QUADRICS AND HIGHLY DIVISIBLE ARCS IN FINITE PROJECTIVE GEOMETRIES

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1. The extension problem for linear codes and arcs

◇ **Linear** $[n, k]_q$ **code**: $C < \mathbb{F}_q^n$, $\dim C = k$, $(\mathbb{F}_q = \mathrm{GF}(q))$

◇ $[n, k, d]_q$-**code**: $d = \min\{d(\boldsymbol{u}, \boldsymbol{v}) \mid \boldsymbol{u}, \boldsymbol{v} \in C, \boldsymbol{u} \neq \boldsymbol{v}\}$.

- n - the **length** of C;
- k - the **dimension** of C;
- d - the **minimum distance** of C.

◇ A_i – number of codewords of (Hamming) weight i

◇ $(A_i)_{i \geq 0}$ – the **spectrum** of C

Given the positive integers k and d and the prime power q, find the smallest value of n for which there exists a linear $[n,k,d]_q$-code. This value is denoted by $n_q(k,d)$.

The Griesmer bound: $n_q(k,d) \geq g_q(k,d) := \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

the Hamming code $[7,4,3] \rightarrow$ the extended Hamming code $[8,4,4]$

Definition. A linear $[n,k,d]_q$-code C is said to be extendable if there exists a $[n+1,k,d+1]_q$-code C' such that C is obtained from C' by puncturing.

Definition. A linear code over $\mathbb{F}_q$ is said to be divisible with divisor $\Delta > 1$ if the weight of every codeword is a multiple of Δ.

Theorem. (H. N. Ward) Let C be a Griesmer code over $\mathbb{F}_p$, p a prime. If p^e divides the minimum weight of C, then p^e is a divisor of the code.

Definition. A linear $[n,k,d]_q$-code is said to be t-quasidivisible modulo Δ if $d \equiv -t \pmod{\Delta}$ and all weights in the code are congruent to $-t,\ldots,-1,0$ modulo Δ.

Theorem. (R. Hill, P. Lizak, 1995) Every linear $[n,k,d]_q$-code with weights 0 and d modulo q, where $(d,q)=1$, is extendable to a $[n+1,k,d+1]_q$-code.

The most common case is $d \equiv -1 \pmod{q}$.

Equivalently: every 1-quasidivisible code is extendable.

Theorem. (T. Maruta, 2004) Let $q \geq 5$ be an odd prime power. If an $[n,k,d]_q$-code with $d \equiv -2 \pmod{q}$ has only weights $-2,-1,0 \pmod{q}$ then it is extendable.

Equivalently: every 2-quasidivisible code over a field of order $q \geq 5$, q odd, is extendable.

Theorem. (H. Kanda, 2020) Let $\mathcal{C}$ be an $[n,k,d]_3$ code with $(d,3)=1$ whose possible weights of codewords satisfy $A_i = 0$ for all $i \not\equiv 0,-1,-2 \pmod{9}$. Then $\mathcal{C}$ is doubly extendable.

Definition. A **multiset** in $\mathrm{PG}(k-1,q)$ is a mapping

$$\mathcal{K}:\left\{\begin{array}{lcl}\mathcal{P} & \rightarrow & \mathbb{N}_0,\\ P & \rightarrow & \mathcal{K}(P).\end{array}\right.$$

$\mathcal{K}(P)$ – the **multiplicity** of the point P.

$\mathcal{Q}\subset\mathcal{P}$: $\mathcal{K}(\mathcal{Q})=\sum_{P\in\mathcal{Q}}\mathcal{K}(P)$; $\mathcal{K}(\mathcal{P})$ – the **cardinality** of $\mathcal{K}$.

a_i – the number of hyperplanes of multiplicity i

$(a_i)_{i\geq 0}$ – the **spectrum** of $\mathcal{K}$

Definition. (n, w)-**arc** in $\mathrm{PG}(k-1, q)$: a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \leq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. (n, w)-**blocking set with respect to hyperplanes** in $\mathrm{PG}(k-1, q)$: a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \geq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. An (n, w)-arc $\mathcal{K}$ in $\mathrm{PG}(k-1, q)$ is called t-extendable, if there exists an $(n+t, w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(k-1, q)$ with $\mathcal{K}'(P) \geq \mathcal{K}(P)$ for every point $P \in \mathcal{P}$.

Definition. An arc $\mathcal{K}$ in $\mathrm{PG}(k-1, q)$ with $\mathcal{K}(\mathcal{P}) = n$ and spectrum (a_i) is said to be divisible with divisor Δ if $a_i = 0$ for all $i \not\equiv n \pmod{\Delta}$.

Definition. An arc $\mathcal{K}$ with $\mathcal{K}(\mathcal{P}) = n$ and spectrum (a_i) is said to be t-quasidivisible with divisor Δ (or t-quasidivisible modulo Δ) if $a_i = 0$ for all $i \not\equiv n, n+1, \ldots, n+t \pmod{\Delta}$.

