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KOLMOGOROV, STOCHASTICS IN BULGARIA, AND PROBABILISTIC PROBLEMS WITH UNEXPECTED SOLUTIONS*

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In this talk, I am going to share with the readers my memories of personal meetings with Andrey Nikolaevich Kolmogorov (1903–1987), the content of our conversations, and the fruitful consequences. The reader is familiar with, or could read, the timely prepared recent comprehensive paper [34], presented by N.M. Yanev at the 52nd Spring Conference of the UBM. Included here are several new details about Andrey Nikolaevich and the great influence of the Moscow Probability School on the development of Stochastics in Bulgaria. I have used *MathSciNet* and introduced the 'Kolmogorov number', a 'collaboration distance' between a mathematician and Kolmogorov, with a focus on Bulgarian stochasticians. I am writing about Kolmogorov's approach to doing mathematics in general and the role of counterexamples. Discussed are two specific probabilistic problems which, according to him, are with most unusual solutions, 'Skitovich-Darmois theorem' and 'Plackett problem'. Several related short stories and not well-known facts are presented.
Keywords: A. N. Kolmogorov, Stochastics, Probability theory, Kolmogorov number, Normal distribution, Skitovich-Darmois Theorem, Mean sample range, Plackett problem.

КОЛМОГОРОВ, СТОХАСТИКА В БЪЛГАРИЯ, И ВЕРОЯТНОСТНИ ЗАДАЧИ С НЕОЧАКВАНИ РЕШЕНИЯ

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В този доклад ще споделя с читателите мои спомени за лични срещи с Андрей Николаевич Колмогоров (1903 – 1987), съдържанието на наши разговори и последвали резултати. Читателят е запознат, или може да прочете, навременно подготвения и изчерпателен доклад [34]. Тук съм включил редица нови факти за А.Н. и голямото влияние на Московската Вероятностна школа върху развитието на стохастиката в България. Използвах MathSciNet и дефинирах т. нар. число на Колмогоров, съавторско разстояние на математик и Колмогоров, с акцент върху български стохастици. Разказвам за подхода на Колмогоров в математиката и за ролята на контрапримерите. Обсъждам две конкретни вероятностни задачи, които според него са с доста неочаквани решения, Теоремата на Скитович–Дармоа и Задачата на Плакет. Включени са също редица кратки случки по темата и не така добре известни факти.
Ключови думи: А. Н. Колмогоров, Стохастика, Теория на вероятностите, Число на Колмогоров, Нормално разпределение, Теорема на Скитович–Дармоа, Среден размах на извадка, Проблем на Плакет

Mathematics is a synthesis of the two ideals making meaningful human life, the ideal of truth and the ideal of beauty.

A. N. Kolmogorov

Introduction. There were at least five remarkable occasions in the year 2023 related to the great mathematician Andrey Nikolaevich Kolmogorov (1903–1987):

• 120th anniversary of his birth;

• 100th anniversary of the publication of his first paper [10] with a sensational result showing by a counterexample that not always a continuous function has a convergent Fourier series. According to the well-known British mathematician W. K. Hayman (1926–2020), *This result brought him an instant fame*!

• 90th anniversary of the publication of the fundamental book:

Kolmogoroff, A.N. *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Julius Springer, Berlin, 1933.

• 60th anniversary of the opening of the 'Kolmogorov's School-Internat', 18th Physics-Mathematics School at Moscow State University (abbr.: MSU).

• 70th anniversary of Skitovich-Darmois Theorem. [Topic much appreciated by A.N. Kolmogorov.]

Academic events. The 120th anniversary of the birth of Andrey Nikolaevich Kolmogorov (abbr.: ANK) was properly and deservingly celebrated at many scientific institutions worldwide. Among them were special meetings organized by MSU and the Russian Academy of Sciences (abbr.: RAS).

The Principal Seminar in Probability Theory at MSU, chaired by A.N. Shiryaev, dedicated to ANK a series of online talks in the spring of 2023. I was glad to prepare and deliver one of the talks, 5 April 2023, see [31]. This prestigious seminar was initiated and founded by ANK in 1955 and chaired by him until 1967. The next chairman was B.V. Gnedenko (1911–1996) with A.N. Shiryaev succeeding the chairmanship until now. That seminar and its participants played a central role in establishing what is called 'Moscow Probability School'.

Two related events in Bulgaria should be listed here: the talk by N.M. Yanev [34] given at the 52th Spring Conference of the UBM, 10 April 2023, and my talk [32] given at the National Seminar in Stochastics (IMI–BAS), 26 April 2023; see also the Colloquium talk [30].

Comment. Besides MSU and RAS, two more abbreviations were used above: UBM for 'Union of Bulgarian Mathematicians' and IMI–BAS for 'Institute of Mathematics and Informatics, Bulgarian Academy of Sciences'. In what follows we also use FMI-SU for 'Faculty of Mathematics and Informatics, Sofia University' and TPA for the journal 'Theory of Probability and Its Applications'.

ANK's public recognition. Over the last 70 years, the main building of MSU is located on the hill 'Vorobyevy Gory', and is seen from almost any point in the city. On 15 October 2015, the city Mayor, Mr. S. Sobyanin, announced at a ceremony that the name ***Kolmororov Street*** is given to a small and nice street, only for pedestrians, just in a front of Lomonosov's monument, with the main building of MSU seen behind. In Moscow, this is the only street named after Kolmogorov, and notice, there is no number.

1968: First meeting with ANK. In the autumn of 1968, I met ANK for the first time. It happened in the 18th Physics-Mathematics School affiliated with MSU. This is the famous 'Kolmogorov school-internat', founded in 1963 by three academicians, A.N. Kolmogorov, I.G. Petrovskii and I.K. Kikoin.

As a student at MSU, I was lucky to be located in one group consisting of 23 students who had just entered the university directly from the 18-th school. Among them were A. K. Zvonkin[1] and S. V. Matveev.[2] Being already university students, they started 'working as teachers' at their former school. I was curious and they invited me to join them and attend their two-hour lesson. When going to the classroom, we accidentally met ANK, so I was introduced to him, followed by a brief conversation. He promptly asked me if I wished to tell something interesting to the students. My reply was, 'Yes, will try, I have in mind a few topics'. And I did. After that, I put in order my notes on one of the topics and prepared a paper for the Bulgarian magazine 'Matematika', see [26].

