CHAIN RING ANALOGUES OF SOME THEOREMS FROM EXTREMAL SET THEORY

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– National Colloquium in Mathematics, IMI-BAS, 14.10.2020 –

1. Preliminaries

A family $\mathcal{F}$ of subsets of a gound set $\Omega = \{1, 2, \ldots, n\}$ is intersecting if any two sets from $\mathcal{F}$ have at least one element in common.

More generally, it is t-intersecting if any two sets from $\mathcal{F}$ have at least t elements in common.

Theorem. (Erdős,Ko,Rado, 1961)

Let Ω be a finite set with n elements and let $k \leq n/2$ be an integer. If $\mathcal{F}$ is an intersecting family of k-element subsets of Ω then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

If $k < n/2$ and $\mathcal{F}$ meets the bound then it is canonically intersecting.

The reason this theorem is important is that it has many interesting applications. let us restate it as a question in graph theory

The Kneser graph $K(n, k)$

Vertices: all k-subsets of Ω

Edges: (X, Y) is an edge iff $X \cap Y = \varnothing$

An intersecting family: a coclique in the Kneser graph

The EKR-theorem characterizes the maximal co-cliques in the Kneser graph.

Theorem. (Erdős,Ko,Rado,Frankl,Wilson)

Let Ω be a finite set with $|\Omega| = n$ and let k and t, $t < k$, be integers. Let $\mathcal{F}$ be a t-intersecting family of k-subsets of Ω. There exists a constant $f(k,t)$ such that if $n > f(k,t)$

$$|\mathcal{F}| \leq \binom{n-t}{k-t},$$

and $\mathcal{F}$ meets the bound iff it is canonically intersecting.

• In the original EKR-paper from 1961:

If $n > t+(k-t)\binom{k}{t}^3$ and $\mathcal{F}$ is not canonically t-intersecting then $|\mathcal{F}| < \binom{n-t}{k-t}$.

• EKR point out that their bound is not optimal. For $n = 8, k = 4, t = 2$ take the following 2-intersecting set:

1234	1235	1236	1237	1238	1245
1246	1247	1248	1345	1346	1347
1348	2345	2346	2347	2348	

These are all 4-element subsets containing at least three elements from $\{1, 2, 3, 4\}$.

The above family has 17 subsets but $\binom{n-t}{k-t} = \binom{6}{2} = 15$.

• The exact value $f(k,t) = (t+1)(k-t+1)$ is due to Frankl (1978) and Wilson (1984).

• In 1997 R. Ahlswede and L. Khachatrian determined the largest t-intersecting k-set systems for all n.

For each choice of n, k, and t they find the maximum size of the t-intersecting families and the exact structure of the families that reach this size.

Johnson graphs $J(n,k)$, $n \geq 2k$.

Vertices: all k-subsets of Ω

Edges: (X,Y) is an edge iff $|X \cap Y| = k-1$.

$J(n,k)$ has diameter k and two subsets are adjacent in $K(n,k)$ iff they are at maximum possible distance in $J(n,k)$.

Width of a subset of vertices: maximum possible distance between two vertices in the subset.

The first version of the EKR theorem characterizes the subsets of maximum possible size of width $k-1$ in $J(n,k)$.

The second version characterizes the subsets of maximum size of width $k-t$.

Theorem. (Hsieh,Frankl,Wilson,Tanaka)

Let t and k be integers with $0 \leq t \leq k$. Let $\mathcal{F}$ be a set of k-dimensional subspaces in $\mathrm{PG}(n,q)$ pairwise intersecting in at least a t-dimensional subspace.

If $n \geq 2k+1$, then $|\mathcal{F}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$.

Equality holds if and only if $\mathcal{F}$ is the set of all k-dimensional subspaces, containing a fixed t-dimensional subspace of $\mathrm{PG}(n,q)$, or in case of $n = 2k+1$, $\mathcal{F}$ is the set of all k-dimensional subspaces in a fixed $(2k-t)$-dimensional subspace.

If $2k-t \leq n \leq 2k$, then $|\mathcal{F}| \leq \begin{bmatrix} 2k-t+1 \\ k-t \end{bmatrix}_q$. Equality holds if and only if $\mathcal{F}$ is the set of all k-dimensional subspaces in a fixed $(2k-t)$-dimensional subspace.

Consider $\mathrm{PG}(n,q)$.

Let $d \leq e$ be integers with $d+e=n-1$.

Fix a subspace W with $\dim W = e$.

Let $\mathcal{U}$ be the set of all subspaces U in $\mathrm{PG}(n,q)$ with $\dim U = d$, $U \cap W = \varnothing$.

Theorem. (Tanaka,2006)

Let $0 \leq t \leq d$ be an integer and let $\mathcal{F}$ be a family of subspaces from $\mathcal{U}$ with $\dim(U' \cap U'') \geq t$ for every two $U', U'' \in \mathcal{U}$. Then

$$|\mathcal{F}| \leq q^{(d+1-t)(e+1)}.$$

Equality holds iff

(a) $\mathcal{F}$ consists of all subspaces U through a fixed t-dimensional subspace U_0 with $U_0 \cap W = \varnothing$;

(b) in case of $e = d$, $\mathcal{F}$ is the set of all elements of $\mathcal{U}$ contained in a fixed $(2d-t)$-dimensional subspace V with $\dim V \cap W = d - t$.