Equivalence of linear codes and arcs

$[n,k,d]_q$-code C of full length	$\Leftrightarrow$	$(n, w=n-d)$-arc $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$
$\mathbf{0} \neq \boldsymbol{u} \in C$, $\mathrm{wt}(\boldsymbol{u}) = u$	$\Leftrightarrow$	a hyperplane H with $\mathcal{K}(H) = n-u$,
extendable $[n,k,d]_q$-code C	$\Leftrightarrow$	extendable $(n,n-d)$-arc $\mathcal{K}$
divisible $[n,k,d]_q$-code $A_i = 0$ for all $i \not\equiv 0 \pmod{\Delta}$	$\Leftrightarrow$	divisible $(n,n-d)$-arc in $\mathrm{PG}(k-1,q)$ $a_i = 0$ for all $i \not\equiv n \pmod{\Delta}$
t-quasidivisible $[n,k,d]_q$-code $A_i = 0$ for all $i \not\equiv -j \pmod{q}$ $j \in \{0,1,\ldots,t\}$	$\Leftrightarrow$	t-quasidivisible $(n,n-d)$-arc in $\mathrm{PG}(k-1,q)$ $a_i = 0$ for all $i \not\equiv n+j \pmod{q}$

◇ Griesmer arcs: arcs associated with codes meeting the Griesmer bound

2. $(t \mod q)$-Arcs

Definition. Let $t < q$ be a non-negative integer.

An arc $\mathcal{K}$ in $\mathrm{PG}(r, q)$ is called a **$(t \mod q)$-arc** if every subspace S of positive dimension has multiplicity $\mathcal{K}(S) \equiv t \pmod{q}$.

If in addition, every point P has multiplicity at most t, i.e. $\mathcal{K}(P) \leq t$; the $\mathcal{K}$ is called a **strong $(t \mod q)$-arc**.

Remark. It is enough to require the congruence in the definition only for the the subspaces of dimension 1 (i.e. for the lines).

◇ $\mathcal{K}$ - (n, w)-arc in $\Sigma = \mathrm{PG}(r, q)$

◇ for every hyperplane H, we have $\mathcal{K}(H) \equiv n, n+1, \ldots, n+t \pmod{q}$ where $0 < t < q$ is an integer constant, i.e. $\mathcal{K}$ is t-quasidivisible modulo q.

◇ Define an arc $\widetilde{\mathcal{K}}$ in the dual space $\widetilde{\Sigma}$

$$(\star) \qquad \widetilde{\mathcal{K}} : \begin{cases} \mathcal{H} & \rightarrow \quad \mathbb{N}_0, \\ H & \rightarrow \quad \widetilde{\mathcal{K}}(H) := n + t - \mathcal{K}(H) \pmod{q}. \end{cases}$$

where $\mathcal{H}$ is the set of all hyperplanes of Σ.

E.g. maximal hyperplanes become 0-points.

Theorem A.(Landjev,Rousseva,2016) Let $\mathcal{K}$ be an (n, w)-arc in $\Sigma = \mathrm{PG}(r, q)$ which is t-quasidivisible modulo q. Then the arc $\widetilde{\mathcal{K}}$ is a strong $(t \mod q)$-arc.

Theorem B.(Landjev,Rousseva,2016) Let $\mathcal{K}$ be an (n, w)-arc in $\Sigma = \mathrm{PG}(r, q)$ which is t-quasidivisible modulo q, $t < q$. Assume

$$\widetilde{\mathcal{K}} = \sum_{i=1}^{c} \chi_{\widetilde{H}_i} + \widetilde{\mathcal{K}}'$$

for some arc $\widetilde{\mathcal{K}'}$ and c not necessarily different hyperplanes $\widetilde{H_1}, \ldots, \widetilde{H}_c$. Then $\mathcal{K}$ is c-extendable. In particular, if $\widetilde{\mathcal{K}}$ contains a hyperplane in its support then $\mathcal{K}$ is extendable.

Theorem C. (Landjev,Rousseva,2016) Let $t_1 < q$ and $t_2 < q$ be positive integers. The sum of a $(t_1 \mod q)$-arc and a $(t_2 \mod q)$-arc in $\mathrm{PG}(r,q)$ is a $(t \mod q)$-arc with $t = t_1 + t_2 \pmod q$.

