Probability Distributions: New Results on their Moment Determinacy

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Brief Historical Comments:

Chebyshev: proved the C.L.T. for arbitrary seq. of random variables. He showed that the moments of the centered normalized sums convergence to those of the normal distribution $\mathcal{N}(0,1)$. Hence ... job done!

Markov: refined the proof, asking the Teacher if it is possible to have more than one distribution corresponding to a moment sequence.

Chebyshev sent this to **Hermite**, who ... passed to **Stieltjes** all Chebyshev stuff (papers, ideas, continued fractions, integral inequalities)

Stieltjes did a great job! 1894 Name, "iff" conditions, two examples of distributions on $\mathbb{R}_+$ which are non-unique in terms of the moments. Later, strong interest: Hamburger, Hardy, Carleman, Riesz, Fréchet, Cramér, Hausdorff, Akhiezer, Krein, Nudelman, Karlin, ...

Probability Theory ... or Measures and Functions

Probability space $(\Omega, \mathcal{F}, \mathbf{P})$, random events, variables, vectors, etc.

Random variable X, range of values $\mathbb{R}$, $\mathbb{R}_+$ of a subset.

Distribution function $F(x) = \mathbf{P}[\omega : X(\omega) \le x], x \in \mathbb{R}$. Properties, supp.

Density $f(x) = F'(x),\ x \in \mathbb{R}$ (not always !)

Distribution $\mu = \mu_F$, the measure on $(\mathbb{R}, \mathcal{B})$ generated by F.

Characteristic function $\psi(t) := \mathbf{E}[\mathrm{e}^{\mathrm{i}tX}], t \in \mathbb{R}$ (Fourier Transform, FT)

Moment generating function If $M(t) = \mathbf{E}[\mathrm{e}^{tX}] < \infty,\ t \in (-t_0, t_0),\ t_0 > 0$

Bellow we use standard abbreviations: r.v., d.f., ch.f., m.g.f.

Kolmogorov Theorem: Given a d.f. G, then there exists a probability space and a r.v. Y such that the d.f. of Y is exactly G.

Given a r.v. $X \sim F$, expectation (expected value, mean value, ...) is:

$\mathbf{E}[X] = \int_\Omega X(\omega)\,\mathrm{d}\mathbf{P}(\omega) = \int x\,\mathrm{d}F(x) = \int x\,f(x)\mathrm{d}x = \int x\,\mu(\mathrm{d}x).$

Assume X has absolute moments: $\mathbf{E}[|X|^k] < \infty,\ k = 1, 2, \ldots.$ Then $m_k = \mathbf{E}[X^k]$, moment of order k; $\{m_k, k = 1, 2, \ldots\}$ moment seq. of F, X.

Question: We have $\{m_k\}$. Is there a d.f. $G \neq F$ with the same moments?
Answer: Sometime 'yes', sometime 'no'.

Equivalently: If $f(x)$ is a function and $\hat{f}(t)$ its FT, can we recover f if we know all derivatives of $\hat{f}(t)$ at $t = 0$? Recall: $\frac{d^k}{dt^k}\,\hat{f}(t)|_{t=0} = m_k$.
Notice, $\{m_k\}$ is a discrete info, f and F are infinite ... Inverse Fourier. Numerical methods.

Terminology: **Name** of the moment problem, depends on the supp(F):

$[0,1]$ (**Hausdorff**); $\mathbb{R}_+ = [0,\infty)$ (**Stieltjes**); $\mathbb{R} = (-\infty,\infty)$ (**Hamburger**)

Fact 1: For any X, F, f, μ with finite moments $\{m_k\}$ there are two possibilities, (a) or (b):

(a) F is **M-determinate**, or unique with these moments (**M-det**), i.e.

$$\int x^k \, dF(x) = \int x^k \, dG(x) \text{ for all } k = 1, 2, \ldots \Rightarrow F = G.$$

(b) F is **M-indeterminate** (**M-indet**) if there is at least one d.f. G with

$$\int x^k \, dF(x) = \int x^k \, dG(x) \text{ for all } k = 1, 2, \ldots, \text{ but } G \neq F.$$

Fact 2: For any M-indet F, there are infinitely many absolutely continuous distributions, infinitely many discrete distributions, and infinitely many singular distributions, all having the same moments as F.