1969: Kolmogorov's visit to Bulgaria. It is known that ANK attended an International Conference on Problems of Education, Varna, Bulgaria, in the summer of 1969. He gave an invited talk in the section 'Education in Mathematics'. Unfortunately, I was unable to find any specific information and details about his talk. According to N.M. Yanev, our late colleague I. Ganchev (1935–2012) accompanied ANK during a trip in Bulgaria to visit several historical and cultural places. In the sources [17] and [18], dedicated to I. Ganchev and based on his archive, there was nothing said about ANK.

1972: Meeting with ANK in Budapest. In September 1972, together with A. Obretenov (1930–1997), we attended the European Meeting of Statisticians, Budapest, Hungary. Among the well-known participants was AHK. To recall, he was speaking German and French, and of course, his native Russian, but not English (with no problem to read any text in English). Thus, seeing me to walk and talk with ANK, I was asked by a group of PhD students from Harvard to help them communicate with ANK, which I did. Here are, briefly recorded or remembered, some questions (**Q**) and answers (**A**) during the conversations:

[1] Professor Emeritus, University of Bordeaux, France.

[2] Academician, Chelyabinsk State University.

Q1. *How do you usually proceed for discovering and proving a new result?*

A1. *Based on what I already know in a specific area, I arrive at a possible new statement. Either I prove it within a few days, or give a counterexample!*

Q2. *Would you tell us about results in Probability which you have found most unexpected, surprising?*

A2. *There are many. Besides being useful, this kind of problems and solutions bring attractiveness to our science. Among them are 'Skitovich–Darmois theorem' and 'Plackett problem'.*

Q3. *Is there a case of somebody finding an error in your publications?*

A3. *Yes, it happened once. A closer look at the case led to introducing a new notion followed by an interesting and correct development!*

I was amazed by the way ANK answered Q1. I did not miss the chance to share with him details about my 'hobby' to collect counterexamples in Probability and stochastic processes. At that time, my collection was still tiny, but there were already good and impressive items. I was brave to mention that one of them was the Chung's counterexample related to the error, which ANK had in mind in his answer A3 above. The statement that the convolution of two unimodal densities is unimodal, as given originally in the book [5], turned to be incorrect. Later, all was settled.

He strongly encouraged me to continue my work and advised that when having a rich and relatively comprehensive collection, to find a good way to make that treasure available to all probabilists and statisticians worldwide.

1973: First Vilnius Conference. That was the beginning of a long and continuing series of most prestigious international events in Probability and Statistics attended widely by hundreds of scientists from all over the world. Like many others, I benefited from taking part in that and three subsequent conferences. In the photo below, ANK is delivering the opening lecture, 'Randomness and Complexity'.

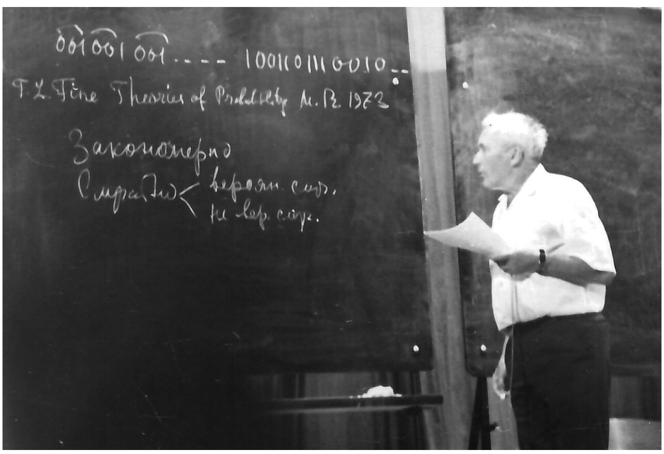

Fig. 1. A. N. Kolmogorov, Vilnius 1973

1973: ANK's special lecture. In the 'Ceremonial Hall' of the MSU, ANK delivered a special lecture, *Recent Advances in Spectral Analysis of Stationary Processes.* The announcement was distributed among all universities and institutions in Moscow and beyond a month in advance. Our group of PhD students of A.N. Shiryaev got a strong advice to prepare ourselves on the topic and take seats on time. There was an audience of more than 500 people, from the whole country, the hall was full. The presentation was impressive. ANK answered several relevant questions. Then, he was asked an unusual question, namely, 'It is well-known that Mathematics in our country is at the highest level. Is this a consequence of the fact that mathematics and politics have nothing in common?' The answer was, 'Not related to the talk, nothing to comment.'

1981: Vilnius Conference, last personal meeting with ANK. Below is a photo taken by A. Lyasoff (Boston) during my last conversations with ANK. In fact, I 'reported' to him on the good progress, which I had made over the last years, and mentioned about the useful comments and suggestions made in person by A.N. Shiryaev, B.V. Gnedenko and D.G. Kendall during the European Meeting of Statisticians, Varna, September 1979. Since some steps were already done, I decided to tell ANK about receiving an invitation from 'John Wiley & Sons' to publish my future book 'Counterexamples in Probability'. He politely wished me 'good luck', by adding, 'if first in English, think of a Russian edition'. My active work continued with frequent discussions with colleagues and friends at conferences and when traveling within Europe. I was in correspondence with leading mathematicians, thus receiving valuable suggestions. Encouragements were expressed in letters to me from J.B. Doob, K.L. Chung and K. Itô. Essential, timely suggestions were coming from friends and 'members' of the Moscow Probability School.

Fig. 2. A. N. Kolmogorov and J. Stoyanov, Vilnius 1981

1982: Bulgarian translation of the ANK's book *Foundations of Probability Theory* was published as a part of the collection [11]. It was my pleasure to be the translator. A copy was sent to ANK, no idea where it is. In a few years, after the ANK's death, I arranged a gift-copy to reach A.N. Shiryaev. The copy is in the archive of ANK in the House-Museum of Alexandrov and Kolmogorov in Komarovka.