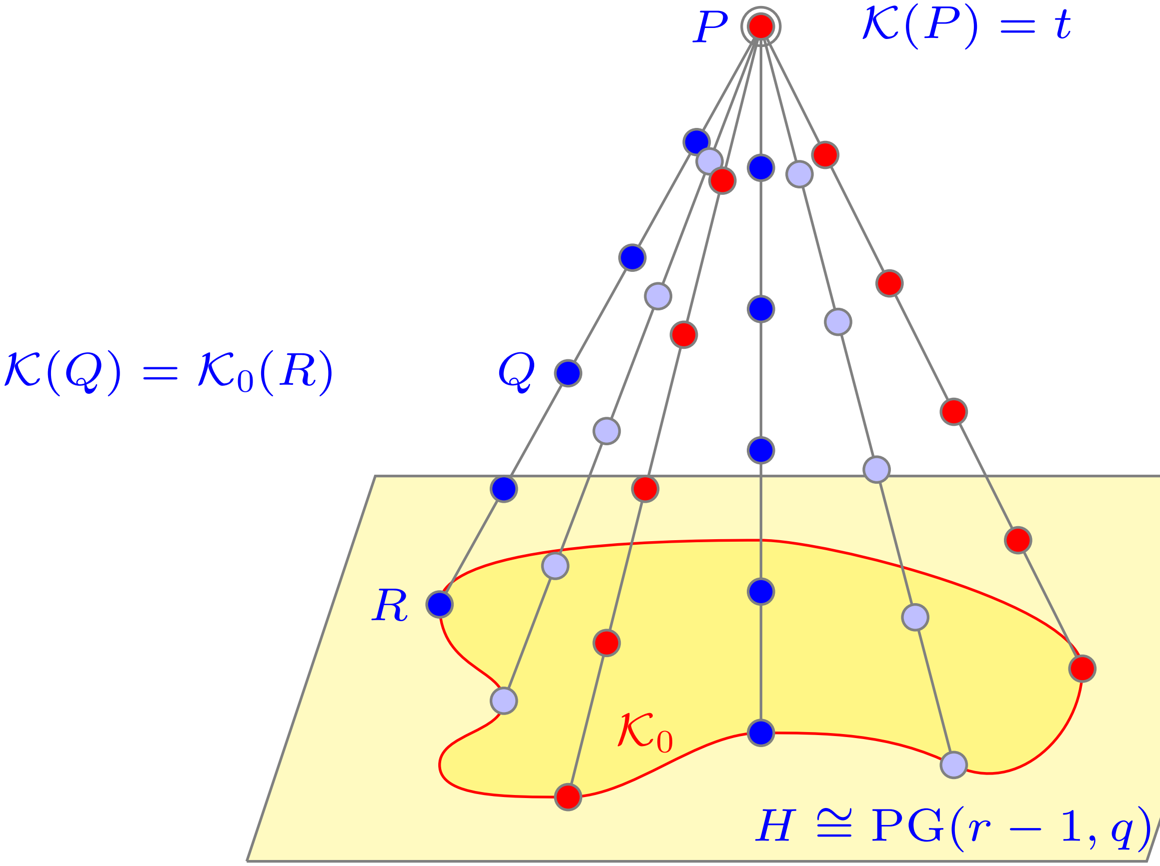
In particular, the sum of t hyperplanes in $\mathrm{PG}(r,q)$ is a strong $(t \mod q)$-arc.

Theorem D. (Landjev,Rousseva,2016) Let $\mathcal{K}_0$ be a $(t \mod q)$-arc in a hyperplane $H \cong \mathrm{PG}(r-1,q)$. of $\Sigma = \mathrm{PG}(r,q)$. For a fixed point $P \in \Sigma \setminus H$, define an arc $\mathcal{K}$ in Σ as follows:

- $\mathcal{K}(P) = t$;
- for each point $Q \neq P$: $\mathcal{K}(Q) = \mathcal{K}_0(R)$ where $R = \langle P, Q \rangle \cap H$.

Then the arc $\mathcal{K}$ is a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$ of size $q|\mathcal{K}_0| + t$.

Definition. $(t \mod q)$-arcs obtained by Theorem D are called **lifted arcs**.


P
$\mathcal{K}(P) = t$
$\mathcal{K}(Q) = \mathcal{K}_0(R)$
Q
R
$\mathcal{K}_0$
$H \cong \mathrm{PG}(r-1, q)$

Theorem E. (Landjev,Rousseva,2016) A strong $(t \mod q)$-arc $\mathcal{K}$ in $\mathrm{PG}(2,q)$ of cardinality $mq+t$ exists if and only if there exists an $((m-t)q+m, m-t)$-blocking set $\mathcal{B}$ with line multiplicities contained in $\{m-t, m-t+1, \ldots, m\}$.