The latter can be extracted from a series of works by C. Berg and co.

Notion: **Stieltjes class**. See next page.

Importance of M-uniqueness: Fréchet-Shohat Theorem:
Sequence of measures $\mu_n, n = 1, 2, \ldots$ s.t. for each k, $\int x^k \, \mathrm{d}\mu_n \to m_k$ as $n \to \infty$. Then $\{m_k\}$ is the moment sequence of a measure, say μ, and if μ is M-det, then μ_n converges weakly to μ.
Wide applications in graph theory, number theory, statistical physics, ..., combinatorics.

Statistical Inference: Recall skewness and kurtosis coefficients. Risky business. Optimization problems, including in Financial Mathematics.

Log-normal distribution:

$Z \sim \mathcal{N}(0,1)$, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$, $m_{2k+1} = 0$, $m_{2k} = (2k-1)!!$.

Then $X = e^Z \sim Log\,\mathcal{N}(0,1)$, density

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left[-\frac{1}{2}(\ln x)^2\right], \; x > 0; \;\; f(x) = 0, \; x \le 0.$$

About X: No m.g.f., HT, $m_k = \mathbf{E}[X^k] = e^{k^2/2}$, $k = 1, 2, \ldots$

Two infinite sets of r.v.s, one absolutely continuous, one discrete:

$X_\varepsilon, \varepsilon \in [-1,1]$: density $f_\varepsilon(x) = f(x)\,[1 + \varepsilon \sin(2\pi \ln x)]$, $x > 0$

Y_a, $a > 0$: $\mathbf{P}[Y_a = ae^n] = a^{-n} e^{-n^2/2}/A$, $n = 0, \pm 1, \pm 2, \ldots$

Shocking property: $\mathbf{E}[X_\varepsilon^k] = \mathbf{E}[Y_a^k] = \mathbf{E}[X^k] = e^{k^2/2}$, $k = 1, 2, \ldots$.

Conclusion: LogN is M-indet! So 'many' others, the same moments.

Classical Conditions: Back to Stieltjes and Hamburger: moment seq. $\{m_k,\ k = 1, 2, \ldots\}$, write an infinite seq. of Hankel matrices:

H_N involves $m_0 = 1, m_1, \ldots, m_N, \ldots, m_N, \ldots, m_{2N},\ N = 1, 2 \ldots.$

If all H_N are non-negative, or positive-definite, then the is a measure μ with these moments, and it is unique. Advanced case (O.K., Ark. Math. 2010): complex-valued, pseudo-positive-definite ...

Recently, Berg, Chen, Ismail (Math. Scand. 2002): If λ_N is the smallest eigenvalue of H_N, then $\lambda_N \searrow$ as $N \to \infty$. Moreover, there is a unique measure μ with the moments $\{m_k\}$ **iff** $\lambda_N \to 0$ as $N \to \infty$.

Not easy to check. For a century, find checkable conditions, even if they are only sufficient, or only necessary for either M-det or M-indet.

All is done!? MathSciNet, 2017="moment problem", 226 new papers.

Cramér condition: For a r.v. $X \sim F$ on $\mathbb{R}$, let the m.g.f. exists:

$$M(t) = \mathbf{E}[e^{tX}] < \infty, \ t \in (-t_0, t_0), \ t_0 > 0 \ (\textbf{light tails}).$$

• X has all moments finite; • X, i.e. F, is M-det.

Remark: Existence of m.g.f. $\Leftrightarrow$ analytic ch.f. More, strong Cramér: $M(t) < \infty$ for all $t \in \mathbb{R} \Leftrightarrow$ real analytic ch.f.