Objects	Definition of intersection
k-element sets blocks in a design multisets	a commont element
vector space over a field subspaces in a finite geometry lines in a partial geometry subspaces of fixed shape in $\mathrm{PHG}(n, R)$	a common 1-dim subspace a common point a common point a common subspace of fixed shape
permutations permutations	both map the element i to the element j or $\sigma\tau^{-1}$ has a fixed point a common cycle
set partitions	a common class
tilings	a tile in the same place
cocliques in a graph	a common vertex
triangulations of a polygon	a common triangle

Object	Atoms
Sets	elements from $\{1, \ldots, n\}$
Integer sequences	pairs (a, i), entry a in position i
Permutations	pairs (i, j) ($i \rightarrow j$)
Permutations	cycle
Set partitions	subsets (cells in the partition)
Subspaces in $\mathrm{PG}(n, q)$	points
	subspaces of a fixed dimension
Subspaces in $\mathrm{PHG}(n, R)$	points
	subspaces of a fixed shape

- Two objects intersect if they share a common atom.

- A canonically intersecting set is the set of all objects that contain a fixed atom.

- Objects have the EKR-property if a canonically intersecting set is the only largest intersecting set.

2. The Structure of Projective Hjelmslev Geometries

Theorem.

Let R be a finite chain ring of length m. For any finite module ${}_RM$ there exists a uniquely determined sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with

$$m \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0,$$

such that ${}_RM$ is a direct sum of cyclic modules:

$${}_RM \cong R/(\operatorname{rad} R)^{\lambda_1} \oplus R/(\operatorname{rad} R)^{\lambda_2} \oplus \ldots \oplus R/(\operatorname{rad} R)^{\lambda_k}.$$

The sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ is called the **shape** of ${}_RM$.

The sequence $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$, where λ'_i is the number of λ_j's with $\lambda_j \geq i$ is called the **dual shape** of ${}_RM$.

The integer k is called the **rank** of ${}_RM$.

The integer λ'_m is called the **free rank** of ${}_RM$.

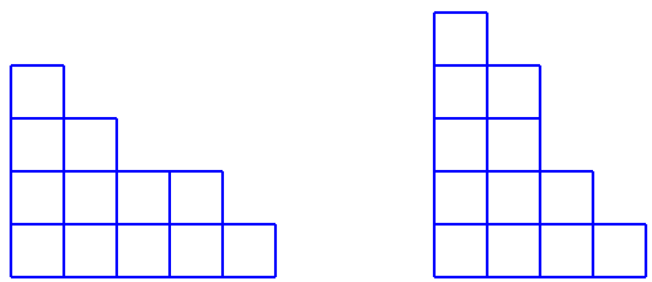
$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$

$N = \lambda_1 + \lambda_2 + \ldots + \lambda_n,$

the **conjugate partition** $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is defined by

$\lambda'_i =$ number of parts in λ that are greater or equal to i

$N = \lambda'_1 + \lambda'_2 + \ldots,$



$\lambda = (4, 3, 2, 2, 1) \quad \lambda' = (5, 4, 2, 1)$

Theorem.

Let R be a chain ring of length m with residue field of order q. Let ${}_RM$ be an R-module of shape $\lambda = (\lambda_1, \ldots, \lambda_n)$. For every sequence $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_1 \geq \ldots \geq \mu_n \geq 0$, satisfying $\mu \leq \lambda$ (i.e. $\mu_i \leq \lambda_i$ for all i) the module ${}_RM$ has exactly

$$\begin{bmatrix}\lambda \\ \mu\end{bmatrix}_{q^m} = \prod_{i=1}^{m} q^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \cdot \begin{bmatrix}\lambda'_i - \mu'_{i+1} \\ \mu'_i - \mu'_{i+1}\end{bmatrix}_q$$

submodules of shape μ. Here

$$\begin{bmatrix}n \\ k\end{bmatrix}_q = \frac{(q^n - 1)\ldots(q^{n-k+1} - 1)}{(q^k - 1)\ldots(q - 1)}.$$

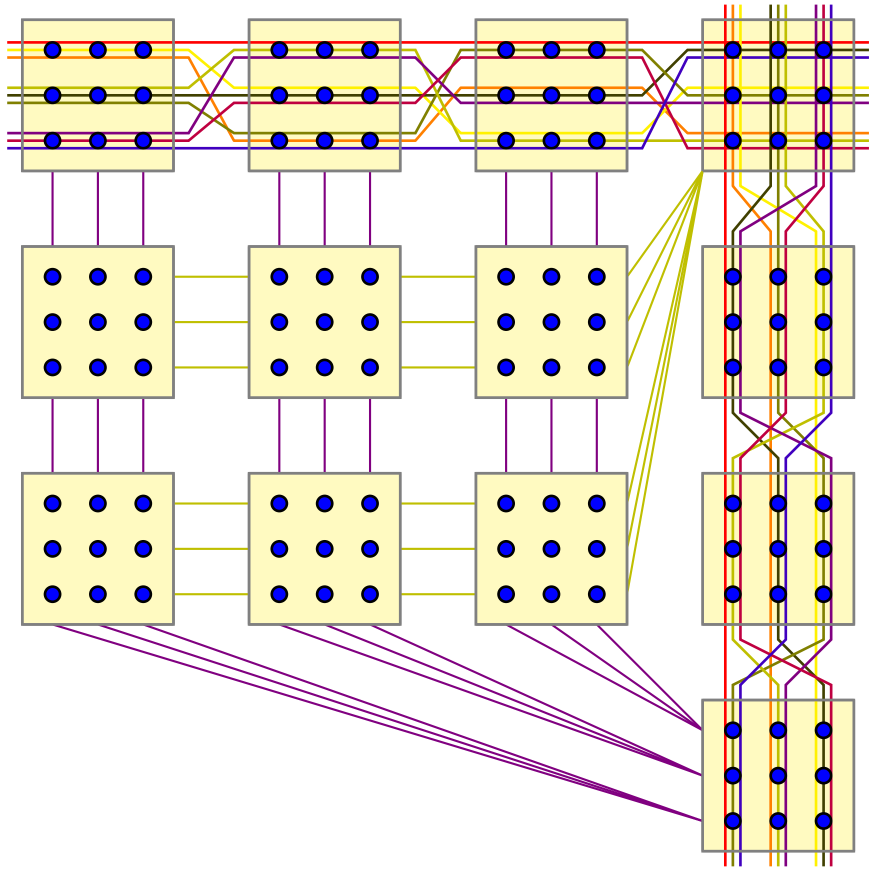
are the Gaussian coefficients.