1983: A paper on the 80th birthday of ANK. The Bulgarian magazine 'Matematika' invited me to write the paper, what I did; see [26].

20 October 1987. The agency TASS announced: 'Today is a sad day for our nation and science. The greatly respected and worldwide recognized mathematician, Academician Andrey Nikolaevich Kolmogorov, passed away . . . '

1987: The Book *Counterexamples in Probability*, 1st edition. It was 01 December 1987 when I received from the publisher, John Wiley & Sons, my author's copies. Gift-copies were given to the Libraries of IMI–BAS, FMI–SU and the Bulgarian National Library 'St.St. Cyril and Methodius'. I was planning to write a proper dedication and sign an autograph in a copy of my book and hand it personally to ANK during a visit to Moscow in 1988. The circumstances changed. The best I could do was to donate the copy to the Library of the Statistical Laboratory, MSU, whose founder and Director was ANK. Ironically, a year later it was found that the copy of my book was stolen! However, somebody's reaction was, 'Be happy, nobody steels bad books!'

ANK's *Will.* According to the Will, the House in Komarovka was inherited by A.N. Shiryaev. This included both the ANK's part and the P.S. Alexandrov's part. For the latter there had been a requirement for the future owner to support financially three PhD students in Topology, Mech-Math, MSU. All has been done accordingly! Moreover, A.N. Shiryaev used his royalty from Springer to completely renovate the whole property. Now, officially, it is ***House-Museum of Alexandrov and Kolmogorov***. The archives of the two great scientists are there, the place is marked as 'protected by the authorities'. I have been there twice, the feeling was unusual, the memories forever!

Kolmogorov's influence on Mathematics. Well-known is the role of Kolmogorov as the founder of, and a great contributor to, fundamental scientific schools within mathematics and its numerous applications. Knowing well the state of arts in several areas of mathematics and based on his own discoverings, ANK initiated new research themes which quickly became popular directions for further development.

The books [4] and [12], and the articles [15], [21], [22] contain a lot of details about the extraordinary life and work of ANK.

Perhaps one of the ANK's greatest and widely recognized achievements is the axiomatization and building up the foundation of Probability theory as a branch of Modern mathematics. Around 1930 he published a series of papers, each containing a new and significant result. Amazingly, each result was finding its natural place as a stone laid in the foundation of the future palace of Probability theory. Remarkably, these results were invented/discovered at the right time and were placed in the right location within the theory. However, the culmination was reached by the appearance of his *Grundbegriffe.* Notice, he wrote the book in German. In 1936 the book was translated into Russian, in 1956 into English. There is a 2nd Russian edition of 1974. There are editions in other languages, including in Bulgarian, see [11].

2003: Conference *Kolmogorov and Contemporary Mathematics.* That was among the greatest scientific events in the area of Mathematics in 2003. More than 1200 participants from 41 countries attended the event. Most of the ANK's direct academic pupils, also grand and great-grand pupils, came from all over the world. The scientific program was intensive. There were a large number of highest level presentations, most of them related to topics originated in works by ANK and further developed into modern scientific directions. Full information is contained in a carefully prepared booklet [7].

2016: Complete Bibliography of ANK. The book [1], of size 160 pp., was the first attempt to collect and put together the bibliographic data of all works by ANK: books, monographs and textbooks, papers in scientific journals, encyclopedic articles, prefaces to translated books, articles in periodicals and newspapers, biographies of other scientists. Included is also a List of publications dedicated to ANK. The material was compiled and edited by three ANK's academic pupils and collaborators. A new updated edition was published recently, in 2023.

Here is a remarkable detail: Mentioned in the Bibliography are 897 book titles. These are titles of books written or co-authored by ANK, titles of books edited by him, titles of books for which he had written the Preface, titles of books on which ANK had written reviews. It is not easy to imagine how such a gigantic amount of work has been completed by one person only.

ANK's Selected works. There are three volumes of his selected works (author's choice), with detailed comments by his academic pupils, all being top specialists in their specific areas. The volumes are published in Russian, and later translated into English. It is amazing to read the ANK's original papers, written compactly, telling clearly what is given, what we are looking for, and what we are proving under specified conditions. As mentioned at the beginning of my report, it was taking a couple of days to ANK to complete the proof of a possible statement. And, there are articles in which the main goal is to construct counterexamples to 'possible' statements. Obviously, for ANK to prove a statement, or give a counterexample, are equally valuable for the development of any area of mathematics.

A special issue of *Annals of Probability*, vol. **17**, no. 3 (1989), is dedicated to the memory of Kolmogorov. This great initiative came from A.N. Shiryaev who wrote a comprehensive paper of 79 printed pages, see [21].

Another great gesture of respect to ANK was made by the well-known British mathematical statistician D.G. Kendall (1918–2007) who organized a tribute to Kolmogorov's memory, a special issue of *Bulletin of LMS* **22**:1 (1990); see [15].

Kolmogorov as a person. Among everything said, written and published about ANK, the best is to refer to A.N. Shiryaev, [22]:

Everything about Kolmogorov was unusual.

From what I have seen and witnessed over the years, ANK was a quiet person, kind and attentive to anyone wishing to talk with him. He was a 'good listener' at seminar talks and when speaking with somebody. Catching easily what is the point, he was politely making comments or expressing his opinion, sometimes correctly telling that one needs a closer look at the topic.

Everybody knew about the deep knowledge and interest of ANK in areas such as history, poetry, music and arts. Thus, if you wished to talk with him, it was better for you to be well-prepared, just in case. Once, during a general conversation, ANK turned to me with a question about the ancient history of the Balkans. I was glad to reply adequately by mentioning a few relevant facts about the Thracians and their treasures.

Over several decades, being a world recognized authority in Mathematics, ANK was receiving gift copies of published books from scientists from all over the world, always with dedication and autograph. ANK was reading the books and giving advice of which book was good to be translated, and for which it was better to wait for the 2nd improved edition. All books were available in the Library of Mech-Math, MSU, or in the Library of the Statistical Laboratory. Hundreds of people benefited a lot by the opportunity to read new books and learn what other scientists were working on.