Examples: Bounded supp(F) (Hausdorff case), *Exp*, $\mathcal{N}$, *Laplace*.

If F has **heavy tail(s)** (no m.g.f.), F is either M-det or M-indet

Question: When is a heavy tailed F unique? And when non-unique?

We will see that we may have uniqueness even if the tail(s) are 'heavy', but not too much, and non-uniqueness comes out if the tail(s) of the distribution is 'very heavy'. Later, example with *Exp*.

"New" Condition: **G. Hardy (1917/1918).** *The Math. Messenger*

Statement (Hardy): r.v. $X > 0$, $X \sim F$. Suppose $\sqrt{X}$ has m.g.f.:

$\mathbf{E}[e^{t\sqrt{X}}] < \infty$ for $t \in [0, t_0)$, $t_0 > 0$ (H)=Hardy's condition; $\frac{1}{2}$-Cramér.

Then all $m_k = \mathbf{E}[X^k] < \infty$, $k = 1, 2, \ldots$ and F is the only d.f. with $\{m_k\}$.

Hint: Details in S&L TPA (2012/2013). The main, F is arbitrary. More, condition (H) $\Longleftrightarrow m_k(X) \leq c^k\,(2k)! \Rightarrow$ $\mathrm{C}[\{m_k\}] = \infty \Rightarrow$ X is M-det.

Notice: The condition is on $\sqrt{X}$ but the conclusion is for X.

Corollary: If a r.v. $X > 0$ has a m.g.f., then its square X^2 is M-det.

Result: In (H), $\frac{1}{2}$ is the best possible constant for X to be M-det.
For each $\rho \in (0, \frac{1}{2})$ there is a r.v. Y with $\mathbf{E}[e^{tY^{\rho}}] < \infty$ s.t. Y is M-indet.

Comment: Hardy's condition is sufficient but not necessary for M-det.

Carleman: Quasi-analytic functions, book 1926.

Known are all moments $m_k = \mathbf{E}[X^k], \; k = 1, 2, \ldots$

For X in $\mathbb{R}$ or $\mathbb{R}_+$, define the following **Carleman quantity**:

$$C = \sum_{k=1}^{\infty} \frac{1}{(m_{2k})^{1/2k}}, \qquad C = \sum_{k=1}^{\infty} \frac{1}{(m_k)^{1/2k}}.$$

Theorem 1 (Carleman): $C = \infty \Rightarrow F$ is M-det. (Only sufficient!)

Remark: If F is M-indet, then necessarily $C < \infty$ (converse Carleman's).

Theorem 2: (Koosis and Pakes) Suppose $C < \infty$. If F has a density f such that $-\ln f(e^x)$ is convex for $x > x_0 > 0$, then F is M-indet.

Krein: Let $X \sim F$, density $f > 0$, finite moments. For X in $\mathbb{R}$ or $\mathbb{R}_+$, define the **Krein quantity** (Krein 1944 and Slud 1993):

$$\mathrm{K}[f] \equiv \int_{-\infty}^{\infty} \frac{-\ln f(y)}{1+y^2} \mathrm{d}y, \quad \mathrm{K}[f] \equiv \int_{a}^{\infty} \frac{-\ln f(y^2)}{1+y^2} \mathrm{d}y, \ a \geq 0.$$

Theorem 1: $\mathrm{K}[f] < \infty \ \Rightarrow \ F$ is M-indet. (Only sufficient!)

Remark: If F is M-det, then necessarily $\mathrm{K}[f] = \infty$ (converse Krein's).

We say that **Lin's condition** is satisfied if $f > 0$, f is smooth and

$$L_f(x) := \frac{-x f'(x)}{f(x)} \nearrow \infty \ \text{ as } \ x_0 < x \to \infty.$$

Theorem 2: (Lin 1997) $\mathrm{K}[f] = \infty$ + Lin's condition $\Rightarrow F$ is M-det.

