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THE EVOLVING NOTION OF MULTIPLICITY AS AN INVARIANT IN SINGULARITY THEORY

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To Steven Kleiman on the occasion of his 80th birthday

We survey results on the multiplicity of a complex analytic variety at a point and its generalizations as numerical invariants in singularity theory.

1. Multiplicity of a point. Let $X \subset \mathbb{C}^n$ be a reduced equidimensional complex analytic variety of dimension d. Let x be a point in X and let U be a sufficiently small neighborhood of x. The *multiplicity* at a point $x \in X$ is defined as

$$e(x) := \min |L \cap U \cap X|$$

where L runs over all $n-d$-dimensional linear subspaces of $\mathbb{C}^n$ such that x is an isolated point of $L \cap X$. When X is a hypersurface defined as the zero locus of a holomorphic function f, then locally at x the function f has the Taylor expansion $f = f_m + f_{m+1} + \cdots$ where each f_i is homogeneous polynomial of degree i and $f_m \neq 0$. Then $e(x) = m$ by [22, Prp. 5.10]. A principal goal of *equisingularity theory* is to decide if two complex analytic germs *look alike* in some sense by means of numerical invariants. We say that the germs (X_0, x_0) and (X_1, x_1) in $\mathbb{C}^n$ are *topological equisingular,* or they have the same embedded topological type, if there exists a homeomorphism $\phi\colon (\mathbb{C}^n, x_0) \to (\mathbb{C}^n, x_1)$ and neighborhoods U_0 and U_1 of x_0 and x_1 in $\mathbb{C}^n$, respectively, such that $\phi(U_0 \cap X_0) = U_1 \cap X_1$.

For $x \in X \subset \mathbb{C}^n$, the *link* of x in X is defined as $K := X \cap S_\epsilon^{2n-1}$ where S_ϵ^{2n-1} is a sphere of sufficiently small radius ϵ around x in $\mathbb{C}^n$. If x is an isolated singularity, then K is a compact smooth real manifold. By [21, 2.20] X is locally homeomorphic to a cone over K with vertex x. Understanding the topological type of X near x is the same as understanding K and its embedding in S_ϵ^{2n-1}. One can easily see from here that the cusp $X_0 = \mathbb{V}(y^2 - x^3)$ and the node $X_1 = \mathbb{V}(y^2 - x^3 - x^2)$ have different topological types at $0 \in \mathbb{C}^2$, yet they both have multiplicity 2 at the origin. In the other direction

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we have the famous Zariski conjecture stating that if two reduced hypersurface germs are topologically equisingular, then they have the same multiplicity. The conjecture is still an unsettled problem.

In general, it is a hard problem to decide if two germs are equisingular, unless the germs are part of a family. Then it is somewhat easier to predict when the members of the family are the "same" (equisingular) [29]. The conditions that will guarantee equisingularity depend on the total space of the family. Nevertheless, we would like to control these conditions by fiberwise dependent numerical invariants (multiplicities). In the present paper we will consider families of isolated singularities and conditions that ensure their topological equisingularity.

2. Differential equisingularity. The main notion of equsingularity theory that will be addressed in this work is that of *Whitney equisingularity*, also known as *differential equisingularity*. We will use its algebro-geometric formulation and interpret the problem of finding numerical control for it as a problem of intersection theory. Variations of this problem appear in resolutions of singularities, equiresolutions, numerical control of flatness and existence of simultaneous canonical models.

By a classical result of Whitney [32] every complex analytic variety V can be partitioned into a locally finite family of submanifolds, called *strata*, so that any pair of incident strata gives rise to a family of singularities obtained as the fibers of a retraction to the lower dimensional stratum and such that the total space of the family satisfies certain geometric compatibility conditions which we state below.

More precisely, let $X \subset V$ be a complex analytic variety and Y – a smooth subvariety of X of dimension k such that $X - Y$ is smooth and $(X - Y, Y)$ is a pair of strata. Embed X in $\mathbb{C}^{n+k}$ so that Y contains 0 and $(Y, 0)$ is a linear subspace of dimension k. We say H is a *tangent hyperplane* at $x \in X - Y$ if H is a hyperplane in $\mathbb{C}^{n+k}$ that contains the tangent space $T_x X$. Whitney [32] introduced the following conditions on the regularity of the pair $(X - Y, Y)$ at 0.

Whitney Condition A. Let (x_i) be a sequence of points from $X - Y$ that converges to 0, and suppose that the sequence $\{T_{x_i} X\}$ of tangent hyperplanes has a limit T in the corresponding Grassmannian. Then $T_0 Y \subset T$.