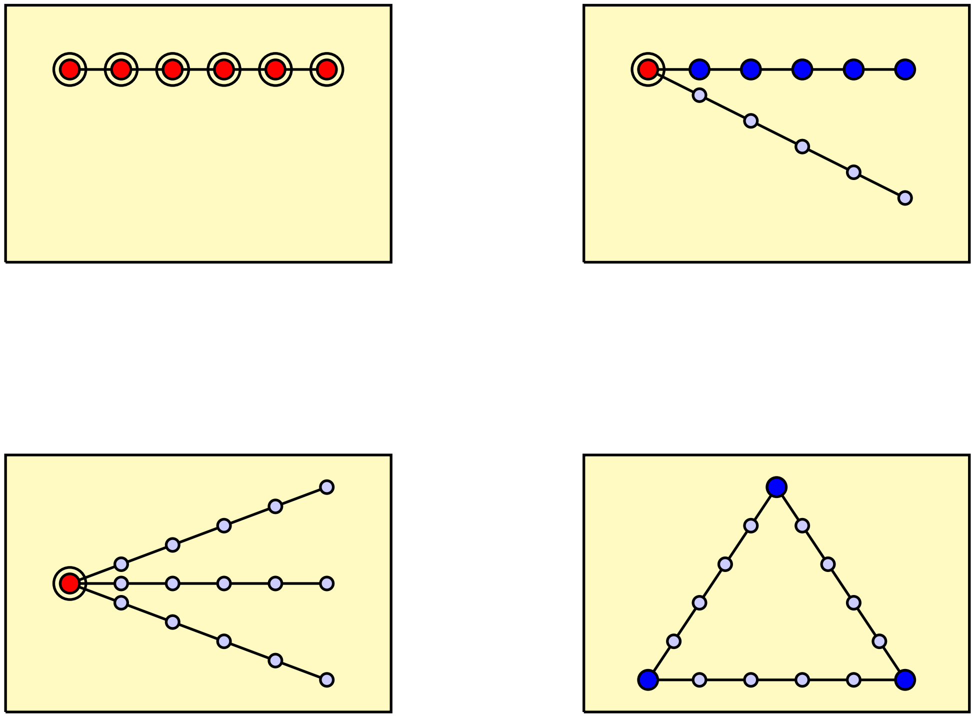
$(1 \mod q)$	$\mathrm{PG}(r,q)$	a hyperplane
$(2 \mod q)$	$\mathrm{PG}(2,q)$	lifted from a $2, q+2$, or $2q+2$-line, or an oval $+$ a tangent $+2\times$the internal points (T. Maruta, 2004, S.Kurz, 2021)
	$\mathrm{PG}(r,q)$, $r \geq 3$	lifted from a $(2 \mod q)$-arc in $\mathrm{PG}(r,q)$ (I. Landjev, A. Rousseva, 2019)
$(3 \mod 5)$	$\mathrm{PG}(2,5)$	185 arcs
	$\mathrm{PG}(3,5)$	lifted and three sporadic $(3 \mod 5)$-arcs of sizes $128, 143$, and 168 (S. Kurz, I. Landjev, A. Rousseva, 2023)
	$\mathrm{PG}(r,5)$, $r \geq 4$	lifted and ???

3. Strong $(3 \mod 5)$-Arcs in $\mathrm{PG}(2,5)$

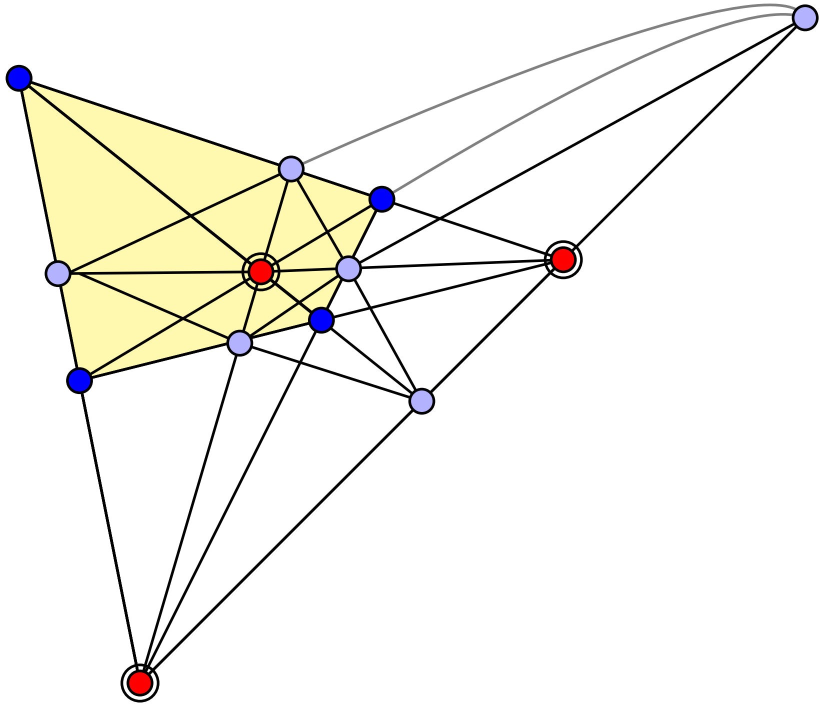
$\vert\mathcal{K}\vert$	BS	# arcs	$\vert\mathcal{K}\vert$	BS	# arcs
18	$(3,0)$	4	48	$(39,6)$	49
23	$(9,1)$	1	53	$(45,7)$	17
28	$(15,2)$	1	58	$(51,8)$	11
33	$(21,3)$	10	63	$(57,9)$	9
38	$(27,4)$	23	68	$(63,10)$	6
43	$(33,5)$	53	93	$(93,15)$	1

- Ivan Landjev & Assia Rousseva (computerfree)
- Sascha Kurz (computer search)

