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FROM CONSTRUCTION TO COMPUTATION TO SYMBOLIC CALCULATION: AN EXAMPLE IN PLANE GEOMETRY*

Boyko Bantchev

On the example of a construction problem, we compare synthetic, computational, and calculational approaches, and demonstrate the advantages of vector-based calculation.

A problem statement. The following problem is one of the three given recently to the participants of a programming competition. On Fig. 1 a particular instance and the respective result are depicted.

Let AA', BB', and CC' be known segments in the plane and no two of them are parallel. Construct a triangle KLM with sides parallel to the given segments and such that the sum of their lengths equals that of the segments. $\triangle KLM$ must be anticlockwise oriented, its centroid must be at a given point Z, and the side KL must be the same direction as AA'.

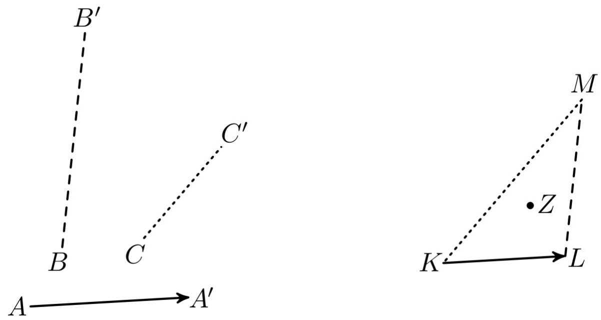

Fig. 1

The participants were professional computer programmers, a significant part of them being also higher-school students. The problem was supposed to be easy, especially considering that the competition took two weeks (so that everyone could pick their own amount and particular periods of time to spend on the problems). However, the number of people that eventually sent a working solution was small, and about 3/4 of their

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programs were inadequately, even absurdly large. This, along with some of the questions asked by the participants upon being introduced to the problem, suggests that perceiving the problem as easy, as we did initially, was at least partially unjustified.

So how do we solve this problem? It appears that there are three ways to answer this, each related to a different understanding of what a solution is. The three different ways emerge from the synthetic, computational, and (symbolic) calculational approaches to solving problems in geometry.

In fact, the reason that the author chose to discuss the particular problem here is not because the problem itself is especially interesting — which perhaps it is not — but because it seems good enough to illustrate the differences between the said three approaches, and how vector based symbolic calculation is highly advantageous.

Solving the problem. Let $K'L'M'$ be a triangle, such that $K'L'$ is the same length and direction as AA', while one of $K'M'$ and $L'M'$ is parallel to BB' and the other to CC' in such a way that $\triangle K'L'M'$ is positively (i.e. anticlockwise) oriented. If we manage to build $\triangle K'L'M'$, in order to obtain the needed $\triangle KLM$ it remains to submit the former triangle to a suitable pair of homothety and translation. For example, one can do the translation first, making Z become the centroid of the triangle, and then stretch or shrink the resulting figure with Z as a centre, so that its perimeter becomes equal to $AA'+BB'+CC'$.

Building $\triangle K'L'M'$ can be done as follows. Let $K' \equiv A$ and $L' \equiv A'$. The line (A, CC'), which passes through A and is parallel to CC', intersects the line (A', BB') through A' and parallel to BB' at a point P. Similarly, the line (A, BB') intersects the line (A', CC') at a point P'. Thus $APA'P'$ is a parallelogram and precisely one of P and P' is on the left of AA' (Fig. 2). M' is chosen to be either P or P', depending on which of them is the one on the left.

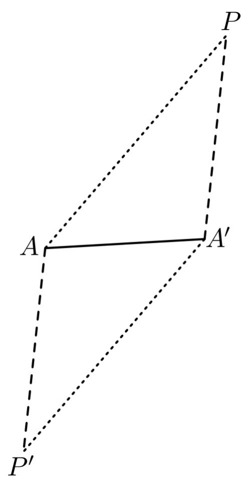

Fig. 2

Can this be considered a solution? Only in a rather limited sense the answer is 'yes'. Following the synthetic style of reasoning, the solution is essentially incomplete unless a geometric process for finding K, L, and M is presented. However, it seems very unlikely that such a process can be found. Lacking this, the best we can do is to point out that the amount of stretching to be applied on $\triangle K'L'M'$ in order to obtain KLM is $\dfrac{K'L'+L'M'+M'K'}{AA'+BB'+CC'}$. But this is unsatisfactory, because no geometric or algebraic relation between the sides of the two triangles has been identified and made use of, although such a relation does exist. In a sense, we are cheating by just referring to the sides of $\triangle K'L'M'$ instead of finding them in terms of AA', BB', and CC'.