For ANK to do mathematics was both strictly individual work and collective activity. The progress, however, comes by sharing knowledge and keeping mutual respect within the professional community.

From friends who were very close to ANK, I learned that if there was an occasion to celebrate, ANK preferred it done in a quiet way, without big noise and artificial speeches. Still, on the occasion of his 50th, 60th, 70th and 80th birthday, there were articles written by his colleagues and published usually in the journals TPA and Uspehi Matematicheskih Nauk. Interestingly, these jubilee articles, resemble in their style the typical compact style of ANK's writings.

As I mentioned before, ANK established in 1955 and chaired the Principal Seminar of the Department of Probability theory, Mech-Math, MSU. The organization was remarkable: for many years it is the same day, the same hours, and even the same room. Usually the starting time was 16:00, thus giving chance to many colleagues working at other institutions to come to MSU and attend the seminar. In fact, that was a city seminar involving scientists from Moscow with participants coming even from far away. The seminar program was prepared reasonably in advance and widely distributed. After ANK's death, the chairmanship was taken by B.V. Gnedenko and then by A.N. Shiryaev. The Kolmogorov's spirit and atmosphere of that seminar, and of other seminars, has been kept for the decades to come.

It is well-known, that ANK had been awarded, many times, nationally and internationally, at the highest level. Full details are given in the paper [22]. Among the foreign awards are the prestigious 'Wolf Prize' and 'Balzan Prize'. In those and other cases, ANK's decision and advice was to use all money, around hundred thousands for each, to buy books for the Library and make subscription to journals. For some journals a pre-subscription was made for a period of 10 years to come. All books and journals are available in the Libraries.

ANK: Great supervisor of PhD students. The system ***MGP*** (Mathematics Genealogy Project) shows 82 names (with no info for a few) of PhD students who had been supervised by him and defended their dissertations for PhD degree, and some for Doctor of Sciences degree. Only three of his PhDs were foreigners. It is proper, however, to mention the name[3] of N. Dmitriev, of Bulgarian origin, who was a PhD student of ANK and defended his dissertation most likely in 1948, see [35]. Referring again to MGP we see that ANK is among the top 25 supervisors. Moreover, the pair ANK and A.N. Shiryaev, is a unique pair in the top 100 of MGP!

ANK: Bulgarian academic grand-pupils. Bulgarian PhD students have been supervised by the following ANK 'academic pupils':

B.V. Gnedenko, B.A. Sevastyanov, A.N. Shiryaev, V.M. Zolotarev,
L.N. Bolshev, V.A. Uspensky.

Let me list the names of the ANK's Bulgarian academic grand-pupils:

Boyan Dimitrov, Penka Mayster, Nikolay Yanev, Jordan Stoyanov, Hristo Pavlov, Atanas Geshev, Dimitar Khadzhiev, Milan Petkov, Svetlozar Rachev, Elisaveta Pancheva,

[3] Thanks to N. Yanev for bringing my attention to this remarkable case.

Georgi Gargov, Vladimir Sotirov, Rossitza Dodunekova, Stoyan Poryazov, Rumyana Lukanova.

Comment. I have to add the names of other Bulgarian stochasticians with doctoral degrees, *Dimitar Vandev, Plamen Mateev, Mitko Dimitrov* and *Doncho Donchev*, supervised, respectively, by *A. Vershik, M. Malyutov, A. Solovyev* and *A. Yushkevich.*

ANK: Academic great-grand-pupils worldwide. It is not easy to count them, but there is a very large number of ANK's academic grand pupils and academic great-grand pupils. It is curious to tell that besides those in Bulgaria, ANK had, via Bulgaria, academic great-grand pupils in countries such as Mexico, Saudi Arabia, Brazil, Northern Macedonia, USA, Australia, New Zealand, Vietnam, Brazil, and possibly others.

ANK: Chairman of the Scientific Council at MSU. In the beginning of the 70s last century, ANK was the chairman of the scientific council at Mech-Math, MSU, for defense of dissertations and rewarding young scientists with PhD degree ('candidate of science', 'doctor') and Doctor of Sciences. Thus, ANK was the person signing the final Council decision for the doctoral degrees of five Bulgarian stochasticians.

Bulgarian Academic Rewards for ANK's academic pupils. Two ANK's academic pupils have been deservingly rewarded with honors by BAS and IMI for their essential contribution to the development of Stochastics in Bulgaria. They are B.A. Sevastyanov and A.N. Shiryaev.

Bulgarians publishing papers in TPA. The journal TPA was founded by ANK in January 1956. SIAM Society organized its translation from Russian into English for distributing it in the USA and elsewhere.

Here is another historical fact: Twelve Bulgarian probabilists have published papers in TPA. In a random order, they are: Tz. Ignatov, P. Mateev, G. Yamukov, N. Yanev, P. Mayster, D. Khadzhiev, S. Rachev, E. Pancheva, P. Jordanova, B. Penkov, P. Kopanov, J. Stoyanov.

'Kolmogorov number' for Bulgarian stochasticians. For any mathematician one can define a non negative integer number, denoted here by K, named 'Kolmogorov number', which is easy to find from MathSciNet. The number K can be considered as the 'collaboration distance' of a mathematician from AHK.

First, introduce the notations $\mathcal{S}_0$ and $\mathcal{S}_n$ for $n \in \bar{\mathbb{N}} = \{1, 2, \ldots, \infty\}$ for the sets (groups) of mathematicians. Thus, we deal with a finite number of sets each with a finite number of members (elements). We may think of a finite graph where each mathematician is a 'node', and if two mathematicians, X and Y, are co-authors, they are connected by an 'edge'. Every mathematician belongs to one of these sets depending on his/her Kolmogorov number. Roughly speaking, K is the number of 'edges' connecting X and ANK. The sets $\mathcal{S}_n$ can be defined inductively.

K = 0. It is clear that ANK is the only member of the set $\mathcal{S}_0$. Any other X belongs to exactly one of the sets $\mathcal{S}_n, n \in \bar{\mathbb{N}}$: X $\in \mathcal{S}_n$ if and only if X has its K $= n$. Notice, if all works of X are single-authored, then X belongs to the set $\mathcal{S}_\infty$.