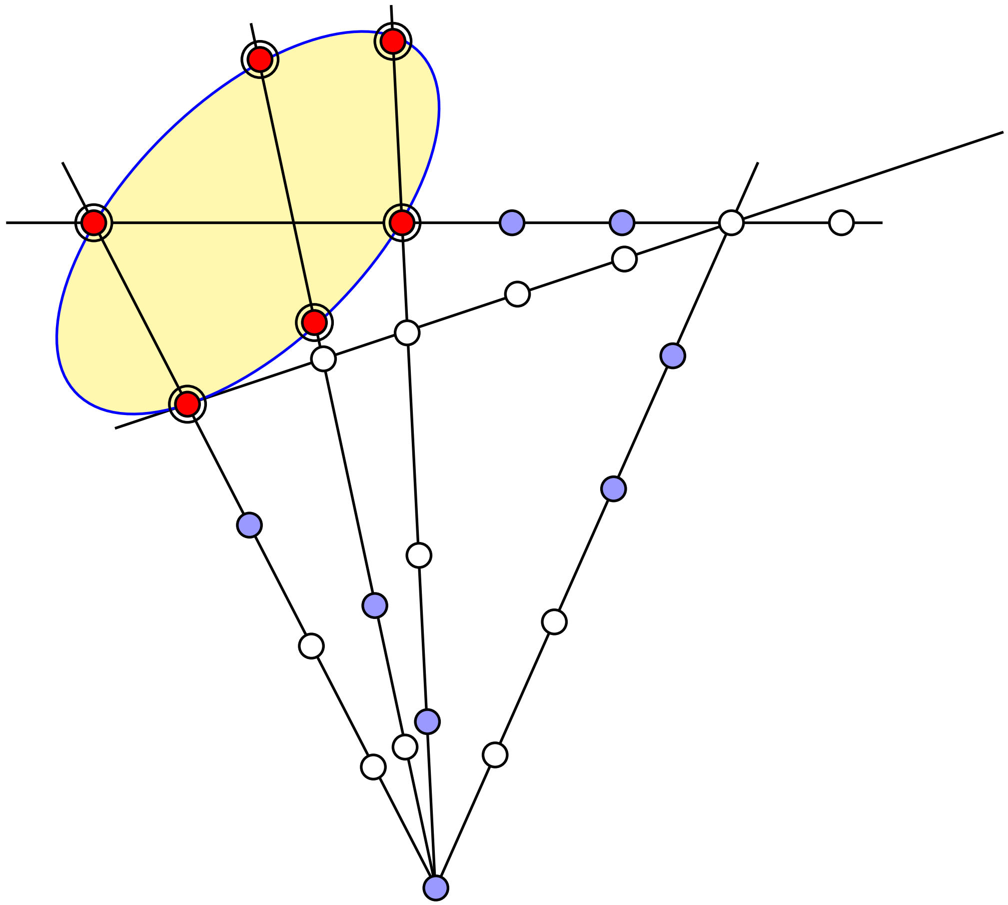
$(18, \{3, 8, 13, 18\})$-arcs



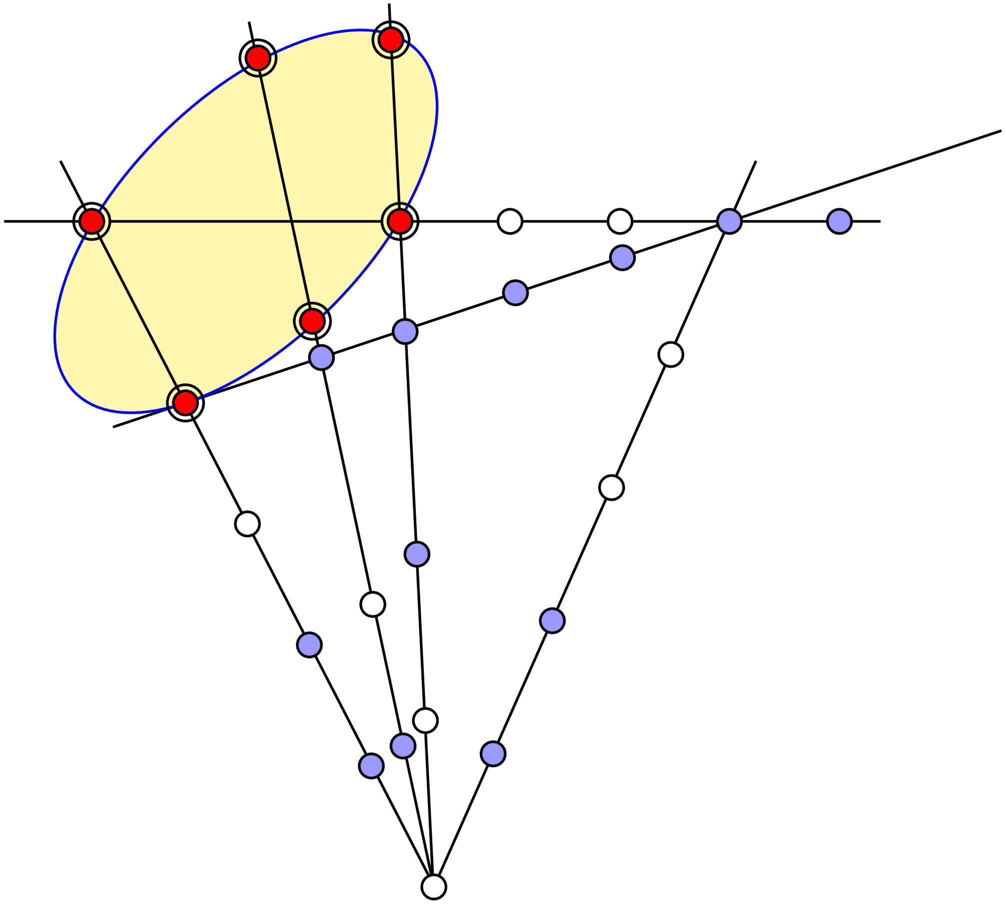
$(23, \{3, 8\})$-arc in $\mathrm{PG}(2, 5)$