Also unsatisfactory is the fact that we are choosing between P and P' as if by way of experimenting and seeing which of the two fits, while in fact we should expect to be able to make the choice again in terms of AA', BB', and CC'. But how can we tell which of P and P' is to the left of AA' without producing the four lines as we did?

In view of the unlikeliness of achieving a truly satisfactory solution to the problem while limiting ourselves to only synthetic reasoning, it might seem that a better option is the computational approach, traditionally embodied by analytic geometry.

What one usually does is write equations of the lines (A, CC'), (A', BB'), (A, BB'), and (A', CC') and solve them in pairs to find P and P' as points of intersection. If coordinates are given to the points, the equations are solved in terms of those coordinates. Thus the coordinates of P and P' are expressed using those of A, A', B, B', C, and C', which means finding four unknown variables in terms of twelve known ones.

The task is easy to solve numerically, but then again this is experimenting: producing numbers from other numbers and using them to find out whether the resulting P is on the left of AA'.

Instead, it is desirable to solve the equations symbolically, so that the condition for P being on the left of AA' can be obtained in symbolic form. By doing so, the easy but dumb numeric computation is replaced by calculation with symbols, which is attractive in principle as it implies generality. But in practice, dealing with twelve variables is very cumbersome, including the expected result. And it is extremely unlikely that in the process of solving the problem an insight can be gained with regard to what properties of the three given segments actually affect the choice between P and P'.

A much preferrable symbolic calculation is one where vectors instead of coordinates are dealt with. Let $\mathbf{a} = \mathbf{AA'}$, $\mathbf{b} = \mathbf{BB'}$, and $\mathbf{c} = \mathbf{CC'}$. As the point P lies on both lines $(A, \mathbf{c})$ and $(A', \mathbf{b})$, there must be true $\mathbf{AP} = s\,\mathbf{c}$ and $\mathbf{A'P} = t\,\mathbf{b}$ for some numbers s and t. Through finding the scalar products with $\mathbf{b}$ and $\mathbf{c}$ of each side of each of these equations, a pair of linear numeric equations results from which s and t can be obtained, each one effectively leading to $\mathbf{P}$. For instance,

$$\mathbf{P} \;=\; \mathbf{A} + s\,\mathbf{c} \;=\; \mathbf{A} + \frac{(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}^2 - (\mathbf{a}\cdot\mathbf{b})\,(\mathbf{b}\cdot\mathbf{c})}{\mathbf{b}^2\,\mathbf{c}^2 - (\mathbf{b}\cdot\mathbf{c})^2}\,\mathbf{c}.$$

As for P', it is readily known from $\mathbf{AP} + \mathbf{A'P'} = \mathbf{0}$.

A much simpler expression for s can be arrived at in the following way. Let $\mathbf{b_p}$ be any non-zero vector perpendicular to $\mathbf{b}$. Rewriting the equation $\mathbf{A'P} = t\,\mathbf{b}$ in the form $\mathbf{AP} = \mathbf{a} + t\,\mathbf{b}$ and comparing this with $\mathbf{AP} = s\,\mathbf{c}$ eliminates $\mathbf{AP}$ and, upon multiplying by $\mathbf{b_p}$, leads to $s = \dfrac{\mathbf{a}\cdot\mathbf{b_p}}{\mathbf{c}\cdot\mathbf{b_p}}$.

Now we have symbolic expressions for both P and P' in terms of the given segments, but it is still not clear how to choose between P and P', again in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

For this we need to be able to algebraically express orientation in the plane in terms of vectors, as effectively as it is done in coordinates.

Let $\mathbf{u}\times\mathbf{v}$ be the oriented area of the parallelogram built on (representatives of) the vectors $\mathbf{u}$ and $\mathbf{v}$ as its sides. The oriented area is positive when $\mathbf{v}$'s direction is to the left of $\mathbf{u}$'s — i.e., if (representatives of) $\mathbf{u}$ and $\mathbf{v}$ are taken with the same origin, $\mathbf{v}$ is in the left half-plane with respect to $\mathbf{u}$'s direction. If any of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$ or they are parallel, it is assumed that $\mathbf{u}\times\mathbf{v}=0$.

We should note that the choice of which of the two half-planes is 'left' is arbitrary in principle, but here it has to be consistent with our understanding of 'anticlockwise' in the formulation of the problem being discussed. The same holds of the coordinate-based criteria of 'anticlockwise' or 'left' when analytic geometry is used.

In the following we make use of the fact that the $\times$ operator is linear with respect to each of its arguments: $(k\,\mathbf{u}+k'\mathbf{u}')\times\mathbf{v}=k(\mathbf{u}\times\mathbf{v})+k'(\mathbf{u}'\times\mathbf{v})$ and similarly for $\mathbf{v}$.