K = 1. For $n = 1$, the set $\mathcal{S}_1$ includes all direct co-authors of ANK, so X $\in \mathcal{S}_1$ means that X has K = 1, there is one 'edge' connecting X and ANK. In this group there is one Bulgarian: N. Dmitriev, see [35].

K = 2. How is the set $\mathcal{S}_2$ built up? Suppose that X and Y are co-authors such that X $\in \mathcal{S}_1$. If Y $\notin \mathcal{S}_1$, then Y has K = 2, hence Y $\in \mathcal{S}_2$. In this case there is one mathematician, it is X, the 'node' between Y and ANK, and there are 2 consecutive 'edges' connecting Y with ANK. For clarity, after Y we write in brackets the name of X. There are no Bulgarian stochasticians in the group $\mathcal{S}_2$.[4]

K = 3. We easily extend the idea used for the set $\mathcal{S}_2$. Namely, if X and Y are co-authors such that X $\in \mathcal{S}_2$ and Y is not in any of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$, then Y $\in \mathcal{S}_3$. Equivalently, Y has K = 3, there will be 2 'nodes' (mathematicians) and 3 consecutive 'edges' connecting Y with ANK. After Y we can write the 2 specific names.

Now one can describe inductively any set $\mathcal{S}_n$ for $n > 3$. Indeed, if X and Y are co-authors such that X $\in \mathcal{S}_{n-1}$ and Y is not in any of the sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{n-1}$, then Y $\in \mathcal{S}_n$. Equivalently, Y has K = n, there are $n-1$ 'nodes' (mathematicians) and n consecutive 'edges' connecting Y with ANK.

Bulgarian stochasticians with K = 3:[5] A. Obretenov (Galambos-Rényi), B. Penkov (Sendov-Nikolskii), B. Dimitrov (Klebanov-Linnik), J. Stoyanov (Liptser-Shiryaev), R. Dodunekova (Khasminskii-Sinai), S. Rachev (Zolotarev-Sinai), D. Donchev (Yushkevich-Dynkin).

Bulgarian stochasticians with K = 4: N. M. Yanev (Vatutin-Zolotarev-Sinai), M. Tanushev (J Stoyanov-Liptser-Shiryaev), M. Savov (Doney-Ibragimov-Fadeev), A. Geshev (Kostadinov-Zabreiko-Fadeev), P. Kopanov (J. Stoyanov-Liptser-Shiryaev), E. Stoimenova (Balakrishnan-Kotz-Ostrovskii), L. Boneva (Kendall-Lorenz-Kantorovich), E. Pancheva (Balkema-Embrechts-Shiryaev), Tz. Ignatov (Rachev-Zolotarev-Sinai), L. Mutafchiev (Jaworski-Götze-Prokhorov), M. Bojkova (Bruss-Yor-Vershik), V. Kaishev (Krug-Tsypkin-Voronov), D. Vandev (Obretenov-Galambos-Rényi), L. Minkova (Omey-Korolev-Gnedenko), O. Enchev (= A. Lyasoff) (Stroock-Bresis-Arnold), S. Penev (Roszczynski-Vanderbei-Dynkin), N. Neykov (Cizek-Spokoiny-Shiryaev), G. Yanev (Nevzorov-Ibragimov-Fadeev), V. Stefanov (Manca-Korolyuk-Gnedenko), V. Stoimenova (Yakovlev-Gordon-Gorin), N.I. Nikolov (Kateri-Sturmfels-Gelfand), S. Stoev (Taqqu-Y. Meyer-Arnold), D. Vladeva (J. Stoyanov-Liptser-Shiryaev), Ch. Pirinsky (J. Stoyanov-Liptser-Shiryaev), Tz. Zaevski (Kyurkchiev-Andreev-Nikolskii), E. Petkova (Vannuci-Pfeiffer-Freidlin), R. Robeva (Pitt-Shepp-Shiryaev).

While the value of the Kolmogorov number for an individual can be considered just for curiosity, much more interesting and valuable is to see the large number of well-known co-authors of Bulgarian stochasticians. This is a clear indication of the strong relationships between Bulgarian stochasticians and colleagues worldwide.

Stochastics in Bulgaria. The paper [25] is a a good source of information about the status and development of stochastics in Bulgaria for a long period, from the end of 19th century to the end of 2014.

[4]However, with K = 2 are the following Bulgarian mathematicians, not from the area of Stochastics: Bl. Sendov (Nikolskii), V. Drensky (Ershov), P. Kenderov (Archangelskii), P. Popivanov (Egorov), B. Bojanov (Nikolskii), L. Stoyanov (Smirnov), P. Petrushev (Nikolskii), A. Andreev (Nikolskii), D. Bainov (Mitropolskii).

[5]Any Bulgarian mathematician not from Stochastics, who is a co-author of somebody with K = 2, see their names above, will have K = 3. Here are: L. Iliev, J. Revalski, S. Troyanski, K. Ivanov, O. Mushkarov, R. Lazarov, V. Petkov, V. Popov, S. Markov, N. M Nikolov, G. Nikolov, N. Ribarska, M. Konstantinov , N. Kyurkchiev, I. Kortezov. Some colleagues have K = 3 in a different way, e.g., S. Dodunekov (Bassalygo-Sinai), N. Popivanov (Moiseev-Oleinik), N. Kutev (P Lions-Arnold), L. Katzarkov (Itenberg-Viro).

In this talk, I would like to share with the audience my personal observations and details about my involvement and work in the years after 1970. That was the year when I graduated from MSU and joined the Mathematical Institute of BAS. Being well-familiar with the traditions, observed and experienced at MSU, and keeping that as my personal philosophy, I was well-'equipped' to collaborate with other colleagues and we started implementing novelties and good academic practices. As a result of our collective work, a visible progress was achieved by Bulgarian stochasticians. The following positive aspects have to be mentioned:

(a) *Staff.* A good number of new staff were appointed at both BAS and SU, people who got their university education and/or doctoral degrees mainly at MSU, and also at other institutions.

