tag{9}$$

A proof of (9) was given by Foster and Krasikov in [5]. Again, they were not the first to discover this inequality, as in 1963 Academician Nikola Obreshkov, in his last published work [13], proved (9), among others. Foster and Krasikov should not be blamed for being unaware of Obreshkov's work, as it was written in Bulgarian (now an English translation of [13] is included as Addendum in [14]). In their interesting paper [5], Foster and Krasikov proposed a conjecture, which contains the inequalities (7) and (9) as particular cases. In order to formulate this conjecture, let us set

$$U_{2m}^n(f;x) := \sum_{j=0}^{2m} (-1)^{m+j} \binom{2m}{j} \frac{(n-j)!(n-2m+j)!}{(n-m)!(n-2m)!} f^{(j)}(x) f^{(2m-j)}(x). \tag{10}$$

Conjecture 2.2 (Foster and Krasikov [5]). *Let $f \in \mathcal{RP}_n$; then for any integer m, $0 \le 2m \le n$,*

$$U_{2m}^n(f;x) \ge 0 \quad \text{for every} \quad x \in \mathbb{R}.$$

Conjecture 2.2 has been validated by Nikolov and Uluchev in [12], where thr following theorem was proved.

Theorem 2.3 ([12]). (i) *Let $f \in \mathcal{RP}_n$; then for any integer m, $0 \le 2m \le n$,*

$$U_{2m}^n(f;x) \ge 0 \quad \text{for every} \quad x \in \mathbb{R}. \tag{11}$$

(ii) *If $f(x) = c(x-x_1)(x-x_2)\dots(x-x_n)$, then*

$$U_{2m}^n(f;x) = (2m)!f^2(x)\sum\Big(\frac{1}{x-x_{i_1}}-\frac{1}{x-x_{i_2}}\Big)^2\dots\Big(\frac{1}{x-x_{i_{2m-1}}}-\frac{1}{x-x_{i_{2m}}}\Big)^2,$$

where the sum on the right-hand side is symmetric and is extended over all m-tuples $\{\{i_1,i_2\},\dots,\{i_{2m-1},i_{2m}\}\}$ of pairs of distinct indices among $\{1,\dots,n\}$.

(iii) *For $m \ge 1$, the inequality in* (11) *becomes an identity if and only if either f is of degree less than m or f has a zero of multiplicity at least $n-m+1 \ge 0$, i.e., $f(x) = (x-a)^{n-m+1}f_1(x)$, where $f_1 \in \mathcal{RP}_{m-1}$.*

For the reader's convenience, we present a short proof of Conjecture 2.2, based on a beautiful result of Academician N. Obreshkov. In his last published paper [13], Obreshkov showed that if f_1 and f_2 are polynomials of degree p and q, respectively, and m is a positive integer, not exceeding p and q, then the polynomial

$$g_m(z) = \sum_{j=0}^{m}(-1)^j\binom{p-j}{m-j}\binom{q-m+j}{j}f_1^{(j)}(z)f_2^{(m-j)}(z) \tag{12}$$

is their covariant. Let us point out that, although at a first sight the degree of g_m seems to be $p+q-m$, actually it does not exceed $p+q-2m$. Obreshkov proved the following theorem:

Theorem 2.4 ([13, Theorem 6]). *Let the zeros of the polynomial $f_1(z)$ of degree p be located in a circular domain K_1, and let the zeros of the polynomial $f_2(z)$ of degree q be located in a circular domain K_2, which has no common points with K_1. Let m be a positive integer, not exceeding p and q. Then the polynomial* (12) *has exactly $p-m$ zeros in K_1, exactly $q-m$ zeros in K_2, and there are no zeros of* (12) *outside K_1 and K_2.*

(As usual, by a circular domain it is meant the interior or the exterior of a circle in the complex plane $\mathbb{C}$, or one of the two open half-planes, determined by a straight line in $\mathbb{C}$).