Whitney Condition B. Let (x_i) be a sequence of points from $X - Y$ and let (y_i) be a sequence of points from Y both converging to 0. Suppose that the sequence of secants $(\overline{x_i y_i})$ has limit l and the sequence of tangents $\{T_{x_i} X\}$ has limit T. Then $l \subseteq T$.

Choose an embedding of $(X, 0)$ in $\mathbb{C}^n \times \mathbb{C}^k$, so that $(Y, 0)$ is represented by $0 \times U$, where U is an open neighborhood of 0 in $\mathbb{C}^k$. Let $\mathrm{pr} : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^k$ be the projection. Set $h := \mathrm{pr}_{|X}$. View X as the total space of the family $h : (X, 0) \to (Y, 0)$. For each closed point $y \in Y$ set $X_y := X \cap \mathrm{pr}^{-1}(y)$. Whitney conjectured that these conditions would ensure the existence of an ambient homeomorphism h that maps $X_0 \times Y$ onto X. In other words the family X_y is *topologically equisingular*. The conjecture is nowadays known as the Thom–Mather [27] first isotopy theorem. Thom and Mather proved it by considering constant tangent vector field to Y and lifting it carefully to X so that the lift is integrable. The integral provides a continuous flow on X which is $\mathbb{C}^\infty$ away from Y.

The Whitney stratifications play an important role in the classification of differentiable maps ([19], [20], and [8]), in D-module theory and the solution of the Riemann–Hilbert problem [17], and in the Goresky–McPherson theory of intersection homology

[9], to mention few.

Hironaka [11] introduced an intrinsic modified verson of the Whitney conditions. We say that X *satisfies the strict Whitney condition A* if the distance from Y to the tangent space T_xX approaches 0 as x approaches 0 along any analytic path $\phi : (\mathbb{C}, 0) \to (X, 0)$ such that $\phi(u) \subset X - Y$ for any $u \neq 0$, i.e. the inequality

$$\operatorname{dist}(Y, T_xX) \leq c \operatorname{dist}(x, Y)^e \tag{1}$$

holds where the exponent e and the constant c depend on the path ϕ. Hironaka proved that if the pair $(X - Y, Y)$ satisfies both Whitney condition A and B at 0, then it satisfies the strict Whitney condition A with exponent e independent of the path ϕ. The strict Whitney condition with exponent $e = 1$ and c independent of the path ϕ is called Verdier's *W condition.* Verdier [31] showed that his condition implies Whitney A and B. Teissier (Thm. 1.2 in Chap. V of [25]) showed that in the complex-analytic case W is in fact equivalent to Whitney A and B.

Define the absolute conormal space $C(X)$ as the closure in $X \times \check{\mathbb{P}}^{n+k-1}$ of the set of pairs (x, H) such that x is a smooth point in X and H is a hyperplane tangent at x, i.e. a hyperplane that contains T_xX. Teissier inspired by some work of Hironaka, considered the following normal-conormal diagram

$$\begin{array}{ccc}
\mathrm{Bl}_{c^{-1}(Y)}C(X) & \xrightarrow{a'} & C(X) \\
\Big\downarrow{\scriptstyle c'} & & \Big\downarrow{\scriptstyle c} \\
\mathrm{Bl}_Y X & \xrightarrow{a} & X
\end{array}$$

Set $\xi := c \circ a'$. Teissier [25] in the case of Whitney B obtained the following equivalence.

Theorem 2.1 (Teissier)**.** *Preserving the setup from above, the following conditions are equivalent:*

(i) *The pair $(X - Y, Y)$ satisfies Whitney B at* 0*.*

(ii) *Let $\xi_Y : D_Y \to Y$ be the restriction of ξ to the exceptional divisor D_Y of* $\mathrm{Bl}_{c^{-1}(Y)}C(X)$*. Then ξ_Y is equidimensional and* $\dim \xi_Y^{-1}(0) = n - 2$*.*

Next we provide a concrete algebraic description of the conormal variety $C(X)$ using the Jacobian module of X. Suppose X is reduced. Let X be defined by the vanishing of some analytic functions $f_1, \ldots, f_p$ on a Euclidean neighborhood of 0 in $\mathbb{C}^{n+k}$. Consider the following exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega^1_{\mathbb{C}^{n+k}}|X \longrightarrow \Omega^1_X \longrightarrow 0 \tag{2}$$

where I is the ideal of X in $\mathcal{O}_{\mathbb{C}^{n+k},0}$ and the map δ sends a function f vanishing on X to its differential df. Dualizing we obtain the following nested sequence of torsion-free sheaves:

$$\mathrm{Image}(\delta^*) \subset (\mathrm{Image}\ \delta)^* \subset (I/I^2)^*.$$