(b) *Research.* Gradually, the research output was increasing in the number of papers, reports at seminars and conferences, and publications in top professional journals. The range of topics covered in those papers was quite wide: Markov chains and processes, Queuing theory, Survival analysis, Characterization of distributions, Brownian motion, Diffusion processes, Stochastic calculus and SDEs, Branching processes, Discrete distributions, Martingales, Limit theorems, Statistical inference, Stochastic models in Biology. In more recent times, we see successful work in Stochastic financial models, Lévy processes, and others.

(c) *Publications.* A large number of first class papers by Bulgarian stochasticians were published in leading international journals. Besides in Bulgarian, books on modern stochastics appeared in English language by top international publishers.

(d) *Courses at SU.* New modern courses and programs were established for the students at FMI-SU. For nearly 25 years, staff from both BAS and SU were taking actively part in the education at SU, including supervision of Master and Doctoral students. Several books at good international level were published. Unfortunately, the involvement of staff at BAS in SU business became negligible.

(e) *Events.* We organized a series of International Summer Schools in Probability and Statistics, Workshops in Data Analysis, European Meeting of Statisticians, First World Congress in Branching Processes, and twice, Sections in SDEs during conferences in differential equations. Several other national seminars and meetings were held.

(f) *Seminars.* In the autumn of 1970 we organized the 'General Seminar in Probability and Statistics', by fixing that talks would be given on every Wednesday, at 15:00. The chairman was B. Penkov for a few months and then A. Obretenov for many years to come. I delivered two talks, and was appointed as a Seminar Secretary for 10 years, until 1980, being succeeded by S. Rachev. We managed to consolidate almost all Bulgarian stochasticians. Around 180 seminar talks were given by staff members at BAS and SU, colleagues from other institutions, and foreign visitors. It was a great time of dedicated individual and collective work. I was collecting real paper copies of all announcements. A folder containing that invaluable treasure was given in person to a known colleaguessa with the words, 'this is unique, please, preserve it for the history'. It is sad to report today that the folder has not be found. The search continues.

(g) *Dissertations.* Over the years, a large number of mathematicians were attracted to perform studies in Stochastics and its applications. They earned their scientific degrees, thus joining the Stochastics guild in Bulgaria.

(h) *Visits.* Bulgarian stochasticians were finding good positions at prestigious universities abroad. Impressive is the list of leading foreign stochasticians who were quests of BAS or SU. The international cooperation went up.

Comment. A separate article could and should be written on any of the above items. The paper [34] is a useful source about ANK and covers developments in Branching processes, in general, and in Bulgaria. I am appealing to other colleagues also to share appropriately their own experience and memories.

It deserves to tell good words about the modern trend of the courses at SU. A. Obretenov was teaching Probability theory based on Kolmogorov's axiomatics. Following the advice by ANK, given in Budapest, the textbook [10] was prepared and published as the first modern book in Probability in Bulgaria. Another advice by ANK was to have at hands a collection of exercises, from easy to difficult, with solutions or hints. Together with colleagues, after a few years of working on that project, we produced the manual [33], a rich collection of diverse exercises, used for university courses at all stages. The material was class-tested proving to be more than useful. Perhaps in the 70s-80s of the last century, [33] was one of the best available collections of the sort not only nationally but also internationally. We were glad to see that the manual was translated into Polish, there have been two editions: Warsaw, PWN, 1982 and 1992. The 2nd Bulgarian edition of 1985, containing already 777 exercises, was translated into English. That edition of 1989 was dedicated to the 100th anniversary of the foundation of SU. There is a 3rd Bulgarian edition of 2012, still in use at SU and other universities.

In the academic year 1973/1974, I started teaching regularly a special modern course in 'Stochastic processes'. As a result, after giving the course for a few years, I prepared and published the book [28]. The book was used for many years, in Bulgaria and other countries. It proves useful and fresh even today.

In the 70s and 80s, for more than 10 years, I was conducting a 'Circle in Analysis, Probability, Combinatorics', attended usually by 10 good students at FMI-SU. It was a work for pleasure, not for marks. It was a great satisfaction to see that most of the attendees grew up and became professional mathematicians.

Additionally to our institutional duties, there was a system for staff at BAS and SU to travel to other cities of the country and give lectures on selected mathematical problems addressed to school students and their teachers. That was also part of the students' training for national and international competitions in mathematics, such as the National Olympiad and the IMO. I was among the not too many colleagues who wrote and published popular papers in the area of Probability and Combinatorics in the magazine 'Matematika', in Bulgarian. All that 'public/society activity' was and is quite useful in general, but also for attracting young boys and girls to study mathematics.

Let me tell more about my book 'Counterexamples', see [29] for details of all three editions in English and two editions in Russian. I was satisfied to learn from foreign colleagues that the book became a useful and irreplaceable supplementary source for young researchers, university teachers of undergraduate and graduate courses, including for training of PhD students. Based entirely on the book, special courses/seminars have been organized at Yale University, MSU, Technische Universität Dresden. I have given a large number of talks on the topic.

I met with interest a series of questions from colleagues during my trips and visits worldwide. I was asked, e.g., how the idea was born, how I managed to complete such an enormous amount of work over so many years, how I was allowed to deal with a Western publisher, how the book was met in Bulgaria and abroad, what kind of reviews appeared, did I become a millionaire, etc.

To answer the above and other related questions, I found it appropriate to include details in the present talk. Let me add a few words more.

First, 'Counterexamples in Probability' is a volume in the most prestigious series, 'Probability and Mathematical Statistics' of the Publisher 'John Wiley & Sons'. For 'Wiley' successful is a book receiving 5–6 reviews. Surprisingly to me, there were 24 reviews on the 1st edition of my book. This was and still is an absolute record for a book by 'Wiley'. Some reviewers explicitly wrote that, for good reasons, the book is a great achievement, it was listed among the most useful sources in Stochastics, and it was predicted that the book will 'live' at least half a century. Below is a picture of the covers of some editions of the book.[6]

Fig. 3. Covers of the editions 1987, 1997, 2012, 2013

Second, for this book, the 1st edition, I received an award by the Union of Scientific Workers in Bulgaria. Later, the book was explicitly mentioned in the citations on two occasions, when I became 'Elected Member of the International Statistical Institute' and 'Fellow of the Institute of Mathematical Statistics'.