To prove Conjecture 2.2, in Theorem 2.4 we replace m by $2m$, put $p = q = n \ge 2m$ and set $f_1(z) := f(z+i\varepsilon)$, $f_2(z) := f(z-i\varepsilon)$, where $f \in \mathcal{RP}$ is of degree n and $\varepsilon > 0$. We may think that K_1 and K_2 are the half-planes $\Im(z) < -\varepsilon/2$ and $\Im(z) > \varepsilon/2$, respectively. In this situation the polynomial (12) becomes

$$g_{2m}(z) = \frac{(-1)^m}{(2m)!}\sum_{j=0}^{2m}(-1)^{m+j}\binom{2m}{j}\frac{(n-j)!\,(n-2m+j)!}{(n-m)!\,(n-2m)!}f_1^{(j)}(z)f_2^{(2m-j)}(z).$$

According to Theorem 2.4, g_{2m} has exactly $n-2m$ zeros in K_1, exactly $n-2m$ zeros in K_2, and has no zeros in the strip $-\varepsilon/2 \le \Im(z) \le \varepsilon/2$, in particular, g_{2m} has no real zeros. Since $g_{2m}(x)$ is real for $x \in \mathbb{R}$, by virtue of the definition of f_1 and f_2, we conclude that $g_{2m}(x)$ has a permanent sign on $\mathbb{R}$. Moreover, this sign does not depend on the concrete positions of $x_1, \dots, x_n$, the zeros of f, as $g_{2m}(x)$ is a continuous function of $x_1, \dots, x_n$. Therefore, by choosing f of the form $f(x) = (x-x_1)\dots(x-x_k)(x/M-1)^{n-k}$ and letting $M \to \infty$, we see that the above conclusion remains true for any $f \in \mathcal{RP}_n$, i.e., *for any* $f \in \mathcal{RP}_n$, $\operatorname{sign} g_{2m}(x)$ *is a constant, independent of the specific positions of the zeros of* f. By substituting $f(x) = x^m$, we find the sign in question:

$$\operatorname{sign} g_{2m}(x) = (-1)^m \quad \text{for every} \quad f \in \mathcal{RP}_n \quad \text{and} \quad x \in \mathbb{R}.$$

Finally, passage to the limit $\varepsilon \to 0$ implies that

$$\operatorname{sign} U_{2m}^n(f;x) = (-1)^m \operatorname{sign} g_{2m}(x) \geq 0 \quad \text{for every} \quad f \in \mathcal{RP}_n \quad \text{and} \quad x \in \mathbb{R}.$$

This completes the proof of Conjecture 2.2.

We are going now to show that if $f \in \mathcal{RP}_{n_0}$ with $n_0 \geq 2m$, then for every $n \geq n_0$ the inequality $U_{2m}^n(f;x) \geq 0$ is indeed an improvement of the Jensen inequality $\mathcal{L}_m(f;x) \geq 0$.

Theorem 2.5. *Let* $m \in \mathbb{N}$ *and* $f \in \mathcal{RP}_{n_0}$, *where* $n_0 \geq 2m$. *Then, for every fixed* $x \in \mathbb{R}$, *the sequence*

$$\Big\{ n^{-m}\, U_{2m}^n(f;x) \Big\}_{n=n_0}^{\infty}$$

is monotonically increasing

Proof. Let us point out that, under the assumptions of Theorem 2.5, $U_{2m}^n(f;x) \geq 0$ and $U_{2m-2}^{n-1}(f';x) \geq 0$ for every $n \geq n_0$. From the easily verified identity

$$(n+1-m)U_{2m}^{n+1}(f;x) = (n+1)U_{2m}^n(f;x) + 2m(2m-1)U_{2m-2}^{n-1}(f';x)$$

(see [12, Lemma 4]), we conclude that

$$U_{2m}^{n+1}(f;x) \geq \frac{n+1}{n+1-m}\, U_{2m}^n(f;x),$$

and consequently

$$\frac{U_{2m}^{n+1}(f;x)}{(n+1)^m} \geq \frac{n^m}{(n+1-m)(n+1)^{m-1}}\, \frac{U_{2m}^n(f;x)}{n^m}.$$