$(28, \{3, 8\})$-arc in $\mathrm{PG}(2, 5)$



$(33,\{3,8\})$-arc in $\mathrm{PG}(2,5)$



4. Strong $(3 \mod 5)$-Arcs in $\mathrm{PG}(3,5)$

Theorem F. (S. Kurz, I. Landjev, A. Rousseva, 2023) Let $\mathcal{K}$ be a strong $(3 \mod 5)$-arc in $\mathrm{PG}(3,5)$ that is neither lifted nor contains a full hyperplane in its support. Then $|\mathcal{K}| \in \{128, 143, 168\}$ and in each case the corresponding arc is unique up to isomorphism.

Remark.

The nonexistence of $(104, 22)$-arcs in $\mathrm{PG}(3,5)$, or, equivalently, the nonexistence of $[104, 4, 82]_5$-codes

d	$g_q(k,d)$	$n_q(k,d)$	d	$g_q(k,d)$	$n_q(k,d)$
⋮	⋮	⋮	⋮	⋮	⋮
81	103	103–104	161	203	203–204
82	104	104–105	162	204	204–205
83	105	106	163	205	206
84	106	107	164	206	207
85	107	108	165	207	208
⋮ ⋮	⋮ ⋮	⋮ ⋮	⋮ ⋮	⋮ ⋮	⋮ ⋮

$(104, 22)$-arc:

Cardinality of a plane:		18	13	8	3		
	22	17	12	7	2	$\longrightarrow$	**0**
	21	**16**	**11**	**6**	**1**	$\longrightarrow$	**1**
	20	**15**	**10**	**5**	**0**	$\longrightarrow$	**2**
	19	**14**	**9**	**4**		$\longrightarrow$	**3**
Maximal number of points on a line: in such plane:	5	4	3	2	1		

The 128-Arc in $\mathrm{PG}(3,5)$

V. Abatangelo, G. Korchmáros, B Larato, 1996

There exist two 20-caps in $\mathrm{PG}(3,5)$ that do not extend to the elliptic quadric.

We denote these two caps by K_1 and K_2.

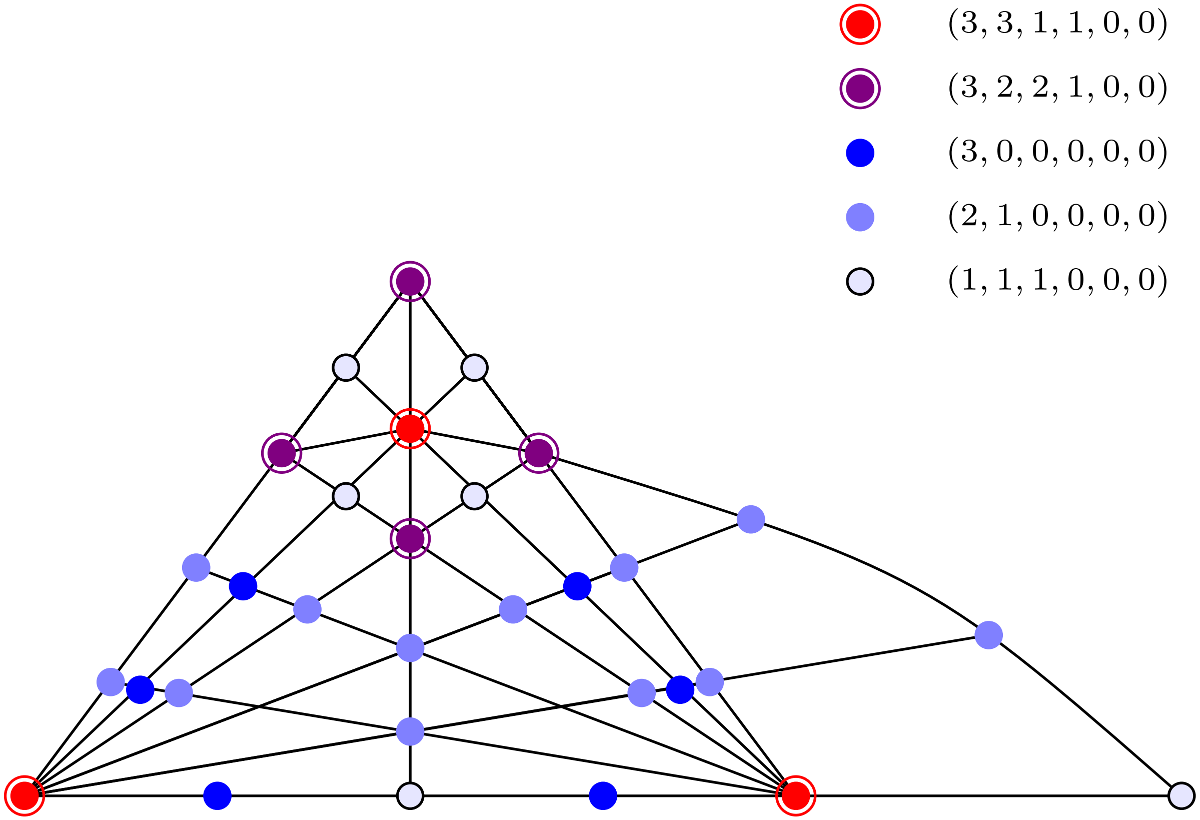
The collineation group G of K_1 is a semidirect product of an elementary abelean group of order 16 and a group isomorphic to S_5. Hence $|G| = 16 \cdot 120 = 1920$.