Observe that locally the sheaf $\mathrm{Image}(\delta^*)$ can be viewed as the column space of the *(absolute) Jacobian module* of X which we denote by $J(X)$. It is a module contained in the free module $\mathcal{O}^p_{X,0}$. Define the *Rees algebra* $\mathcal{R}(J(X))$ to be the subalgebra of $\mathrm{Sym}(\mathcal{O}^p_{X,0})$ generated by $J(X)$ placed in degree 1 (for a general definition see [5]). Then the conormal space $C(X)$ is given by

$$C(X) = \mathrm{Projan}(\mathcal{R}(J(X))).$$

To see that equality holds observe that both sides are equal over $X-Y$ to the set of pairs (x, H) where H is a tangent hyperplane to the simple point x. The left side is the closure of this set, and so is the right side simply because the Rees algebra is by construction a subalgbebra of the symmetric algebra $\mathrm{Sym}(\mathcal{O}^p_{X,0})$.

Consider the family setup $h : (X, 0) \to (Y, 0)$ from the beginning of the section. Assume each f_i is analytic function on two sets of variables: the *fiber* variables $(x_1, \ldots, x_n)$ and the *parameter* variables $y_1, \ldots, y_k$. Form the Jacobian matrix of X with respect to the fiber variables:

$$\begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_p/\partial x_1 & \cdots & \partial f_p/\partial x_n \end{pmatrix}.$$

The column space of this matrix generates a module over the local ring $\mathcal{O}_{X,0}$, which we denote by

$$J_{\mathrm{rel}}(X) := JM_x(f_1, \ldots, f_p) \subset \mathcal{O}^p_{X,0}$$

and call the *relative Jacobian module* of X. Given $y \in Y$ form the image of the module $J_{\mathrm{rel}}(X)$ in the free module $\mathcal{O}^p_{X_y}$ to get the Jacobian module of X_y:

$$J(X_y) := JM_x(f_1, \ldots, f_p)|X_y \subset O^p_{X_y}.$$

Thus for the conormal space $C(X_y)$ of X_y in $\mathbb{C}^n$ we have $C(X_y) = \mathrm{Projan}(\mathcal{R}(J(X_y)))$. Define the *relative conormal space*

$$C_{\mathrm{rel}}(X) := \mathrm{Projan}(\mathcal{R}(J_{\mathrm{rel}}(X))).$$

It is the closure of the set of pairs (x, H) where x is a smooth of X and H is tangent hyperplane containing a parallel to Y. A major result of Gaffney, Kleiman and Teissier (see [7, Thm. 1.8 and Prp. 6.1]) obtained via Teissier's Principle of Specialization of Integral Dependence is that in Thm. 3.1 one can replace $C(X)$ by $C_{\mathrm{rel}}(X)$.

Theorem 2.2 (Gaffney–Kleiman–Teissier)**.** *Whitney B is equivalent to asking that the exceptional divisor of* $\mathrm{Bl}_{c^{-1}(Y)}C_{\mathrm{rel}}(X)$ *has equidimensional fibers, i.e. it has no vertical components.*

Algebraically, $\mathrm{Bl}_{c^{-1}(Y)}C_{\mathrm{rel}}(X)$ is given by $\mathrm{Projan}(\mathcal{R}(m_Y J_{\mathrm{rel}}(X)))$ where m_Y is the ideal of Y in $\mathcal{O}_{X,0}$. The numerical invariants we will use to control for the presence of vertical components in the exceptional divisor will depend on the $\mathcal{O}_{X_y}$-modules $m_y J(X_y)$.

Denote by X^i the section of X by a general linear space of codimension i in $\mathbb{C}^{n+k}$ containing Y. We say that the fibers X^i_y are *topologically equisingular* if there exists a homeomorphism $l^i \colon (\mathbb{C}^{n+k-i}, 0) \to (\mathbb{C}^{n+k-i}, 0)$ which induces a homeomorphism $(\mathbb{C}^{n+k-i}, X^i) \cong (\mathbb{C}^{n-i} \times Y, X^i_0 \times Y)$. The following is a fundamental result of Lê and Teissier [16].