Hence, to prove Theorem 2.5, it suffices to show that

$$\frac{n^m}{(n+1-m)(n+1)^{m-1}} \geq 1,$$

or, equivalently,

$$\Big(1 - \frac{1}{n+1}\Big)^{m-1} \geq 1 - \frac{m-1}{n}.$$

This is obviously true when $m = 1$, and for $m \geq 2$ it follows from the Bernoulli inequality $(1+x)^{m-1} \geq 1 + (m-1)x$, $x > -1$. Theorem 2.5 is proved. $\square$

It is readily seen that

$$\lim_{n\to\infty} \Big\{ n^{-m}\, U_{2m}^n(f;x) \Big\} = \mathcal{L}_m(f;x),$$

moreover, if $f \in \mathcal{RP}_{n_0}$, then, by Theorem 2.5, the sequence $\Big\{ n^{-m}\, U_{2m}^n(f;x) \Big\}_{n=n_0}^{\infty}$ tends towards $\mathcal{L}_m(f;x)$ *monotonically increasing*. Hence, the inequality $U_{2m}^n(f;x) \geq 0$ implies the inequality $\mathcal{L}_m(f;x) \geq 0$, which shows that for real-root polynomials inequalities (11) are sharper than the corresponding Jensen inequalities.

3. Applications. The inequalities for real-root polynomials (11) can be applied to the derivation of estimates for the extreme zeros of the classical orthogonal polynomials of Jacobi (and, in particular, of ultraspherical polynomials), Laguerre and Hermite ([18, Chapter 4]). This is possible thanks to two facts: 1) they satisfy homogeneous ordinary differential equations of second order and 2) their derivatives are orthogonal polynomials from the same family. The inequality (11) with $m = 2$ implies the following necessary condition for real-root polynomials (see also [18, eqn. (6.2.16)]):

Lemma 3.1. *If f is a real-root polynomial of degree $n \geq 4$ and $f(x_0) = 0$, then*

$$3(n-2)\big[f''(x_0)\big]^2 - 4(n-1)f'(x_0)f'''(x_0) \geq 0. \tag{13}$$

Below we briefly demonstrate the application of Lemma 3.1 for obtaining bounds for the zeros of the classical orthogonal polynomials.

3.1. Ultraspherical polynomials. Recall that the ultraspherical polynomials $\{P_m^{(\lambda)}\}_{m=0}^{\infty}$, $\lambda > -1/2$, are orthogonal in $[-1, 1]$ with respect to the weight function $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$. Particular cases of ultraspherical polynomials, corresponding to $\lambda = 0, 1$ and $1/2$, respectively, are the Chebyshev polynomials of the first and the second kind, and the Legendre polynomials. The derivatives of ultraspherical polynomials are also ultraspherical polynomials, more precisely, $\dfrac{d}{dx}\big\{P_m^{(\lambda)}(x)\big\} = 2\lambda\, P_{m-1}^{(\lambda+1)}(x)$ for $\lambda \neq 0$. They satisfy the second order ODE

$$(1-x^2)f''(x) - (2\lambda+1)x\,f'(x) + n(n+2\lambda)f(x) = 0, \qquad f = P_n^{(\lambda)}. \tag{14}$$

Let $f = P_n^{(\lambda)}$ and assume that $f(x_0) = 0$. We make use of (14) and the differential equation for f',

$$(1-x^2)f'''(x) - (2\lambda+3)x\,f''(x) + (n-1)(n+2\lambda+1)\,f'(x) = 0,$$

to express $f'(x_0)$ and $f'''(x_0)$ in terms of $f''(x_0)$ as follows:

$$f'(x_0) = \frac{1-x_0^2}{(2\lambda+1)x_0}\,f''(x_0),$$

$$f'''(x_0) = \Big[\frac{2\lambda+3)x_0}{1-x_0^2} - \frac{(n-1)(n+2\lambda+1)}{(2\lambda+1)x_0}\Big]\,f''(x_0).$$