Remark: There are variations and extensions (Pedersen, Berg, ...).

Rate of Growth of the Moments and (In)Determinacy:

Stieltjes case: $X \sim F$, unbounded $\text{supp}(F) \subset \mathbb{R}_+$, $m_k = \mathbf{E}[X^k]$.

Property 1: (i) $\{m_k\}$ is log-convex. (ii) As $k \to \infty$, $m_k \nearrow \infty$.

How much lager m_{k+1} is compared with m_k? Involve the ratio

$$\Delta_k = \frac{m_{k+1}}{m_k}, \; k = 1, 2, \ldots \Rightarrow \Delta_k \nearrow \text{ strictly } \Rightarrow \text{ unique } \lim_{k\to\infty} \Delta_k.$$

Property 2: Case $\lim_{k\to\infty} \Delta_k = c_* = const < \infty \;\Leftrightarrow\; X$ is bounded.

Assume now $\Delta_k \nearrow \infty$ is $RV(\delta)$, some $\delta \geq 0$ (in Karamata's sense):

$\Delta_k \approx k^\delta \, \ell_k$ for large k, with a monotone $\{\ell_k\} \in SV$.

Def.: $\delta =$ **rate of growth of the moments** of X. We have $0 \leq \delta \leq \infty$.

Hamburger case: Now $X \sim F$, $\text{supp}(F) \subset \mathbb{R}$, we work with even order moments. As above we have $m_{2k} \nearrow \infty$, etc., involve

$$\Delta_k = \frac{m_{2k+2}}{m_{2k}} \approx k^\delta \, \ell_k \text{ as } k \to \infty.$$

Remark: Two boundary cases. If $X \in [0,1]$ or $[-1,1]$, moments $\searrow$, easy to see $\delta = 0$. Take $X \sim Log\,\mathcal{N}$, $m_k = e^{k^2/2}$, $\Delta_k = m_{k+1}/m_k = e^{k+1/2} \nearrow +\infty$ as $k \to \infty$ faster than any power of k, hence $\delta = \infty$.

Three statements for both cases (Stieltjes and Hamburger):

Statement 1: If $\delta \le 2$, then X is M-det.

Statement 2: $\delta = 2$ is the best possible constant for which X is M-det.
Equiv: There is a r.v. Y with $\Delta_k \approx k^{2+\varepsilon}\ell_k$, $\varepsilon > 0$, s.t. Y is M-indet.

Statement 3: If $\delta > 2$ **and** Lin's condition holds, then X is M-indet.

Remark: Interestingly, Lin's condition is essentially involved with the converse Krein to get M-det, and here, with 'fast' rate δ, to get M-indet.

Example 1: $\xi \sim Exp(1)$, e^{-x}, $x > 0$, $m_k = k!$, m.g.f. $= 1/(1-t)$, $t < 1$.

Clearly, ξ is M-det. And we know everything for any power ξ^r, $r > 0$.

r = 2: ξ^2, $m_k = (2k)!$, fast $\nearrow$, still ξ^2 is M-det. Notice, no m.g.f.!
How to show that $f(x) = \frac{1}{2}x^{-1/2}\exp(-x^{-1/2})$, $x > 0$ is the only solution of

$$\int_0^\infty x^k f(x)\,\mathrm{d}x = (2k)!,\ k = 1, 2, \ldots ?$$

r = 3: ξ^3, $m_k = (3k)!$, very fast $\nearrow$, ξ^3 is M-indet. How to show that

$$\int_0^\infty x^k f(x)\,\mathrm{d}x = (3k)!,\ k = 1, 2, \ldots$$

has infinitely many solutions? Take, e.g. the Stieltjes class

$$f_\varepsilon(x) = f(x)[1 + \varepsilon h(x)],\ x > 0,\ \varepsilon \in [-1, 1],$$

where $f(x) = \frac{1}{3}x^{-2/3}e^{-x^{1/3}}$ and $h(x) = \sin\left(\frac{\pi}{6} - \sqrt{3}x^{1/3}\right)$.