Corollary 2.3 (Lê–Teissier)**.** *The fibers* X^i_y *are topologically equisingular if and only if* $(X - Y, Y)$ *satisfies Whitney B at* 0.

3. Hilbert–Samuel multiplicity. Let (Z, z) be an equidimensional complex analytic germ of dimension d. Denote the local ring of functions at z by $\mathcal{O}_{Z,z}$ and by $\mathfrak{m}_z$ the maximal ideal of $\mathcal{O}_{Z,z}$. Let J be an $\mathfrak{m}_z$-primary ideal in $\mathcal{O}_{Z,z}$. For $l \gg 0$ we have

$$\dim_{\mathbb{C}} \mathcal{O}_{Z,z}/J^l = \frac{e(J)}{d!} l^d + \text{lower degree terms}$$

where $e(J)$ is a positive integer called the *Hilbert–Samuel (HS) multiplicity* of J. When $J = \mathfrak{m}_z$, then $e(\mathfrak{m}_z)$ coincides with the definition of the multiplicity of Z at z given in Sct. 1 (see [22, Chp. 5]). Furthermore, $e(J)$ has the following geometric interpretation. Denote by D_J the exceptional divisor of $\mathrm{Bl}_J Z$. Then $e(J) = (-1)^{d-1} D_J^d$ (see [23]).

In the setup of Sct. 2 suppose $(X, 0)$ is a hypersurface in $(\mathbb{C}^{n+k}, 0)$, i.e. $(X, 0) = \mathbb{V}(f_1)$ where f_1 is a holomorphic function on $(\mathbb{C}^{n+k}, 0)$. Then the conormal space $C(X)$ is the blowup of X with respect to the Jacobian ideal $J(f_1)$. Similarly, the relative conormal space $C_{\mathrm{rel}}(X)$ is the blowup of X with respect to the relative Jacobian ideal $J_x(f_1)$, i.e. the ideal in $\mathcal{O}_{X,0}$ generated by the partials of f_1 with respect to the fiber coordinates $x_1, \ldots, x_n$. We will assume that X_y have isolated singularities. Our goal is to find appropriate HS mutiplicities that depend only on the fibers X_y whose constancy across Y is equivalent to Whitney B. To do this let us analyze first in more general terms the problem of finding numerical invariants that control the presence of vertical components of the exceptional divisor of $\mathrm{Bl}_{J_x(f_1)} X$.

To simplify the exposition assume $(Y, 0)$ is smooth of dimension one. Suppose $h\colon X \to Y$ is a morphism with equidimensional fibers of positive dimension. Let $S \subset X$ be proper over Y such that S_y is nowhere dense in X_y for each point $y \in Y$. Let C be an equidimensional reduced complex analytic variety, projective over X such that the structure morphism $c\colon C \to X$ maps each irreducible component of C to an irreducible component of X. Set $D := c^{-1}S$ and $\dim C := r + 1$.

Denote by D_{vert} the union of components of D of top dimension r that map to $0 \in Y$ under $h \circ c$. We call these components *vertical*. Our goal is to control numerically the presence of D_{vert}.

Suppose $C := \mathrm{Bl}_S X$. In this case D is the exceptional divisor of the blowup. For each y denote by $C(y)$ the blowup of X_y by S_y and by $D(y)$ the exceptional divisor. Set $l := c_1 \mathcal{O}_C(1)$ and $l_y := c_1 \mathcal{O}_{C(y)}(1)$. Let U be a small enough affine neighborhood of 0. Then we have the following *Excess–Degree Formula* (EDF) ([24, Thm. 2.1]):

$$\int_{S_0} l_0^{r-1}[D(0)] - \int_{S_y} l_y^{r-1}[D(y)] = \int_{S_0} l^r [D_{\mathrm{vert}}]$$

for $y \in U - \{0\}$. Suppose $\mathcal{O}_C(1)$ is ample on D_{vert}. Then D_{vert} is empty if and only if the intersection number $\displaystyle\int_{S_y} l_y^{r-1}[D(y)]$ is constant for $y \in Y$. A typical situation when $\mathcal{O}_C(1)$ is ample on D_{vert} is when S is finite over Y, for example.

