Continuation across small sets in complex analysis

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Plan

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- Joint work with Żywomir Dinew

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- If $lim_{z \to z_0} f(z) = \infty$ then f has a **pole of finite order** $(=$ Laurent series expansion has finite negative part);
- If neither of the above happens then f has an **essential singularity**.

What is true in higher dimensions?

$\mathbf{z_0} \in \Omega \Subset \mathbb{C}^n,\ n \geq 2,\ \mathbf{f} \in \mathcal{O}(\Omega \setminus \{\mathbf{z_0}\})$.
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THEOREM (RIEMANN)

Let M be a closed subset of $\Omega \subset \mathbb{C}^n$, and $f \in \mathcal{O}(\Omega \setminus M)$. Then:
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-If M is conatined in a complex submanifold of **codimension at least 2** *then f extends* **unconditionally** *to Ω.*

Applications include: extension of analytic sets, extensions of holomorphic vector bundles, extension of closed currents..

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More assumptions on f yield **larger** sets over which extension is possible.

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No assumptions on the size of the zero set- only its **image** has to be small!

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Let $U \in \mathbb{R}^n$ be open. $u : U \to [-\infty, \infty)$ is called **subharmonic** if it is upper semicontinuous and for every domain $D \subset\subset U$ and every harmonic function h in D continuous up to the boundary one has

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If $U \subset \mathbb{C}$ and $f \in \mathcal{O}(U)$ then $Re(f), |f|$ and $log|f|$ are subharmonic. This links potential theory in the plane to complex analysis.Does **not** work in $\mathbb{C}^n,\ n \geq 1$!

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Lelong conjecture Is the uppersemicontinuity really needed in the definition?

Extension results for subharmonic functions. What is known?

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Other results: Harvey-Polking' 70, Kaufman-Wu' 80, Tamrazov' 86, Riihentaus-Tamrazov' 91 and ' 93, Yarmetov' 94, Abdullaev-Imomkulov' 97, Pokrovskiĭ' 17

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Other results: Grauert-Remmert' 56, Pflug' 80, Favorov' 81, Abidi' 99 and ' 10

Extension through bigger sets (but with compatibility assumptions). What is known?

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General setting: $\Omega \subseteq \mathbb{C}^n$- a domain, $E \subseteq \Omega$- a subset, bigger than a removable singularity for a class of *PSH* functions but still somehow small, $u \in PSH(\Omega \setminus E)$, u subject to compatibility conditions $\Longrightarrow$ u can be extended to a $\tilde{u} \in PSH(\Omega)$ if:

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u can be extended to a $\tilde{u} \in PSH(\Omega)$ if:

- $u \in C^0(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$, E hypersurface of class C^1 which divides Ω to two subdomains Ω_1 and Ω_2, $u|_{\Omega_i} = u_i \in C^1(\Omega_i \cup E)$,

$$\frac{\partial u_i}{\partial \vec{\mathbf{n}}_k} \geq \frac{\partial u_k}{\partial \vec{\mathbf{n}}_k} \quad \text{on } E, \quad i \neq k, i = 1, 2, k = 1, 2,$$

where $\vec{\mathbf{n}}_k$ are the outward unit normal vectors of Ω_k on E (Blanchet' 95)

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$$u(z) := \begin{cases} \|z\|^2 & \text{if } \|z\| \leq 1 \\ 1 & \text{if } \|z\| > 1 \end{cases}$$

Definition

Let Ω be a domain in $\mathbb{C}^n$ and $E \subseteq \Omega$ be a closed subset of Lebesgue measure zero. Let u be a subharmonic function in Ω which is furthermore plurisubharmonic in $\Omega \setminus E$. Then u is said to be a plurisubharmonic function with **subharmonic singularities** along E.

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Problem of Chirka:

Is it true that any plurisubharmonic function with subharmonic singularities along E extends as a plurisubharmonic function, provided that E is a closed subset of Ω *with a locally finite* $(2n-1)$*-dimensional Hausdorff measure?*

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Theorem (D.-D.'21)

Let $E \subseteq \Omega$ be a closed subset of Lebesgue measure zero. Then any subharmonic function u in Ω which is plurisubharmonic in $\Omega \setminus E$ is actually plurisubharmonic in the whole Ω.

A Radó- type theorem reads:

Theorem

Let Ω be open in $\mathbb{R}^n$ ($\mathbb{C}^n$) and $E \subseteq \Omega$ be a Borel set. If f is subharmonic (respectively plurisubharmonic) in some neighborhood of $\Omega \setminus E$ and f belongs to some family $\mathcal{F}(\Omega)$ then f is actually subharmonic (respectively plurisubharmonic) in Ω if and only if $G = f(E) \subseteq \mathbb{R}$ is of specified type.

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Proof.

The function $f(x_1, \ldots, x_n) = -|x_1|$ is locally affine (hence subharmonic and plurisubharmonic in the complex case) outside the set $E = \{0\} \times \mathbb{R}^{n-1}$, f is Lipschitz and $f(E) = 0$. □

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This shows that Radó-type theorem fails for $C^{0,\alpha}$, $0 < \alpha \leq 1$ (pluri)subharmonic functions even if the image of E is a point.

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The same is true when f is more regular than C^2.

THEOREM (D.-D.21)

Let Ω be open in $\mathbb{R}^n$ (respectively in $\mathbb{C}^n$) and $E \subseteq \Omega$ be a Borel set. If $f \in C^{1,p}(\Omega)$, $p \in (0,1]$ is subharmonic (respectively plurisubharmonic) in some open neighborhood of $\Omega \setminus E$ and the Hausdorff measure $\mathcal{H}^p(f(E)) = 0$ then f is actually subharmonic (respectively plurisubharmonic) in Ω. If $f \in C^1(\Omega)$ then the same conclusion holds if $f(E)$ is at most countable. The results are optimal with respect to the size of the image of E.

Remark

No (pluri)potential proof of the above results is known. Instead we make use of **viscosity** theory for elliptic PDEs due to the following result:

THEOREM

A function u is subharmonic if and only if it is a **viscosity subsolution** *to the equation*

$$\Delta w = 0$$

(i.e. $\Delta u \geq 0$ in viscosity sense.) A function v is plurisubharmonic if and only if it satisfies $i\partial\bar{\partial}v \geq 0$ in viscosity sense.