A necessary and sufficient condition for $\triangle AA'P$ to be anticlockwise oriented is $\mathbf{AA'}\times\mathbf{AP}>0$, i.e. $\mathbf{a}\times\mathbf{AP}>0$. The vector $\mathbf{AP}$ can be found by solving together

$$\left|\begin{array}{rcl}\mathbf{c}\times\mathbf{AP} & = & 0\\ \mathbf{b}\times\mathbf{A'P} & = & 0,\end{array}\right.$$

which are algebraic expressions of $\mathbf{AP}\|\mathbf{c}$ and $\mathbf{A'P}\|\mathbf{b}$ and thus none less but equations of the lines $(A,\mathbf{c})$ and $(A',\mathbf{b})$.

By substituting $\mathbf{A'P}=\mathbf{AP}-\mathbf{AA'}=\mathbf{AP}-\mathbf{a}$ the second equation becomes

$$\mathbf{b}\times\mathbf{AP} \;=\; \mathbf{b}\times\mathbf{a}.$$

Systems of the kind

$$\left|\begin{array}{rcl}\mathbf{u}\times\mathbf{p} & = & k\\ \mathbf{v}\times\mathbf{p} & = & l\end{array}\right.$$

can easily be solved in general, but in our particular case it is suffices to note that due to the fact that $\mathbf{AP}\|\mathbf{c}$, $\mathbf{AP}=s\,\mathbf{c}$ holds as above, and then from the second equation we obtain $s=\dfrac{\mathbf{b}\times\mathbf{a}}{\mathbf{b}\times\mathbf{c}}$, resulting in $\mathbf{AP}=\dfrac{\mathbf{b}\times\mathbf{a}}{\mathbf{b}\times\mathbf{c}}\,\mathbf{c}$.

Hence

$$\mathbf{AA'}\times\mathbf{AP} \;=\; \frac{(\mathbf{b}\times\mathbf{a})(\mathbf{a}\times\mathbf{c})}{\mathbf{b}\times\mathbf{c}} \;=\; \frac{(\mathbf{a}\times\mathbf{b})(\mathbf{b}\times\mathbf{c})(\mathbf{c}\times\mathbf{a})}{(\mathbf{b}\times\mathbf{c})^2}.$$

The sign of the expression on the right, if positive, shows that $\triangle AA'P$ is anticlockwise oriented and therefore M' must be P. The sign is that of the expression

$$(\mathbf{a}\times\mathbf{b})(\mathbf{b}\times\mathbf{c})(\mathbf{c}\times\mathbf{a}). \tag{1}$$

Thus we have obtained a general, simple and symmetric criterion for choosing between P and P'.

From $\mathbf{AP}=\dfrac{\mathbf{b}\times\mathbf{a}}{\mathbf{b}\times\mathbf{c}}\,\mathbf{c}$, by exchanging $\mathbf{b}$ and $\mathbf{c}$, we obviously obtain an expression for $\mathbf{AP'}$, namely $\mathbf{AP'}=\dfrac{\mathbf{a}\times\mathbf{c}}{\mathbf{b}\times\mathbf{c}}\,\mathbf{b}$. Thus we find the perimeter of $\triangle K'L'M'$ to be

$$p \;=\; AA'+AP+A'P \;=\; AA'+AP+AP' \;=\; \frac{a\,|\mathbf{b}\times\mathbf{c}|+b\,|\mathbf{c}\times\mathbf{a}|+c\,|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}\times\mathbf{c}|}$$

and $\dfrac{a+b+c}{p}$ is the scaling factor of the homothety that has to be applied on $\triangle K'L'M'$

in order to obtain $\triangle KLM$.

We have solved the problem symbolically and with minimum effort. The use of vectors and properly chosen operations leads to a result that is generally and simply expressed and exposes how the obtained construction is related to the given data with utmost clarity.

The solution to the above problem can be expressed even more directly, without intermediately constructing $\triangle K'L'M'$.

Consider the equation $p\,\mathbf{a}+q\,\mathbf{b}+r\,\mathbf{c}=\mathbf{0}$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are non-zero and non-collinear vectors. Apart from the trivial solution $p=q=r=0$, the equation holds if and only if $p:q:r = \mathbf{b}\times\mathbf{c} : \mathbf{c}\times\mathbf{a} : \mathbf{a}\times\mathbf{b}$. This is easy to prove but for lack of space let it be taken for granted here.