- $M = {}_R R^{n+1}$;
- $\mathcal{P}$ – all free submodules of M of rank 1;
- $\mathcal{L}$ – all free submodules of M of rank 2;
- $I \subseteq \mathcal{P} \times \mathcal{L}$ – incidence relation;
- $\bigcirc_i$ - **neighbour relation**:

  $X \bigcirc_i Y$ iff $\eta_i(X) = \eta_i(Y)$, where η_i is the canonical epimorphism $\eta_i : R \to R/(\mathrm{rad}\, R)^i$.

- Hjelmslev subspaces of dimension k – free submodules of rank $k+1$;
- subspaces of shape λ – submodules of shape λ;
- Notation: $\mathrm{PHG}({}_R R^{n+1})$, or $\mathrm{PHG}(n, R)$.

$\mathrm{PHG}(2, \mathbb{Z}_9)$



S – a Hjelmslev subspace with $\dim S = k$

Points: $[X]^{(i)} \cap T$, where $T \in [S]^{(i)}$ is a k-dimensional Hjelmslev subspace

Subspaces: the sets of points in $T \cap [S]^{(i)}$, where T is a subspace in $\mathrm{PHG}(n, R)$

Incidence: the incidence inherited from $\mathrm{PHG}(n, R)$;

Theorem.

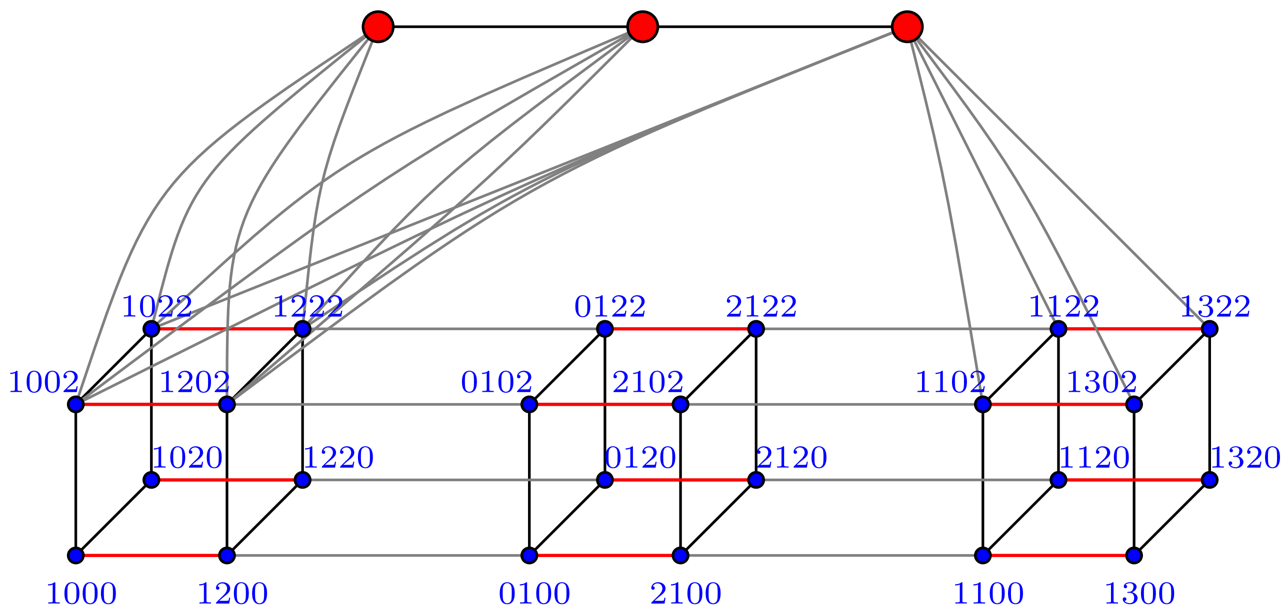
The obtained structure can be imbedded isomorphically into

$$\mathrm{PHG}(n, R/(\mathrm{rad}\, R)^{m-i}).$$

.