Third, what I said above, confirms that the book 'Counterexamples in Probability' is considered as a contribution to Stochastics in general, hence to Stochastics in Bulgaria. Notice, the two leading Bulgarian institutions, BAS and SU, are listed as my affiliation in all five editions of the book.

In conclusion, a lot has been done in the area of Stochastics in Bulgaria over the last half a century. The research was transformed from traditional classical topics to most modern topics from Contemporary stochastics.

Probabilistic problems with unusual solutions. The terms 'unusual', 'non-standard', 'unexpected', 'surprising', or 'strange but true', are used to name something which does not correspond to what we believe we know, it may even contradict our intuition. However, in general, the human knowledge is not unlimited and as for our intuition, it is frequently wrong.

[6] There are five illustrations by A.T. Fomenko (MSU) used in the text of all editions. One of them is used for the cover of the 1st and the 2nd 'Wiley' editions, another one, for the MCCME edition.

I have mention above about ANK's indication of two problems with unexpected solutions, see his answer A2. I will give details below and describe some real and relevant stories.

Characterization of the normal distribution. Because of the Central Limit Theorem, the importance of this distribution for both theory and applications is out of doubt. Hence, one of the important questions is: How to characterize the normal distribution? Equivalently, how to find a property, preferably easy to check, which is true only for the normal distribution?

One of the first results was found by G. Polya and S.N. Bernstein: If X and Y are independent random variables with the same unknown distribution function, say F, and the variables $X+Y$ and $X-Y$ are independent, then necessarily F is normal. The converse is clearly also true. Thus, by using two symbols, $\mathcal{N}$ for the normal distribution, and $\perp\!\!\!\perp$ for the property 'independence', we can write:

Given are $X \sim F$, $Y \sim F$ with $X \perp\!\!\!\perp Y$. Then $X+Y \perp\!\!\!\perp X-Y \iff F \in \mathcal{N}$.

There are several and diverse extensions of this result. Here is a remarkable case and a few related stories. Suppose $X_1, \ldots, X_n, \ldots$, are independent random variables with values in $\mathbb{R} = (-\infty, \infty)$ and arbitrary distributions, and $a_1, \ldots, a_n, \ldots$ and $b_1, \ldots, b_n, \ldots$, are real numbers. Consider two linear forms:

$$L_n^{(a)} := a_1X_1 + \cdots + a_nX_n \quad \text{and} \quad L_n^{(b)} := b_1X_1 + \cdots + b_nX_n. \tag{1}$$

Since both forms are functions of the same variables, we may think that they are dependent, or, that 'it is unlikely' for $L_n^{(a)}$ and $L_2^{(b)}$ to be independent. Contrary to our intuition, there is a unique and remarkable case, when such a property, independence, holds!

Skitovich–Darmois Theorem. *With the above notations, we have:*
Claim 1. $L_n^{(a)} \perp\!\!\!\perp L_n^{(b)} \implies$ *all* X_j *with* $a_jb_j \neq 0$ *are* $\mathcal{N}$.

Claim 2. *If all* X_j *with* $a_jb_j \neq 0$ *are* $\mathcal{N}$ *and* $\sum_{i=1}^{n} a_ib_i = 0 \implies L_n^{(a)} \perp\!\!\!\perp L_n^{(b)}$.

Besides the original proofs in [2] and [23], available in the literature are several different proofs, see [6], [8], [9], [20]. One of the ideas is to use the multiplicative property of the characteristic function of sums of independent random variables and arrive at a differential equation whose solution exactly corresponds to the normal distribution.

1953: Story 1: It happened 70 years ago, two papers appeared, Darmois [3] and Skitovich [23], proving the same result. Working under the supervision of Yu.V. Linnik, Skitovich completed his work in 1952, to meet first a skepticism from Linnik, who showed the work to ANK. ANK found that the result was amazingly unusual and suggested, 'check again'. In about 3 months, all was confirmed to be correct. Thus, Skitovich announced his result in 'Doklady', [23], and the full text in 'Izvestiya', [24]. The common name used for decades is Darmois-Skitovich or Skitovich-Darmois Theorem.

However, Skitovich–Darmois Theorem became a center around which a lot of research was performed. One of the great scientists, ***C.R. Rao***[7] (India, USA), who was both a

[7]C.R. Rao was born on 10 September 1920 and died in August 2023 just three weeks before his 103th birthday.

Mathematician and a Statistician, made his own contribution, by publishing papers on that topic, including one in 'Sankhya', see [20].

1970: Story 2. In September 1970, C.R. Rao visited the Mathematical Institute, BAS, Sofia, for a few days. In the ancient building, in Latinka Str., he delivered a talk on Skitovich–Darmois Theorem (by chalk on a small black board). Being a good speaker, Rao provided convincing arguments. During the questions/comments time, I asked him the following question:

The linear forms $L_n^{(a)}$ and $L_n^{(b)}$ in (1) depend on n, a fixed number. What will happen if instead of n we take random Markov times?

Recall, Markov times (also called stopping times) are positive (X_j)-adapted integer-valued random variables. Rao commented, 'No idea, it looks more complicated'. He made a record in his Notebook.

1971: Story 3. In 1971, ***Yu. V. Linnik*** visited Sofia as a guest of the BAS. He delivered a public talk on 'Random Walks'. Then the discussion switched to Skitovich-Darmois Theorem. I told him about C.R. Rao and my question. Linnik's reaction was, 'Looks interesting'. I am keeping a piece of a paper with my brief notes. Somehow, I have forgotten the story, it is time to refresh it.

1973: Story 4. The following fundamental book was published:

A. M. Kagan, Yu. V. Linnik, C. R. Rao. *Characterization Problems of Mathematical Statistics.* New York, John Wiley & Sons, 1973. (Russian, Moscow, Nauka, 1972.)

At the end of the book there is a chapter called 'Unsolved Questions'. Among them is included my question mentioned above, see Problem 4.1 in [8].