To apply the above result to the problem discussed in the text, let again $\mathbf{a}=\mathbf{AA'}$, $\mathbf{b}=\mathbf{BB'}$, and $\mathbf{c}=\mathbf{CC'}$. If it turns out that $(\mathbf{a}\times\mathbf{b})(\mathbf{b}\times\mathbf{c})(\mathbf{c}\times\mathbf{a})<0$, exchange $\mathbf{b}$ and $\mathbf{c}$ to ensure a positive sign and thus an anticlockwise orientation of $\triangle KLM$.

We want $\mathbf{KL}=p\,\mathbf{a}$, $\mathbf{LM}=q\,\mathbf{b}$, and $\mathbf{MK}=r\,\mathbf{c}$, such that $p:q:r = \mathbf{b}\times\mathbf{c} : \mathbf{c}\times\mathbf{a} : \mathbf{a}\times\mathbf{b}$, $|p\,\mathbf{a}|+|q\,\mathbf{b}|+|r\,\mathbf{c}|=a+b+c$, and, in addition, $p>0$ (so that $\mathbf{KL}\upuparrows\mathbf{a}$).

The above conditions immediately suggest that by letting

$$k \;=\; \operatorname{sign}(\mathbf{b}\times\mathbf{c})\,\frac{a+b+c}{a\,|\mathbf{b}\times\mathbf{c}|+b\,|\mathbf{c}\times\mathbf{a}|+c\,|\mathbf{a}\times\mathbf{b}|}$$

and then $\mathbf{MK}=k\,(\mathbf{a}\times\mathbf{b})\,\mathbf{c}$, $\mathbf{KL}=k\,(\mathbf{b}\times\mathbf{c})\,\mathbf{a}$, the vertices of $\triangle KLM$ are obtained by

$$\mathbf{K} \;=\; \mathbf{Z}+\frac{\mathbf{MK}-\mathbf{KL}}{3}, \qquad \mathbf{L} \;=\; \mathbf{K}+\mathbf{KL}, \qquad \mathbf{M} \;=\; \mathbf{K}-\mathbf{MK}.$$

Attitudes and orientation. In the above discussion the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are in fact arbitrary, given that all three are non-zero and no two of them are parallel.

One can observe that the sign of (1) does not change if any of the vectors is replaced by a vector of the opposite direction. It follows that by changing, if necessary, the direction of one of the vectors to the opposite, it is always possible to make the three products $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$, and $\mathbf{c}\times\mathbf{a}$ have the same sign without affecting the sign of (1).

When $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are the sides of a triangle, each of the three products is twice the oriented area of the triangle, its sign indicating the triangle's orientation just as (1) does.

The sign of (1) does change if any two vectors exchange their places.

In geometry, it is customary to associate a direction to figures and other objects. It is no less useful to consider a more general property, sometimes called *attitude*. A direction becomes an attitude by not distinguishing it from its opposite. Thus parallel lines, line segments, beams, or vectors share an attitude.

Our observations regarding (1) reveal that, as far as its sign is concerned, (1) is actually related to attitudes rather than vectors or their directions. The sign of (1) is indeed a function of attitudes — ones represented by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ or any other parallel to them.

Specifically, the sign of (1) is defined by the cyclically ordered triple of the said attitudes. In this sense, any ordered triple of different attitudes defines either a clockwise or anticlockwise orientation in the plane — just as any ordered triple of non-collinear points does so — and (1) is but an algebraic means of finding out, given three attitudes, which one of the two possible orientations they define.

It is a notable virtue of the calculational method of exploration that a result, once obtained, may suggest a more general interpretation, thus enhancing our geometric comprehension of reality. This has been illustrated here by discovering a simple but rather general result about attitudes while solving the seemingly unrelated problem of constructing a triangle.

Final remarks. Vectors, equipped with useful operations on them, are a simple and very effective, albeit neglected, tool for doing calculations in geometry. On the example of a construction problem we have demonstrated and discussed advantages of the vector-based calculational approach to Euclidean geometry.

In another article [1] we have explored the same topic on the example of a proof problem, bringing attention to a number of advantages of the calculational approach, specifically the vector-based one. These advantages apply to construction problems as well.

The teaching of geometry in schools, colleges and universities, as well as the many software applications of geometry, can greatly benefit from studying and applying vectorial calculational methods.

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Boyko B. Bantchev
Institute of Mathematics and Informatics
Acad. G. Bontchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: bantchev@math.bas.bg

ОТ ПОСТРОЯВАНЕ КЪМ ПРЕСМЯТАНЕ И КЪМ СИМВОЛНО РЕШАВАНЕ: ПРИМЕР ОТ ПЛАНИМЕТРИЯТА

Бойко Бл. Банчев

Чрез примерна задача за построяване сравняваме синтетичния, изчислително-аналитичния подходи и този на символните пресмятания, показвайки предимствата на последния като основан на смятане с вектори.