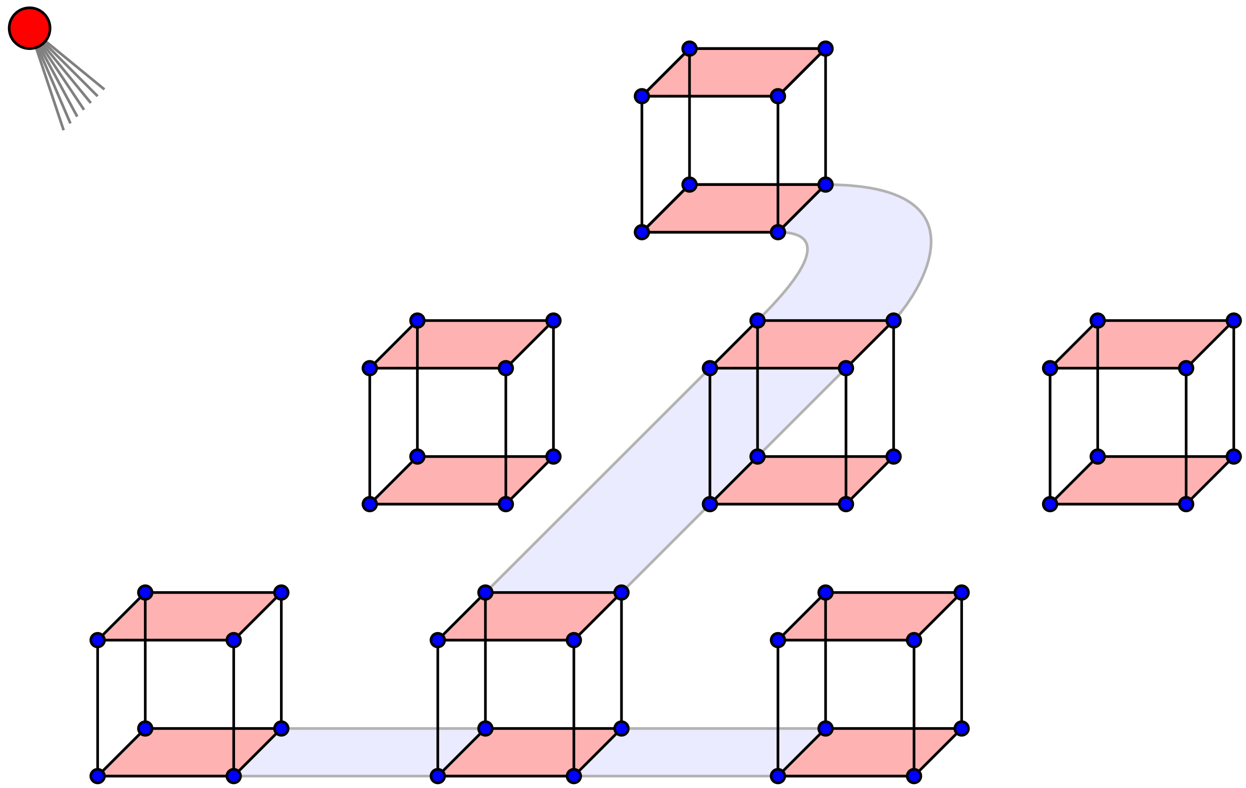
The missing part is isomorphic to $\mathrm{PHG}(n-k-1, R/(\mathrm{rad}\, R)^{m-i})$.

A Neighbour Class of Lines in $\mathrm{PHG}(3, \mathbb{Z}_4)$



The structure is ismorphic to $\mathrm{PG}(3, 2) - \mathrm{PG}(1, q)$.

A Neighbour Class of Planes in $\mathrm{PHG}(3, \mathbb{Z}_4)$



The structure is ismorphic to $\mathrm{PG}(3, 2)$ minus a point.

Problem.

Given $\Sigma = \mathrm{PHG}(n, R)$ and two shapes λ and τ with $\tau \leq \lambda$, what is the maximal number of subspaces of a τ-intersecting family of subspaces in Σ of shape λ?

3. Erdős-Ko-Rado-Type Theorems in Projective Hjelmslev Geometries

Theorem A. Let R be a finite chain ring with nilpotency index m and residue field of order q. Denote by Σ the n-dimensional (left) projective Hjelmslev geometry over R. Let $\mathcal{F}$ be a family of k-dimensional Hjelmslev subspaces every two of which meet in at least one point. If $n \geq 2k+1$ then

$$|\mathcal{F}| \leq \begin{bmatrix} \boldsymbol{m}^n \\ \boldsymbol{m}^k \end{bmatrix}_{q^m}.$$

In case of equality $\mathcal{F}$ is one of the following:

- all the Hjelmslev subspaces through a fixed point,
- in case of $n = 2k+1$, all Hjelmslev k-subspaces in a fixed hyperplane of Σ.

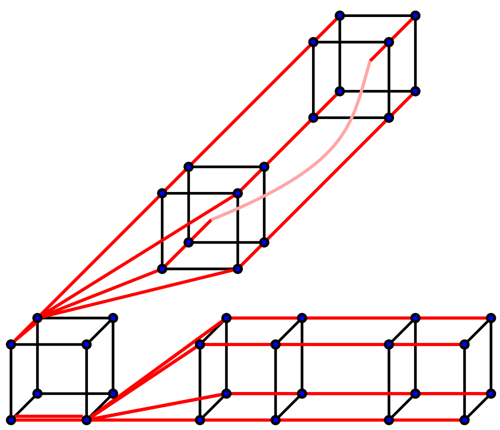
Proof.

W.l.o.g. $n \geq 2k+1$. Let $m = 2$.

$\mathcal{F}$: intersecting family in $\mathrm{PHG}(n, R)$

$\eta(\mathcal{F}) = \{\eta(X) | X \in \mathcal{F}\}$: intersecting family of k-subspaces in $\mathrm{PG}(n, q)$

$[X]$ is $\mathrm{PG}(n, q) - \mathrm{PG}(n-k-1, q)$ and the maximal number is given by Tanaka's theorem.



Further we proceed by induction on m.

$\eta^{(m-1)}(\mathcal{F}) = \{\eta^{(m-1)}(X) | X \in \mathcal{F}\}$: intersecting family of k-dimensional subspaces in $\mathrm{PHG}(n, (R/\operatorname{rad}^{m-1} R))$

$[X]^{(m-1)}$ can be viewed as is $\mathrm{PG}(n, q) - \mathrm{PG}(n-k-1)$ and the maximal number is again given by Tanaka's theorem.

Theorem B. Under the condition of the previous theorem, $\mathcal{F}$ is a family of k-dimensional Hjelmslev subspaces meeting in a Hjelmslev subspace of dimension at least t. Then

$$|\mathcal{F}| \leq \begin{bmatrix} \boldsymbol{m}^{n-t} \\ \boldsymbol{m}^{k-t} \end{bmatrix}_{q^m}.$$

If $n \geq 2k+1$, then $|\mathcal{F}| \leq \begin{bmatrix} \boldsymbol{m}^{n-t} \\ \boldsymbol{m}^{k-t} \end{bmatrix}_{q^m}$. Equality holds if and only if $\mathcal{F}$ is the set of all k-dimensional Hjelmslev subspaces, containing a fixed t-dimensional subspace of $\mathrm{PG}(n,q)$, or $n = 2k+1$ and $\mathcal{F}$ is the set of all k-dimensional subspaces in a fixed $(2k-t)$-dimensional Hjelmslev subspace.