Suppose (X_y, y) is an isolated singularity for each y. Let us apply the EDF with S defined by $m_Y J_x$ where m_Y is the ideal of Y in $\mathcal{O}_X$. Then the blowup of X with respect to $m_Y J_x$ is the blowup of $C_{\mathrm{rel}}(X)$ with respect to the inverse image of Y in $C_{\mathrm{rel}}(X)$. The image of J_x in $\mathcal{O}_{X_y}$ is the Jacobian ideal $J(X_y)$ of X_y in $\mathcal{O}_{X_y}$. Denote by $\mathfrak{m}_y$ the ideal of y in $\mathcal{O}_{X_y}$. Then $e(m_y J(X_y)) := \displaystyle\int_{S_y} l_y^{r-1}[D(y)]$. Thus the EDF along with Thm. 2.2 give us the following result.

Theorem 3.1 (Teissier)**.** *Assume X_y are isolated hypersurface singularities. The* *pair* $(X-Y,Y)$ *satisfies Whitney B at* 0 *if and only if*

$$y \to e(m_y J(X_y))$$

is constant across Y.

Returning to the "cusp and node" example from Sct. 1, note that X_0 and X_1 are part of the family $X = \mathbb{V}(z^2 - x^3 - yx^2) \subset \mathbb{C}^3$. One readily checks that $e(m_0 J(X_0)) = 5$, whereas $e(m_1 J(X_1)) = 4$. Thus $e(m_y J(X_y))$ detects that X_0 and X_1 are not topologically equisingular, whereas the ordinary multiplicity at the origin does not.

The EDF, in the case when the blowup is with center S finite over Y, can be restated as saying that the change of the HS multiplicity from a special fiber to a general nearby fiber equals the degree of the vertical components of the exceptional divisor. In what follows we give two applications of this useful observation due to Teissier (Prp. 3.1 in [26] and Rmk. 5.1.1 in [25]) to the deformation theory of singularities.

Consider an analytic function $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ with an isolated critical point at the origin. Let $\mathbf{x} := (x_1,\dots,x_n)$ be a linear choice of coordinates. Let $J(f) := (\partial f/\partial x_1,\dots,\partial f/\partial x_n)$ be the Jacobian ideal of f in $\mathcal{O}_{\mathbb{C}^n,0}$. Because $J(f)$ is generated by n elements and is primary to the maximal ideal of the regular local ring $\mathcal{O}_{\mathbb{C}^n,0}$, then by [12, Prp. 11.1.10] the Hilbert–Samuel multiplicity $e(J(f),\mathcal{O}_{\mathbb{C}^n,0})$ equals the Milnor number $\mu(f) := \dim_{\mathbb{C}} \mathbb{C}\{x_1,\dots,x_n\}/J(f)$.

Consider $f_y\colon (\mathbb{C}^n \times \mathbb{C},(0,0)) \to (\mathbb{C},0)$ where $f_0 = f$ and f_y for $y \neq 0$ is a generic perturbation of f such that its critical points are nondegenerate, or Morse. Let $J_{\mathbf{x}}(f_y)$ be the relative Jacobian ideal, i.e. the ideal generated by the partials of f_y with respect to the $\mathbf{x}$ coordinates. Let S be the subscheme of $\mathbb{C}^n \times \mathbb{C}$ defined by $J_{\mathbf{x}}(f_y)$. There are no vertical components of the exceptional divisor D of $\mathrm{Bl}_S(\mathbb{C}^n \times \mathbb{C})$ because D_0 is a set – theoretically in $\{0\} \times \mathbb{P}^{n-1}$. Applying the EDF we get that $\mu(f) = \mu(f_y)$ for generic y. But f_y has only Morse critical points and their number is $\mu(f_y)$. Thus $\mu(f)$ equals the number of critical points of a generic perturbation of f.

We remark that whenever $f_y\colon (\mathbb{C}^n \times \mathbb{C},(0,0)) \to (\mathbb{C},0)$ is such that $X = \mathbb{V}(f_y)$ is a family of isolated singularities with $y \to \mu(f_y)$ constant and $n \neq 2$, then Lê and Ramanujam [18] showed that X_y have the same topological type.

As another example, let $(X,0) \to (Y,0)$ be a smoothing of an isolated hypersurface $(X_0,0)$ singularity, defined as the zero locus in $(\mathbb{C}^N,0)$ of a complex analytic function F, and S be defined by the partials of F with respect to the fiber coordinates. Applying the EDF to this setting we get that the first left-hand side term is equal to the Hilbert–Samuel multiplicity of the Jacobian ideal in $\mathcal{O}_{X_0,0}$ associated with X_0, whereas the second term is zero because X_y is smooth. By conservation of number argument, the right-hand side of the EDF is equal to the intersection multiplicity of the *relative polar curve* associated with the total space of the family with a generic fiber. In turn, by [26, Cor. 1.5, pg. 320] this intersection multiplicity is the sum of the Milnor number of $(X_0,0)$ and the Milnor number of a generic hyperplane slice of $(X_0,0)$.