Putting these expressions in (13), canceling out the positive factor $\big[f''(x_0)\big]^2$ and solving the resulting inequality with respect to x_0^2, we arrive at the conclusion that for the largest zero $x_{n,n}(\lambda)$ of $P_n^{(\lambda)}$ there holds

$$x_{n,n}^2(\lambda) \leq \frac{(n-1)(n+2\lambda+1)}{(n+\lambda)^2 + 3\lambda + \frac{5}{4} + 3\frac{(\lambda+1/2)^2}{n-1}}.$$

3.2. Jacobi polynomials The Jacobi polynomials $\{P_m^{(\alpha,\beta)}(x)\}_{m=0}^{\infty}$ are orthogonal in $[-1, 1]$ with respect to the weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, where $\alpha,\ \beta > -1$. They satisfy the second order ODE

$$(1-x^2)f''(x) + \big(\beta-\alpha-(\alpha+\beta+2)x\big)\,f'(x) + n(n+\alpha+\beta+1)f(x) = 0, \quad f = P_n^{(\alpha,\beta)}. \tag{15}$$

The derivatives of Jacobi polynomials are also Jacobi polynomials, more precisely, there holds $\dfrac{d}{dx}\big\{P_m^{(\alpha,\beta)}(x)\big\} = \dfrac{1}{2}(m+\alpha+\beta+1)\,P_{m-1}^{(\alpha+1,\beta+1)}(x)$.

Let $f = P_n^{(\alpha,\beta)}$ and $f(x_0) = 0$, then from (15) we find

$$f'(x_0) = \frac{1 - x_0^2}{\alpha - \beta + (\alpha+\beta+2)x_0}\, f''(x_0).$$

By substituting this expression in the ODE for f' with $x = x_0$, we express $f'''(x_0)$ through $f''(x_0)$ as follows:

$$f'''(x_0) = \left(\frac{\alpha - \beta + (\alpha+\beta+4)x_0}{1 - x_0^2} - \frac{(n-1)(n+\alpha+\beta+2)}{\alpha-\beta+(\alpha+\beta+2)x_0} \right) f''(x_0).$$

Now we apply Lemma 3.1 with $f = P_n^{(\alpha,\beta)}$: by substituting the above expressions in (13) and cancelling out the positive factor $\big[f''(x_0)\big]^2$, we obtain the inequality

$$3(n-2) - \frac{4(n-1)\big[\alpha-\beta+(\alpha+\beta+4)x_0\big]}{\alpha-\beta+(\alpha+\beta+2)x_0} + \frac{4(n-1)^2(n+\alpha+\beta+2)(1-x_0^2)}{\big(\alpha-\beta+(\alpha+\beta+2)x_0\big)^2} \geq 0.$$

After multiplication by $\big(\alpha - \beta + (\alpha+\beta+2)x_0\big)^2$ and rearrangement (the usage of a computer algebra software is strongly recommended), we arrive at the inequality

$$A\,z^2 - 2B\,z + C \leq 0, \qquad z = x_0,$$

where

$$\begin{aligned}
A &= (2n+\alpha+\beta)\big[n(2n+\alpha+\beta) + 2(\alpha+\beta+2)\big] > 0,\\
B &= 2(\beta-\alpha)\big[(\alpha+\beta+6)n + 2(\alpha+\beta)\big],\\
C &= -4n^2(n+\alpha+\beta) + \big[(\alpha-\beta)^2 - 8(\alpha+\beta) + 12\big]n + 2(\alpha-\beta)^2 - 4(\alpha+\beta+2).
\end{aligned}$$