In case of $2k-t \leq n \leq 2k$, we have that $|\mathcal{F}| \leq \begin{bmatrix} \boldsymbol{m}^{2k-t+1} \\ \boldsymbol{m}^{k-t} \end{bmatrix}_{q^m}$. Equality holds if and only if $\mathcal{F}$ is the set of all k-dimensional Hjelmslev subspaces in a fixed $(2k-t)$-dimensional Hjelmslev subspace.

Example.

R, $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$

$\Sigma = \mathrm{PHG}(3, R)$

$\lambda = (2, 2, 1, 0)$, i.e. the subspaces of shape λ are the line stripes consisting of $q^2(q+1)$ points each.

Let $\mathcal{F}$ be an intersecting family of λ-subspaces.

- Let $\mathcal{F}$ be the family of all λ-subspaces through a fixed point in Σ. Then

$$|\mathcal{F}| = q(q+1)(q^2+q+1).$$

- Take a maximal intersecting set in the factor geometry $\mathrm{PG}(3, q)$. It is

a) all lines through a point, or

b) all lines in a plane.

• The maximal number of λ-subspaces in each neighbour class of lines to be chosen is $q^2(q+1)$.

Two λ-subspaces in the same neighbour class of lines do always meet.

Two λ-subspaces in different neighbour classes of lines do not meet exactly when they intersect the common point class (which is $\cong \mathrm{AG}(3,q)$) in parallel planes.

• In the second case, we can take all λ-subspaces in every neighbor class of lines contained in a neighbour class of planes except for those that lie in planes connected in the fixed class. Their number is

$$q^2(q+1)(q^2+q+1) - q^2(q^2+q+1) = q^3(q^2+q+1).$$

We add a set of all λ-subspaces that are contained in a fixed plane from the neighbour class of planes (it forms an intersecting set in $\mathrm{PG}(2, q)$). Their number is: $q^2 + q + 1$. Altogether we have an intersecting set $\mathcal{F}$ of λ-subspaces of size

$$|\mathcal{F}| = (q^3 + 1)(q^2 + q + 1).$$

It can be proved that this is a largest intersecting family of λ-subspaces.

4. The Sperner Theorem

Theorem. (E. Sperner, 1928) If $A_1, A_2, \ldots, A_m$ are subsets of $X = \{1, 2, \ldots, n\}$ such that A_i is not a subset of A_j if $i \neq j$, then $m \leq \binom{n}{\lfloor n/2 \rfloor}$.

Theorem. If $\mathcal{A}$ is an antichain in the partially ordered set of all subspaces of $\mathbb{F}_q^n$, then

$$|\mathcal{A}| \leq \begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_q$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1) \ldots (q^{n-k+1} - 1)}{(q^k - 1) \ldots (q - 1)}.$$

are the Gaussian coefficients.

- Ranked poset $\mathcal{P}$: there exists a function $r : \mathcal{P} \to \mathbb{N}_0$ with $r(x) = 0$ for some minimal element and $r(y) = r(x) + 1$ for all x, y with $x \prec\cdot\, y$.

- We say that the element y of a poset $\mathcal{P}$ covers the element $x \in \mathcal{P}$ if $x \prec y$ and $x \prec y' \preceq y$ implies $y = y'$. This is denoted by $x \prec\cdot\, y$.

- Graded poset: a ranked poset in which all minimal elements have rank 0.

- $L_i(\mathcal{P})$ – the i-th level of $\mathcal{P}$

$$L_i(\mathcal{P}) = \{x \in \mathcal{P} \mid r(x) = i\}.$$

- the i-th Whitney number: $W_i(\mathcal{P}) = |L_i(\mathcal{P})|$

• The Hasse diagram of a partially ordered set is a directed graph $H(\mathcal{P}) = (\mathcal{P}, E(\mathcal{P}))$ where

$$E(\mathcal{P}) = \{(x, y) \mid \text{ where } x \prec\cdot\ y\}.$$

• The underlying nondirected graph is called the Hasse graph.

• The lattice of all submodules of a finitely generated left R-module ${}_RM$ is a graded poset.

• Rank function: $r(L) = \sum_{i=1}^{n} \lambda_i = \log_q |L|$, where ${}_RL < {}_RM$ and has shape $(\lambda_1, \ldots, \lambda_n)$.