4. Buchsbaum–Rim multiplicity. Let (Z,z) be an equidimensional complex analytic germ of dimension d. Let $\mathcal{M}$ be an $\mathcal{O}_{Z,z}$-module contained in a free module $\mathcal{F} := \mathcal{O}_{Z,z}^e$. Denote by $\mathrm{Sym}(\mathcal{F})$ the symmetric algebra of $\mathcal{F}$. It is a polynomial ring in e variables over $\mathcal{O}_{Z,z}$. Denote by $\mathcal{R}(\mathcal{M})$ the Rees algebra of $\mathcal{M}$. It is the subalgebra of $\mathrm{Sym}(\mathcal{F})$ generated by $\mathcal{M}$ placed in degree 1. Denote by $\mathcal{F}^l$ and $\mathcal{M}^l$ the degree l graded pieces of $\mathrm{Sym}(\mathcal{F})$ and $\mathcal{R}(\mathcal{M})$, respectively. Assume that $\mathcal{M}$ and $\mathcal{F}$ are equal locally off

z. In [1] Buchsbaum and Rim showed that for $l \gg 0$

$$\dim_{\mathbb{C}} \mathcal{F}^n/\mathcal{M}^n = \frac{e(\mathcal{M})}{(d+e-1)!} l^{d+e-1} + \text{lower degree terms} \tag{3}$$

where $e(\mathcal{M})$ is a positive integer called the *Buchsbaum–Rim multiplicity (BR)* of $\mathcal{M}$. Obviously, when $\mathcal{M}$ is an ideal, then $\mathcal{F} = \mathcal{O}_{Z,z}$ and $e(\mathcal{M})$ is the HS multiplicity. Thus the BR multiplicity generalizes the HS multiplicity. If $\mathcal{N}$ is an $\mathcal{O}_{Z,z}$-module such that $\mathcal{M} \subset \mathcal{N} \subset \mathcal{F}$, then by replacing $\mathcal{F}$ by $\mathcal{N}$ in (3) we can define the *relative BR multiplicity* $e(\mathcal{M}, \mathcal{N})$. It is a nonnegative integer. It is zero if and only if $\mathcal{N}$ is integrally dependent on $\mathcal{M}$ [12, Chp. 16].

Keeping the setup from Sct. 2 when $(X, 0) \subset (\mathbb{C}^{n+k}, 0)$ is a complete intersection and X_y are isolated singularities, then $J(X_y)$ is equal to $\mathcal{F} := \mathcal{O}^p_{X_y}$ locally off the singular points of X_y. Thus $e(J(X_y))$ and $e(m_y J(X_y))$ are well-defined. Note that $\operatorname{Projan}(\mathcal{R}(m_Y J_{\text{rel}}(X))$ is the blowup of $C_{\text{rel}}(X)$ by the inverse image of Y. As an application of the EDF (see [24, Thm. 2.6] and Gaffney's Multiplicity-Polar Theorem proved in [24, Cor. 2.7]) assuming $\dim Y = 1$ we derive

$$e(m_0 J(X_0)) - e(m_y J(X_y)) = \int l^r [D_{\text{vert}}]$$

where D_{vert} is the union of vertical components of the exceptional divisor of the blowup of $C_{\text{rel}}(X)$ by the inverse image of Y. Gaffney and Kleiman [7, Thm. 6.4] obtained the following generalization of Thm. 3.1.

Theorem 4.1 (Gaffney–Kleiman)**.** *Assume X_y are isolated complete intersection singularities. The pair $(X - Y, Y)$ satisfies Whitney B at 0 if and only if*

$$y \to e(m_y J(X_y))$$

is constant across Y.

By additivity for the BR multiplicity we have $e(m_y J(X_y)) = e(m_y J(X_y), J(X_y)) + e(J(X_y))$. For $e(m_y J(X_y), J(X_y))$ we have an expansion formula in terms of weighted polar multiplicities, whereas the second term $e(J(X_y))$ plays the role of a zero-dimensional polar invariant. We need to recall first the notion of local polar varieties as developed by Teissier in [25, Chp. IV].