Possibly somebody has tried to answer this or a similar question. After nearly 50 years, A. Kagan and L. Klebanov (2021) (Personal communication) have made a progress. They start with an infinite sequence of independent random variables $X_1, X_2, \ldots$, and independent from them all a positive integer-valued random variable τ. Consider two random sums, in each sum the number of terms is τ:

$$\tilde{L}^{(a)} := a_1 X_1 + \cdots + a_\tau X_\tau, \quad \tilde{L}^{(b)} := b_1 X_1 + \cdots + b_\tau X_\tau.$$

Since τ is independent from all X_j, then τ is a Markov time.

Statement. *Suppose that the random variables X_j are all normally distributed and $X_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$. Then the independence property $\tilde{L}^{(a)} \perp\!\!\!\perp \tilde{L}^{(b)}$ may hold only under two conditions: τ is degenerate, i.e., there is a fixed integer n_0 such that $\mathsf{P}[\tau = n_0] = 1$; and, $\sum_{j=1}^{n_0} a_j b_j \sigma_j^2 = 0$.*

Comment. Tools from characteristic functions are intensively used in the proof. Such a statement is a good step, however, further study is needed.

It seems, the most reasonable formulation of the problem is to consider two independent random times, τ_1 and τ_2, being independent from all variables X_j, and define the random sums

$$\tilde{L}_1 := a_1 X_1 + \cdots + a_{\tau_1} X_{\tau_1}, \quad \tilde{L}_2 := b_1 X_1 + \cdots + b_{\tau_2} X_{\tau_2}.$$

Notice, $\tilde{L}_1$ is a sum of a random number, τ_1, of random variables. Similarly, τ_2 is the number of terms in $\tilde{L}_2$.

As before, of interest are the independence of the sums $\tilde{L}_1$ and $\tilde{L}_2$ and the normality of their terms. As far as I know, no progress is reported, yet!

Plackett problem. This is one of the problems indicated by ANK as having an unusual solution, see question Q2 and its answer A2. To be honest, in 1972, I heard the name 'Plackett' for the first time. Later, I identified the exact Plackett's name and found the paper which ANK had had in mind.

Let me give details. R.L. Plackett (1920–2008) was a well-known British mathematical statistician[8]. He considered the following problem. Suppose that X is a random variable defined on an underlying probability space $(\Omega, \mathcal{F}, \mathsf{P})$, it takes values in the set of reals $\mathbb{R} = (-\infty, \infty)$ and let its distribution function denoted by F, $F(x) = \mathsf{P}[X \leq x]$, $x \in \mathbb{R}$, be absolutely continuous, i.e., there is a density, say $f = F'$. It is assumed that the second moment $m_2 := \mathsf{E}[X^2]$ is finite, and let σ^2 be the variance. We have n independent observations, $X_1, \dots, X_n$, and the corresponding order statistics are $X_{(1)} < X_{(2)} < \dots < X_{(n)}$. The difference $X_{(n)} - X_{(1)}$ is the sample range and we are interested in the quantity d_n, the expected sample range normalized by the standard deviation:

$$d_n := \frac{\mathsf{E}[X_{(n)} - X_{(1)}]}{\sigma}.$$

Before turning to this general case, let us start with something simpler.

Particular case, $n = 2$. Suppose that X and Y are independent random variables with the same absolutely continuous distribution F on the real line $\mathbb{R}$ with finite second moment: $\mathsf{E}[X^2] < \infty$. Assume for simplicity that X has zero mean and unit variance: $\mathsf{E}[X] = 0$, $\mathsf{E}[X^2] = 1$. We are interested in the quantity

$d_2 = \mathsf{E}[|X - Y|]$.

Questions. (a) What is the maximal value of d_2? (b) For which distribution F is d_2 attaining its maximum?

Answer. We use $\mathcal{U}(-\sqrt{3}, \sqrt{3})$ for the uniform distribution on the interval $(-\sqrt{3}, \sqrt{3})$, its density is $f(x) = \dfrac{1}{2\sqrt{3}}$ for $x \in (-\sqrt{3}, \sqrt{3})$, otherwise $f(x) = 0$.

Let me write together the answer to both questions, (a) and (b):

$$d_{2,\max} = \max_{F: X \sim F, Y \sim F} \mathsf{E}[|X - Y|] = \frac{2}{\sqrt{3}} \quad \textit{\textbf{if and only if}} \quad F \sim \mathcal{U}(-\sqrt{3}, \sqrt{3}).$$

General case, any $n \geq 2$**.** All is described as above. In 1947, Plackett [19] suggested the following amazing upper bound for d_n.

Theorem. *Under the above assumptions and notations, for any integer* $n \geq 2$,

$$d_n = \frac{\mathsf{E}[X_{(n)} - X_{(1)}]}{\sigma} \leq n\sqrt{\frac{2}{(2n-1)!}\left((2n-2)! - ((n-1)!)^2\right)}. \tag{2}$$

Remark. The original Plackett's proof in [19] is based on arguments from calculus of variations. A refinement of the proof was suggested by Moriguti [14]. The bound for d_n looks indeed unusual. Notice, the distribution F is arbitrary.

[8] He was the first professor in Mathematical statistics at Newcastle University (UK). When I was working at that university, I met him several times. He was pleased by the story with Kolmogorov and appreciated very much my seminar talk on 'Plackett problem', October 2005.

The most delicate question arising is: Which is F bringing an equality in the inequality (2)? Plackett suggested the answer: $d_{n,\max}$ is attained for a uniform distribution which can be exactly specified in terms of the mean and the variance of $X \sim F$. I am not providing details here, see the paper [13]. This is a challenging topic appropriate for separate special seminar talk.

Concluding comments. Being in our profession for more than 50 years, I have accumulated a unique experience and knowledge. They definitely have been useful in the past, are useful now, perhaps will be useful in the future. Hence, such an info, properly written, is of historical value. 'Who should write it down?' The best is to refer to John E. Elliot, 'If not me, then who? If not now, then when?' By writing this paper, I am completing my professional and institutional duty.

I am grateful to Prof. A. K. Zvonkin (University of Bordeaux), one of the famous Kolmogorov's pupils, for our long-standing friendship and collaboration. Our recent Skype discussions on all topics touched in this talk refreshed well our memories and were more than pleasant and useful.

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