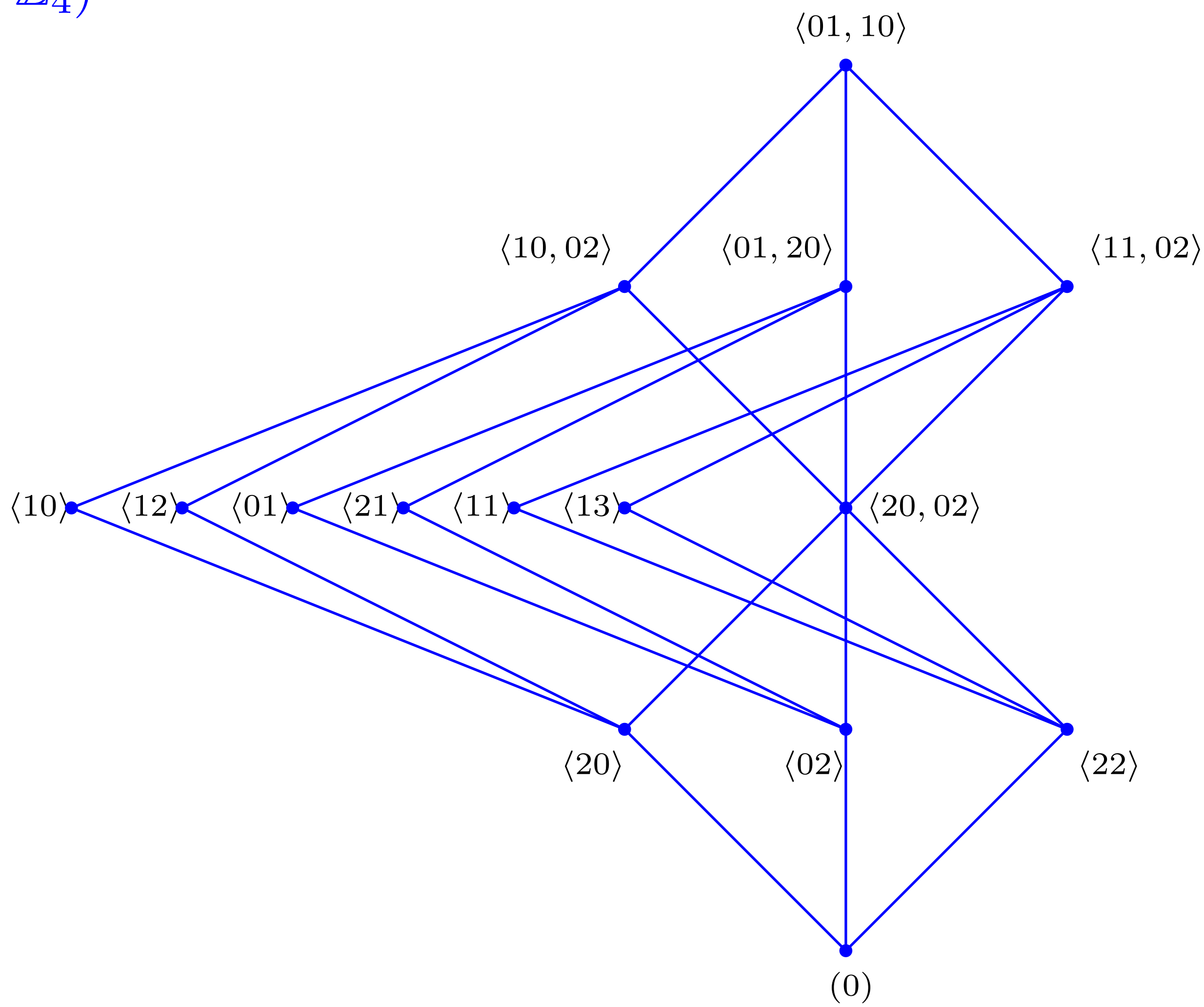
• If $M = R^n$, we have $r(\mathcal{P}_n) = mn$, where m is the length of R.

• The k-th Whitney number:

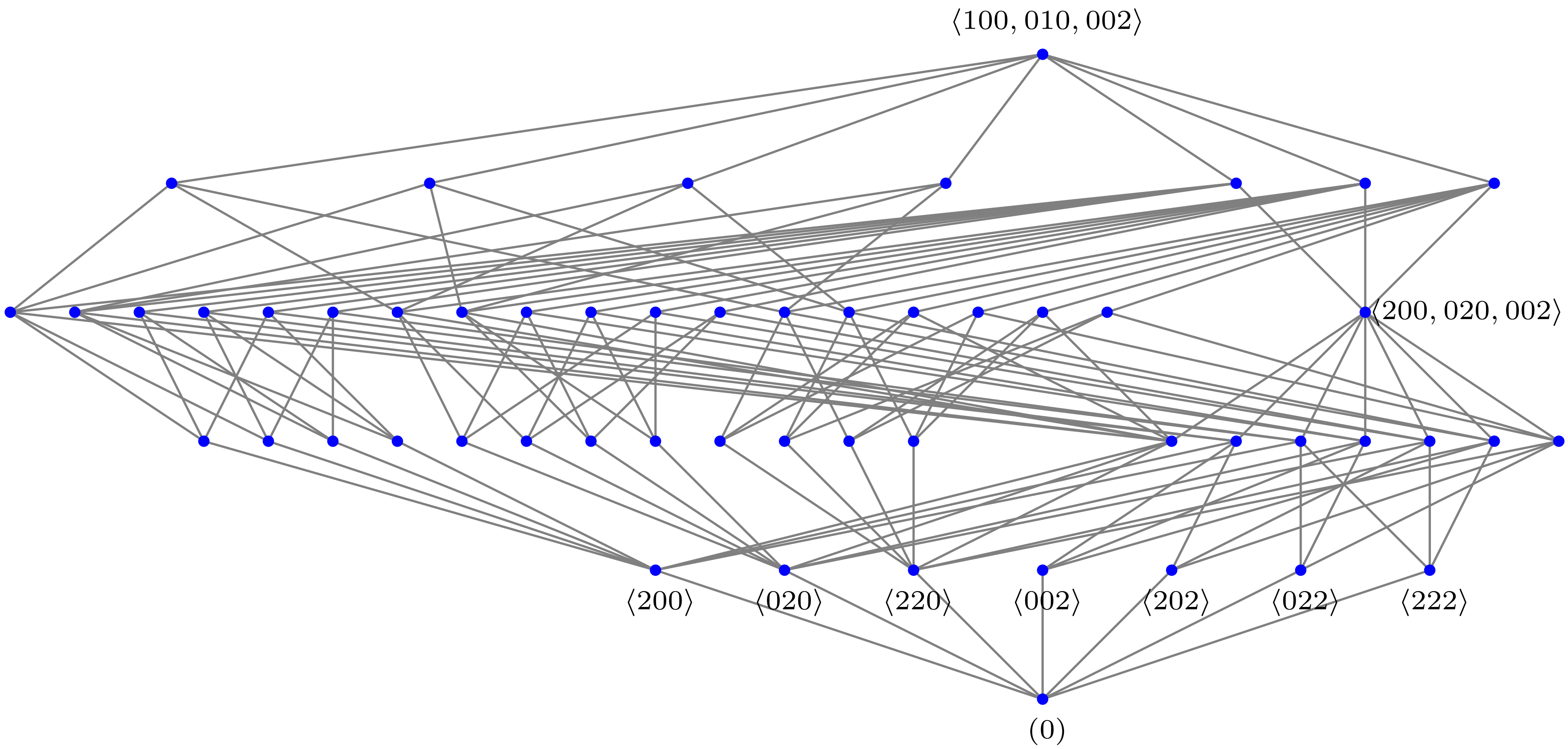
$$W_k(\mathcal{P}_n) = \sum_{\boldsymbol{\mu}} \begin{bmatrix} \boldsymbol{m}_n \\ \boldsymbol{\mu} \end{bmatrix}_{q^m},$$