Example 2: $Z \sim \mathcal{N}(0,1)$, $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $m_{2k+1} = 0$, $m_{2k} = (2k-1)!!$.

We have, e.g., Z is M-det, Z^2 is M-det, Z^4 is M-det, but Z^3 is M-indet.

Strange Case: While $X = Z^3$ is M-indet, it turns out, $|X|$ is M-det.

Hint: X is in $\mathbb{R}$ (Hamburger), $|X|$ in $\mathbb{R}_+$ (Stieltjes). Density of X:

$g_X(x) = \frac{1}{3\sqrt{\pi}} |x|^{-2/3} \exp[-|x|^{2/3}]$, $x \in \mathbb{R} \Rightarrow \mathrm{K}[g_X] < \infty$, X is M-indet.

Or: $m_{2k}(X) = (6k-1)!!$, rate $\delta_X = 3$, Lin's cond. holds $\Rightarrow$ X is M-indet.

Stieltjes class: For some h, $\mathbf{S}(g,h) = \{g_\varepsilon = g[1+\varepsilon h], \varepsilon \in [-1,1]\}$.

Next, for $|Z|$, $m_k = \mathbf{E}[|Z|^k] = \frac{2^{k/2}}{\sqrt{\pi}} \Gamma(\frac{k+1}{2})$, $g_{|X|}(x) = 2\, g_X(x), x > 0$,

$\mathrm{K}[g_{|X|}] = \infty$, $g_{|X|}$ satisfies Lin's condition, hence $|X|$ is M-det.

Also: $m_k(|X|) = m_{3k}(|Z|) = \frac{2^{3k/2}}{\sqrt{\pi}} \Gamma(\frac{3k+1}{2})$, rate $\delta_{|X|} = 3/2 < 2$, $|X|$ is M-det.

Current topics (a few random choices from the boxes):

- Multivariate distributions/measures and related moment problems.
- How to write Lin's condition for discrete distributions?
- Class of distributions F_a, density f_a satisfies 2nd order ODE:

$$f''(x) - a(x)\, f(x) = 0 \quad (+ \text{ initial conditions}).$$

Find $a(\cdot)$ so that f is density, either Krein's condition, or converse Krein's + Lin's condition. Hence, a r.v. $X \sim f$ is either M-indet or M-det.
All above are parts of a joint project with P. Kopanov.

- Distributions whose densities satisfy Kolmogorov PDEs, connection with Itô SDEs. Joint work with M. Zhitlukhin.
- More topics on functions, MaxEnt approach, Lévy processes, etc. with G.D. Lin, A. Tagliani, M. Milev, M. Savov, K. Penson.

Few Final Comments. Open Questions (to me):

Question: Two measures, μ and ν, both with all moments finite. Let

$$\int x^k \, d\mu \le \int x^k \, d\nu \quad \text{for all} \quad k = 1, 2, \ldots \quad \text{(M-domination)}.$$

(1) We know that μ is M-indet, does this imply that ν is also M-indet?
(2) We know that ν is M-det, does this imply that μ is also M-det?

Open Question: In dim. $n \ge 2$, how to write Krein's condition?

Open Question: Find discrete r.v. on $\mathbb{R}_+$ with moments $\{(3k)!\}$.
Equivalently: Find reals, $x_1 < x_2 < \ldots < x_n < \ldots$ (unbounded) and $p_1, p_2, \ldots, p_n, \ldots$, all > 0 with $\sum_{n=1}^{\infty} p_n = 1$ (disc. probab. distr.) such that

$$\sum_{n=1}^{\infty} x_n^k p_n = (3k)! \quad \text{for any} \quad k = 1, 2, \ldots.$$

Open Question: Find discrete r.v. with $m_{2k-1} = 0$, $m_{2k} = (6k-1)!!$.

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