The collineation group G of K_2 is isomorphic to S_5.

Lemma. Let $\mathcal{K}$ be a strong $(3 \mod 5)$-arc in $\mathrm{PG}(3,5)$ of cardinality 128. Let φ be the projection from an arbitrary 0-point in $\mathrm{PG}(3,5)$. Then the arc $\mathcal{K}^{\varphi}$ is unique up to isomorphism and has the structure described below.

- A 0-point is incident only with 3- and 8-lines.
- An 8-line with a 0-point is of type $(3,3,1,1,0,0)$ or $(3,2,2,1,0,0)$.

Projection of a 128-arc from a 0-point



Each 0-point is incident with:

three 8-lines of type $(3, 3, 1, 1, 0, 0)$,
four 8-lines of type $(3, 2, 2, 1, 0, 0)$,
six 3-lines of type $(3, 0, 0, 0, 0, 0)$,
twelve 3-lines of type $(2, 1, 0, 0, 0, 0)$,
six 3-lines of type $(1, 1, 1, 0, 0, 0)$

This implies that

$$\lambda_3 = 16, \lambda_2 = 20, \lambda_1 = 40, \lambda_0 = 80.$$

$$a_{33} = 40, a_{28} = 16, a_{23} = 80, a_{18} = 20.$$

Here λ_i denotes the number of i-points.

The 2-points form a 20-cap C with spectrum:

$$a_6(C) = 40, a_4(C) = 80, a_3(C) = 20, a_0(C) = 16.$$

This cap is **not** extendable to the elliptic quadric. In such case it would have (at least 20) tangent planes, but $a_1(C) = 0$, a contradiction.

Hence the 20-cap on the 2-points in $\mathrm{PG}(3,5)$ is isomorphic to one of the two maximal 20-caps found by Abatangelo, Korchmaros and Larato.

This turns out that this is the cap K_1, since K_2 has a different spectrum.

The action of G on $\mathrm{PG}(3,5)$ gives four orbits on points, denoted $O_1^P, \ldots, O_4^P$ and six orbit on lines, denoted $O_1^L, \ldots, O_6^L$.

The respective sizes of these orbits are

$$|O_1^P| = 40, |O_2^P| = 80, |O_3^P| = 20, |O_4^P| = 16;$$

$$|O_1^L| = 160, |O_2^L| = 240, |O_3^L| = 30, |O_4^L| = 160, |O_5^L| = 120, |O_6^L| = 96.$$

The point-by-line orbit matrix $A = (a_{ij})_{4\times 6}$, where a_{ij} is the number of the points from the i-th point orbit incident with any line from the j-th line orbit is the following

$$A = \begin{pmatrix} 3 & 1 & 4 & 1 & 2 & 0 \\ 3 & 4 & 0 & 2 & 2 & 5 \\ 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}.$$

Let w_i be the multiplicity of any point from O_i^P and let $w = (w_1, w_2, w_3, w_4)$. In order to get a $(3 \mod 5)$-arc we should have

$$wA \equiv 3\boldsymbol{j} \pmod 5,$$

where $\boldsymbol{j}$ is the all-one vector, and $w_i \leq 3$ for all $i = 1, 2, 3$.

The set of all solutions is given by

$$w = \{(w_1, w_2, w_3, w_4) \mid w_i \in \{0, \ldots 4\},$$
$$w_2 \equiv 1 - w_1 \pmod 5, w_3 \equiv 4 - 2w_1 \pmod 5, w_4 = 3\}. \quad (1)$$

Solutions: $w = (3, 3, 3, 3)$ and $w = (1, 0, 2, 3)$.

The second solution gives the desired 128-arc.

The 143- and 168-Arc

Two strong non-lifted $(3 \mod 5)$-arcs in $\mathrm{PG}(3,5)$ were constructed by computer search. The respective spectra are:

$$|\mathcal{F}_1| = 143,\ a_{18}(\mathcal{F}_1) = 26, a_{28}(\mathcal{F}_1) = 65, a_{33}(\mathcal{F}_1) = 65;$$

$$\lambda_0(\mathcal{F}_1) = 65, \lambda_1(\mathcal{F}_1) = 65, \lambda_2(\mathcal{F}_1) = 0, \lambda_3(\mathcal{F}_1) = 26,$$

$$|\operatorname{Aut}(\mathcal{F}_1)| = 62400.$$

$$|\mathcal{F}_2| = 168,\ a_{28}(\mathcal{F}_2) = 60, a_{33}(\mathcal{F}_2) = 60, a_{43}(\mathcal{F}_2) = 36;$$

$$\lambda_0(\mathcal{F}_2) = 60, \lambda_1(\mathcal{F}_2) = 60, \lambda_2(\mathcal{F}_2) = 0, \lambda_3(\mathcal{F}_2) = 36.$$

$$|\operatorname{Aut}(\mathcal{F}_2)| = 57600.$$

There exist two quadrics in $\mathrm{PG}(3,5)$.