where the sum is over all shapes $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ with $\sum_i \mu_i = k$.

$\mathcal{P}(\mathbb{Z}_4 \oplus \mathbb{Z}_4)$



$\mathcal{P}(\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus 2\mathbb{Z}_4)$



Problem. Let R be a finite chain ring and let ${}_RM$ be a (left) module over R. What is the size of the largest antichain in the lattice of all submodules of ${}_RM$?

5. A Sperner-type Theorem

It is said that level L_i can be matched into level L_j, where $j = i - 1$ or $i + 1$, if there is a matching of size W_i in the Hasse graph G_j defined on the elements from $L_i \cup L_j$.

Theorem. Let $\mathcal{P}$ be a graded poset. If there exists an index h such that L_i can be matched into L_{i+1} for all $i = 0, 1, \ldots, h$, and L_i can be matched into L_{i-1} for all $i = h + 1, \ldots, n$ then the size of the largest antichain is $W_h = W_h(\mathcal{P})$.

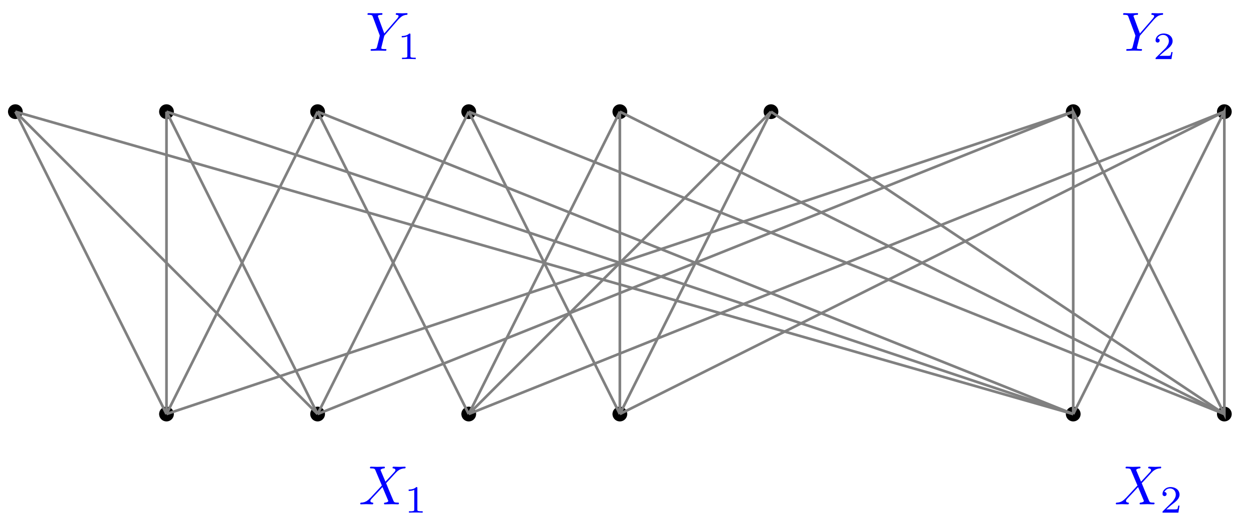
A bipartite graph $G = (X \cup Y, E)$ is called piecewise regular if there exist partitions

$$X = X_1 \cup X_2 \cup \ldots \cup X_s, \quad X_i \cap X_j \neq \varnothing;$$
$$Y = Y_1 \cup Y_2 \cup \ldots \cup Y_t, \quad Y_i \cap Y_j \neq \varnothing$$

such that

- each vertex of X_i is adjacent to exactly x_{ij} vertices of Y_j for all $i = 1, \ldots, s$, $j = 1, \ldots, t$;

- each vertex of Y_j is adjacent to exactly y_{ji} vertices of Y_j for all $i = 1, \ldots, s$, $j = 1, \ldots, t$;



$$|X_1| = 4, \quad |X|_2 = 2, \quad |Y_1| = 6, \quad |Y_2| = 2.$$

$$x_{11} = 3, \quad x_{12} = 1, \quad x_{21} = 3, \quad x_{22} = 2.$$

$$y_{11} = 2, \quad y_{12} = 1, \quad y_{21} = 2, \quad y_{22} = 2.$$

$I \subseteq \{1, \ldots, s\}$

$J = J(I) = \{j \mid x_{ij} > 0 \text{ for some } i \in I\}$

Theorem. Let $G = (X \cup Y, E)$ be a piecewise bipartite graph. A necessary and sufficient condition for the existence of a matching of size $|X|$ in G is the following: for every subset $I \subseteq \{1, \ldots, s\}$

$$\sum_{i \in I} |X_i| \leq \sum_{j \in J(I)} |Y_j|.$$

• $\mathcal{L}(m,n)$: the poset of all n-tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ with $m \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0$ and with partial order defined by

$$\lambda \preceq \mu \Longleftrightarrow \lambda_1 \leq \mu_1, \dots \lambda_n \leq \mu_n.$$

• $\mathcal{L}(m,n)$ can be graded by the rank function $r(\lambda) = \sum_{i=1}^{n} \lambda_i$.