Let $Z \subset \mathbb{C}^N$ be a reduced equidimensional complex analytic varieity. Let $z \in Z$. Consider the germ (Z, z). Consider the following diagram

$$\begin{tikzcd} C(Z) \arrow[r, hook, "i"] \arrow[d, "c"] \arrow[dr, "\lambda"] & Z \times \check{\mathbb{P}}^{N-1} \arrow[d, "\mathrm{pr}_2"] \\ (Z, z) & \check{\mathbb{P}}^{N-1}. \end{tikzcd}$$

Assume (Z, z) is a germ of pure dimension d and of codimension e in $\mathbb{C}^N$. Let H_{e+l-1} be a general plane in $\check{\mathbb{P}}^{N-1}$ of codimension $e+l-1$. Define the *local polar variety* $\Gamma_l(Z, H_{e+l-1})$ with respect to H_{e+l-1} as $c(\lambda^{-1}(H_{e+l-1}))$. Then $\Gamma_l(Z, H_{e+l-1})$ is either empty or of pure codimension l. Teissier [25, Chp. IV, Thm. 3.1] showed that for sufficiently general H_{e+l-1} the multiplicity of $\Gamma_l(Z, H_{e+l-1})$ at z depends only on the analytic type of (Z, z). When writing $\Gamma_l(Z)$ we mean $\Gamma_l(Z, H_{e+l-1})$ for such a general H_{e+l-1}. Denote by $m(\Gamma_l(Z))$ the multiplicity of $\Gamma_l(Z)$ at z, by $J(Z)$ the Jacobian module of Z and by m_z the maximal ideal of $\mathcal{O}_{Z,z}$. Assume z is an isolated singularity of Z. Kleiman and Thorup [15, Thm.

9.8(i)] derived the following formula

$$e(m_z J(Z), J(Z)) = \sum_{i=1}^{d} \binom{N-1}{i} m(\Gamma_{d-i}(Z)).$$

5. Multiplicities as correction terms. Here we mention two instances where the HS and BR multiplicity appear as correction terms.

Let $Z \subset \mathbb{P}^N$ be a d-dimensional irreducible projective variety. The *degree* of Z, denoted by $\deg(Z)$, is defined in the usual way as the number of points of the intersection of Z with a generic linear space of complementary dimension $N-d$. Let $z \in \mathbb{P}^N$ be a point. Let $\mathrm{pr}_z\colon \mathbb{P}^N \to \mathbb{P}^{N-1}$ be the projection from z. Set $W := \mathrm{pr}_z(Z)$. The following formula [22, Thm. 5.11] relates $\deg(Z)$ and $\deg(W)$ in the case when Z is not a cone over z.

$$\deg(Z) = [k(Z) : k(W)]\deg(W) + e(z)$$

where $[k(Z) : k(W)]$ is the index of function fields, which is the number of points in Z above a general point in W, and $e(z)$ is the HS multiplicity of Z at z, which is 0 if $z \notin Z$. This formula played an important role in the resolution of surface singularities (see [28]).

Suppose Z is a curve of degree d in the complex projective plane. In 1834 Plücker showed that the number of tangents with a given direction d' is equal to $d(d-1)-2\delta-3\kappa$, where δ is the number of nodes and κ is the number of cusps of Z. Teissier, in the case of hypersurfaces, and Kleiman, in the case of complete intersections, generalized Plücker's formula as follows. Suppose $Z \subset \mathbb{P}^N$ is a complete intersection. Denote by $C(Z) \subset Z \times \check{\mathbb{P}}^N$ the conormal variety of Z. Let A be a linear space of codimension 2 in general position in $\mathbb{P}^N$. Define $d' := \int l'^{N-1}[C(Z)]$ where $l' = c_1(\mathcal{O}_{\check{\mathbb{P}}^N}(1))$. Then d' is equal to the number of points z in Z_{sm} such that the tangent space T_z and A span a hyperplane.

Suppose Z is cut out by homogeneous $f_1, \ldots, f_k$ with $\deg f_i = d_i$. By Bezout's theorem $\deg(Z) = d_1 \cdots d_k$. Set $\dim Z := d$ and Set $\pi := \sum_{i_1+\cdots+i_k=d} (d_1-1)^{i_1} \cdots (d_k-1)^{i_k}$.