After solving this inequality, we conclude that the zeros $\{x_{k,n}(\alpha,\beta)\}_{k=1}^{n}$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ satisfy the inequalities

$$\frac{B - 4(n-1)\sqrt{\Delta}}{A} \leq x_{k,n}(\alpha,\beta) \leq \frac{B - 4(n-1)\sqrt{\Delta}}{A},$$

with A and B as given above and

$$\Delta = n^2(n+\alpha+\beta+1)^2 + (\alpha+1)(\beta+1)\big[n(n+\alpha+\beta+4) + 2(\alpha+\beta)\big].$$

3.3. Laguerre polynomials The Laguerre polynomials $\{L_m^{(\alpha)}(x)\}_{m=0}^{\infty}$ are orthogonal in $(0,\infty)$ with respect to the weight function $w_\alpha(x) = x^\alpha e^{-x}$, where $\alpha > -1$. Specific properties of the Laguerre polynomials are the ODE

$$x\,f''(x) + (\alpha+1-x)f'(x) + n\,f(x) = 0, \qquad f = L_n^{(\alpha)}, \tag{16}$$

and $\dfrac{d}{dx}\big\{L_m^{(\alpha)}(x)\big\} = -L_{m-1}^{(\alpha+1)}(x)$.

Repeating the reasoning from the ultraspherical and the Jacobi case, we set $f = L_n^{(\alpha)}$ and assume $f(x_0) = 0$, then we find

$$f'(x_0) = \frac{x_0}{x_0 - \alpha - 1}\, f''(x_0)$$

$$f'''(x_0) = \left(\frac{x_0 - \alpha - 2}{x_0} - \frac{n-1}{x_0 - \alpha - 1} \right) f''(x_0).$$

By substituting these values in (13), after some simplification we arrive at the inequality

$$-(n+2)(x_0-\alpha-1)^2 + 4(n-1)n(x_0-\alpha-1) + 4(\alpha+1)(n-1)^2 \geq 0.$$

By solving this inequality, we find that the zeros $\{x_{k,n}(\alpha)\}_{k=1}^{n}$ of the Laguerre polynomial $L_n^{(\alpha)}$ are located in the interval

$$\left[\alpha+1+\frac{2(n-1)n}{n+2}\left(1-\sqrt{1+\frac{(\alpha+1)(n+2)}{n^2}}\right), \alpha+1+\frac{2(n-1)n}{n+2}\left(1+\sqrt{1+\frac{(\alpha+1)(n+2)}{n^2}}\right)\right].$$

3.4. Hermite polynomials The Hermite polynomials $\{H_m(x)\}_{m=0}^{\infty}$ are orthogonal in $(-\infty,\infty)$ with respect to the weight function $w(x)=e^{-x^2}$. Specific properties of the Hermite polynomials are $H_m'(x)=2m\,H_{m-1}(x)$ and the ODE

$$f''(x)-2x\,f'(x)+2n\,f(x)=0, \qquad f=H_n.$$

By applying Lemma 3.1 we deduce that the zeros $\{h_{k,n}\}_{k=1}^{n}$ of the n-th Hermite polynomial H_n satisfy the inequality

$$h_{n,k}^2 \le \frac{2(n-2)^2}{n+2}.$$

Of course, one can use inequality (4) instead of Lemma 3.1 to obtain bounds for the zeros of the classical orthogonal polynomials. However, the resulting bounds are less precise, due to the fact that Laguerre's inequalities are consequences from the corresponding real-root inequalities (11). For further results with this approach we refer to [4].

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ВЪРХУ НЯКОИ НЕРАВЕНСТВА ЗА ПОЛИНОМИ С РЕАЛНИ НУЛИ

Гено Николов

В статията се разглеждат някои неравенства за полиноми с реални нули, които представляват уточнения на неравенствата на Йенсен за цели функции от класа на Лагер–Пойа. Демонстрира се как такива неравенства могат да се прилагат за получаване на оценки за крайните нули на класическите ортогонални полиноми.