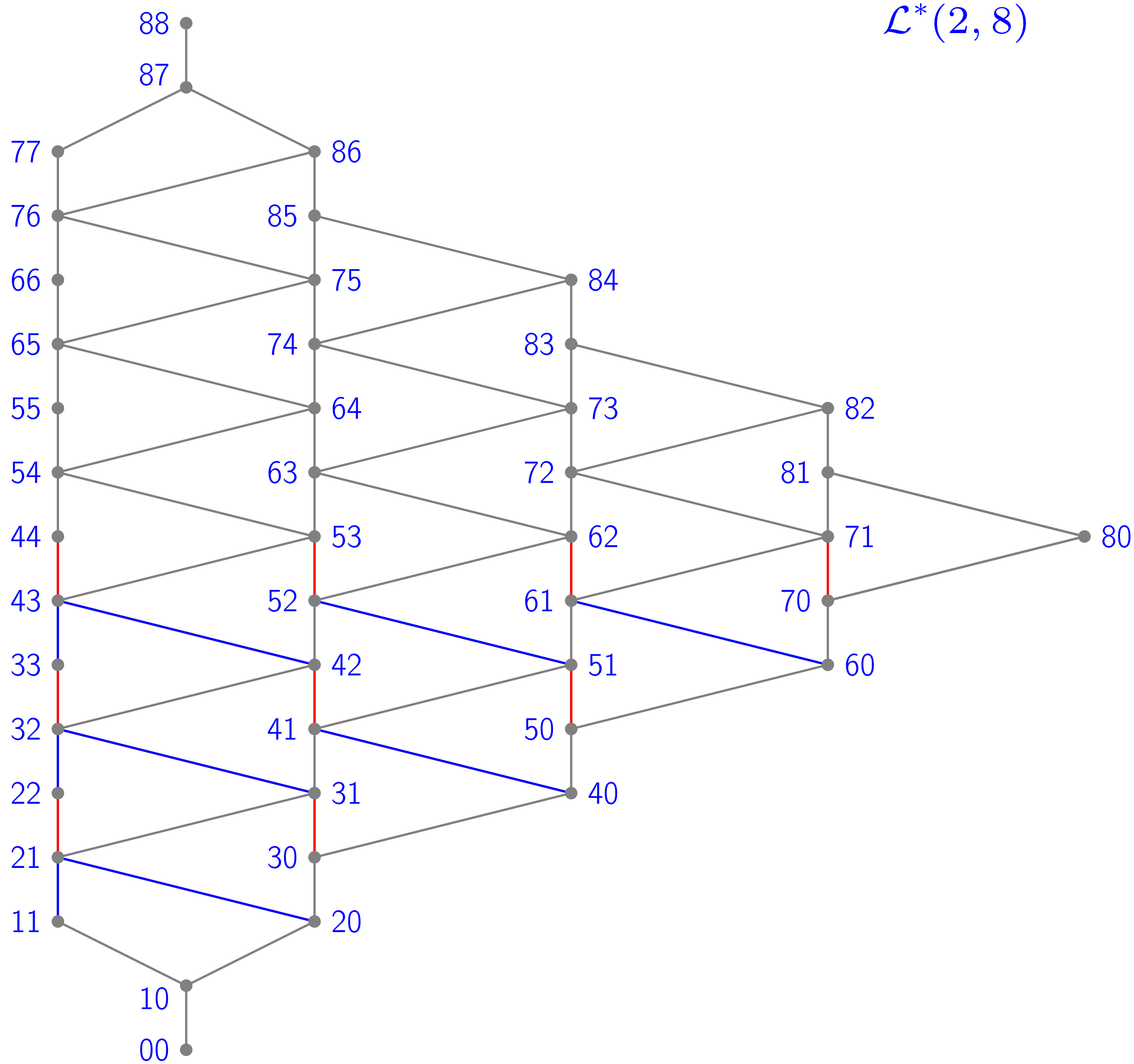
• $\mathcal{L}(m,n)$ is self dual: $(\lambda_1, \dots, \lambda_n) \rightarrow (m - \lambda_n, \dots, m - \lambda_1)$.

• $\mathcal{L}^*(m,n)$: the poset of all conjugate partitions. Then

$$\mathcal{L}^*(m,n) \cong \mathcal{L}(n,m)$$

.

$\mathcal{L}^*(2,8)$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Theorem C. Let R be a chain ring of length m and let $\mathcal{P}_n = \mathcal{P}_n(R)$ be the partially ordered set of all submodules of ${}_RR^n$ with partial order given by inclusion. Then the size of a maximal antichain in $\mathcal{P}$ is equal to

$$\sum_{\mu \prec \boldsymbol{m}^n} \begin{bmatrix} \boldsymbol{m}^n \\ \mu \end{bmatrix}_{q^m},$$

where the sum is over all partitions $\mu = (\mu_1, \ldots, \mu_n) \prec \boldsymbol{m}^n$ with

$$\sum_{i=1}^{n} \mu_i = \lfloor \frac{mn}{2} \rfloor.$$

An open problem.

What is the size of the largest antichain in the lattice of the submodules of a nonfree module over a finite chain ring R?