Denote by $J_z(Z)$ the Jacobian module of Z at z. We have the following formula for d' due to Teissier and Kleiman [14, Thm. 1]

$$d' = \deg(Z)\pi - \sum_{z \in \mathrm{Sing}(Z)} e(J_z(Z)).$$

6. Local volumes. The difficulty of generalizing Thm. 3.1 and Thm. 4.1 to families $X \to Y$ of arbitrary isolated singularities is in finding a suitable generalization of the BR multiplicity in the case when $\mathcal{M}$ fails to have finite colength in $\mathcal{F}$. This occurs when $\mathcal{M}$ is the Jacobian module $J(Z)$ of an isolated singularity (Z, z) which is not a complete intersection. Indeed, the number of equations that cut out Z from $\mathbb{C}^n$, which corresponds to the rank of the free module $\mathcal{F}$ that contains $J(Z)$, is strictly larger than the rank of $J(Z)$ at smooth points. So, $e(J(Z))$ is not defined in this case. Attempts to finding a generalization of $e(J(Z))$ to the more general setup of arbitrary isolated singularities failed until 10 years ago, when the ε-multiplicity and its more general version, the *local volume*, were introduced.

Recall the setup preceding the Excess–Degree Formula in Sct. 3. To simplify the exposition assume again that $(Y, 0)$ is smooth of dimension one. Suppose $h\colon X \to Y$ is a morphism with equidimensional fibers of positive dimension. Let $S \subset X$ be proper over Y such that S_y is nowhere dense in X_y for each $y \in Y$. Let C be an equidimensional reduced complex analytic variety, projective over X such that the structure morphism $c\colon C \to X$ maps each irreducible component of C to an irreducible component of X. Set $D := c^{-1}S$ and $\dim C := r+1$.

From now on assume S is finite over Y. Let $\mathcal{L}$ be an invertible very ample sheaf on C relative to X. Let $\mathcal{A} := \oplus_{n\geq 0}\Gamma(C, \mathcal{L}^{\otimes n})$ be the ring of sections of $\mathcal{L}$. Denote by $\mathcal{A}_n$ the nth graded piece of $\mathcal{A}$. Assume $\dim X_y \geq 2$. Inspired by [6], [30] and [4], for each point $y \in Y$ define the *restricted local volume* of $\mathcal{L}$ at S_y as

$$\operatorname{vol}_{C_y}(\mathcal{L}) := \limsup_{n\to\infty} \frac{r!}{n^r} \dim_{\Bbbk} H^1_S(\mathcal{A}_n) \otimes_{\mathcal{O}_Y} k(y).$$

The existence of the volumes as a limit has been a topic of extensive research (see [2] and Kaveh and Khovanskii [13]). The local volume might be an irrational number as shown by Cutkosky, Herzog and Srinivasan [3].

One of the main results of [24] is the following relation which we refer simply as the *Local Volume Formula* (LVF):

$$\operatorname{vol}_{C_0}(\mathcal{L}) - \operatorname{vol}_{C_y}(\mathcal{L}) = \int_{S_0} l^r[D_{\mathrm{vert}}].$$

Let $C := \operatorname{Projan}(\mathcal{R}(m_Y J_{\mathrm{rel}}(X)))$. Then $\mathcal{A} = \mathcal{R}(m_Y J_{\mathrm{rel}}(X))$. For generic y the restricted local volume $\operatorname{vol}_{C_y}(\mathcal{L})$ in this case is the ε-multiplicity of $m_y J(X_y)$, introduced by Ulrich and Validashti in [30], which we denote by $\varepsilon m(y)$. Note that if X_y is an isolated complete intersection singularity, then by [10, Thm. 3.8] we have $H^1_{S_y}(\mathcal{F}^n/m_y J(X_y)^n) = \mathcal{F}^n/m_y J(X_y)^n$ for each n. Thus $\varepsilon m(y)$ generalizes $e(m_y J(X_y))$. In [24, Sct. 7] we obtain the following ultimate generalization of Thm. 3.1 and Thm. 4.1 assuming $X - Y$ is smooth.

Theorem 6.1. *Assume X_y are isolated singularities. The pair $(X - Y, Y)$ satisfies Whitney B at 0 if and only if*

$$y \to \varepsilon m(y)$$

is constant across Y.

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РАЗВИВАЩОТО СЕ ПОНЯТИЕ ЗА ИНДЕКС НА КРАТНОСТ КАТО ИНВАРИАНТ В ТЕОРИЯ НА ОСОБЕНОСТИТЕ

Антони Рангачев

В тази работа правим обзор на резултати, свързани с индекса на кратност в точка на комплексно аналитично многообразие, негови обобщения и приложението им като числени инварианти в теория на особеностите.