$$\mathcal{E}_3 = \{P(X_0, X_1, X_2, X_3) \mid X_0^2 + 2X_1^2 + X_2X_3 = 0, \} \tag{2}$$

$$\mathcal{H}_3 = \{P(X_0, X_1, X_2, X_3) \mid X_0X_1 + X_2X_3 = 0, \} \tag{3}$$

- $\mathcal{F}_1$: for a point $P(x_0, x_1, x_2, x_3)$ set

$$\mathcal{F}_1(P) = \begin{cases} 3 & \text{if } P \in \mathcal{E}_3, \\ 1 & \text{if } x_0^2 + 2x_1^2 + x_2x_3 \text{ is a square in } \mathbb{F}_5, \\ 0 & \text{if } x_0^2 + 2x_1^2 + x_2x_3 \text{ is a non-square in } \mathbb{F}_5. \end{cases} \tag{4}$$

- $\mathcal{F}_2$: for a point $P(x_0, x_1, x_2, x_3)$ set

$$\mathcal{F}_2(P) = \begin{cases} 3 & \text{if } P \in \mathcal{H}_3, \\ 1 & \text{if } x_0x_1 + x_2x_3 \text{ is a square in } \mathbb{F}_5, \\ 0 & \text{if } x_0x_1 + x_2x_3 \text{ is a non-square in } \mathbb{F}_5. \end{cases} \tag{5}$$

More generally:

$\mathcal{Q}$ – quadric in $\mathrm{PG}(r,q)$, q – odd prime power

$F(x_0, x_1, \ldots, x_r)$ – the quadratic form defining $\mathcal{Q}$

- r – even

  $\mathcal{P}_r = V(x_0^2 + x_1x_2 + \ldots + x_{r-1}x_r)$ – parabolic

- r – odd

  $\mathcal{H}_r = V(x_0x_1 + x_2x_3 + \ldots + x_{r-1}x_r)$ – hyperbolic

  $\mathcal{P}_r = V(f(x_0, x_1) + x_2x_3 + \ldots + x_{r-1}x_r)$ – elliptic
  (f is irreducible over $\mathbb{F}_q$.)

For $r = 2s$:

$$|\mathcal{P}_{2s}| = \frac{q^{2s} - 1}{q - 1}.$$

For $r = 2s - 1$:

$$|\mathcal{H}_{2s-1}| = \frac{(q^{s-1} + 1)(q^s + 1)}{q - 1}.$$

$$|\mathcal{E}_{2s-1}| = \frac{(q^s + 1)(q^{s-1} - 1)}{q - 1}.$$

The points outside $\mathcal{Q}$ split into two classes:

$$\begin{aligned}\mathcal{Q}_1 &= \{P(x_0,\dots,x_r) \mid F(x_0,x_1,\dots,x_r) \text{ is a square }\},\\ \mathcal{Q}_2 &= \{P(x_0,\dots,x_r) \mid F(x_0,x_1,\dots,x_r) \text{ is a non-square }\}.\end{aligned}$$

For a point P of $\mathrm{PG}(r,q)$ set $\mathcal{F}_1(P) = \begin{cases} \frac{q+1}{2} & \text{if } P \in \mathcal{Q},\\ 1 & \text{if } P \in \mathcal{Q}_1,\\ 0 & \text{if } P \in \mathcal{Q}_2.\end{cases}$

For a point P of $\mathrm{PG}(r,q)$ set $\mathcal{F}_2(P) = \begin{cases} \frac{q+1}{2} & \text{if } P \in \mathcal{Q},\\ 0 & \text{if } P \in \mathcal{Q}_1,\\ 1 & \text{if } P \in \mathcal{Q}_2.\end{cases}$

Theorem H. (S. Kurz, I. Landjev, F. Pavese, A. Rousseva, 2023)

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be defined as above. Then $\mathcal{F}_i$, $i = 1, 2$, is a $\left(\frac{q+1}{2} \mod q\right)$ arc in $\mathrm{PG}(r, q)$. Moreover if $\mathcal{Q}$ is non-degenerate, then both arcs are not lifted.

Definition. An arc obained by this construction is called a **quadratic** $(t \mod q)$-**arc**.

Theorem I. (L&R, 2023, unpublished) Assume that every strong $(3 \mod 5)$-arc in $\mathrm{PG}(r,5)$, which does not contain a hyperplane in its support is lifted or obtained from a quadric. Then every strong $(3 \mod 5)$-arc in $\mathrm{PG}(r+1,5)$, is also lifted or a quadratic arc.

Theorem J. (L&R, 2023, unpublished) Every strong $(3 \mod 5)$-arc in $\mathrm{PG}(4,5)$, which does not contain a hyperplane in its support is lifted or a quadratic arc.

Corollary. (L&R, 2023, unpublished) Every strong $(3 \mod 5)$-arc in $\mathrm{PG}(r,5)$, $r \geq 4$, which does not contain a hyperplane in its support is lifted or a quadratic arc.