XII-th International Summer Conference on Probability and Statistics and Seminar on Statistical Data Analysis

Stochastic Monotony and Continuity Properties for the Extinction Time of Age-Dependent Branching Processes

An Application to Epidemic Modelling

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Contents

- Extinction Time of Age-Dependent Branching Processes
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Extinction Time: $T = \sup\{t \geq 0\colon \inf_{0\leq s\leq t} Z(s) > 0\}$

Distribution Function of T: $u(t) = P(T \leq t)$, $t \geq 0$.

$$u(t) = \int_0^t f(u(t-x))\mathrm{d}G(x)$$

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- If $m > 1$, then $q = P(T < \infty) < 1$ and $u(t)$ is the distribution function of an improper random variable.

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Iterations of the integral operator:

$$H^1h = Hh,\ H^{n+1}h = H(H^n h),\ n = 1, 2, \ldots$$

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$$\begin{aligned} u(t) &= \int_0^t f(u(t-x))\mathrm{d}G(t) \\ &= f(0)G(t) + (1-f(0)) \int_0^t \frac{f(u(t-x)) - f(0)}{1-f(0)}\mathrm{d}G(t) \\ &= f(0)G(t) + (1-f(0))(F*G)(t) \end{aligned}$$

$$F(x) = \frac{f(u(x)) - f(0)}{1-f(0)}, \quad x \geq 0$$

Proposition 2. For each $h\colon [0,\infty) \to [0,1]$

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- $G^*(t) = f(0)G(t) \leq Hh(t) \leq G(t) \qquad f(0) \leq f(s) \leq 1.$
- H is a non-decreasing operator:

if $h \leq h^*$, then $Hh(t) \leq Hh^*(t)$, $t \geq 0$

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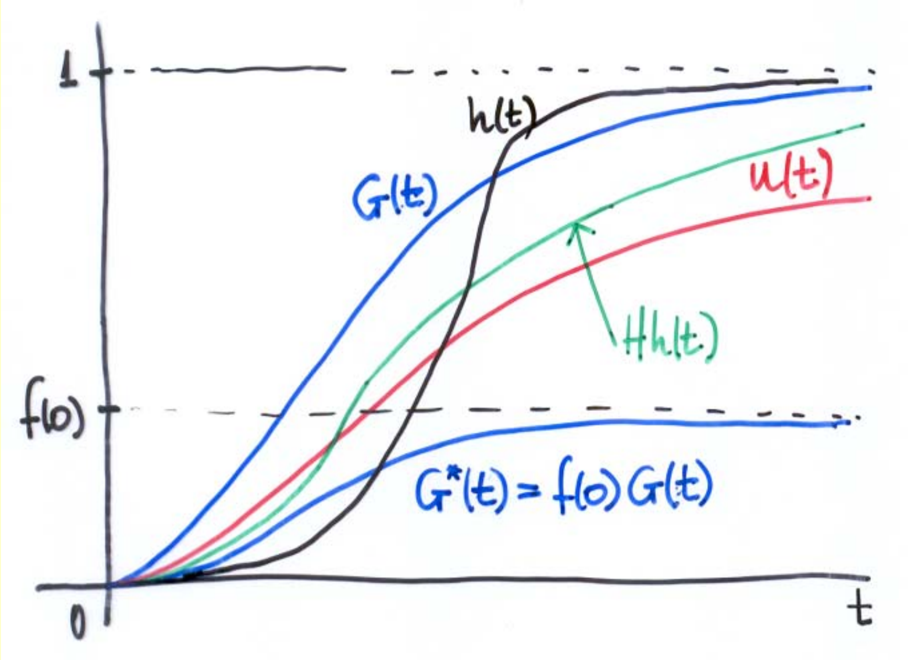

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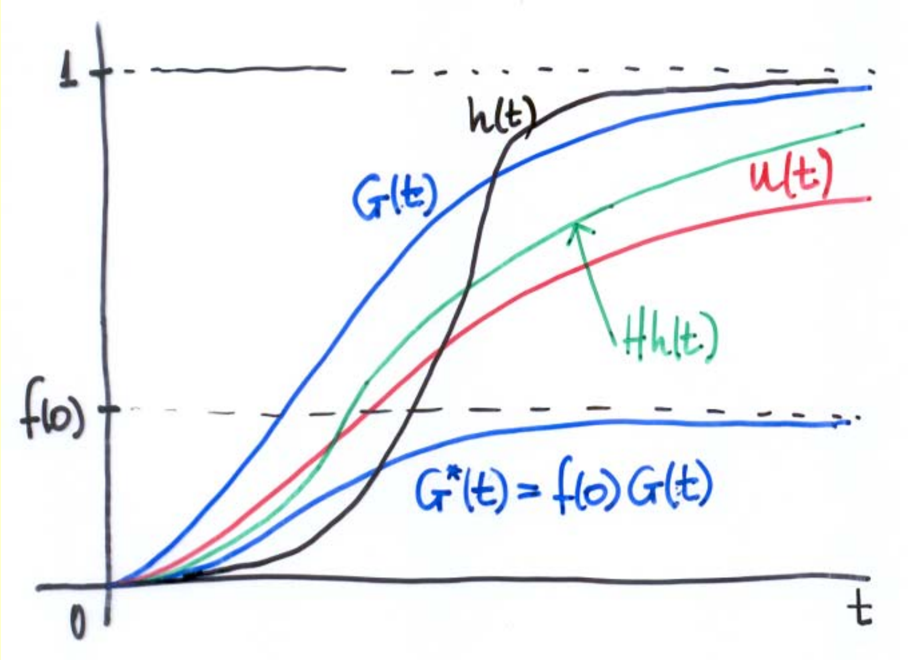

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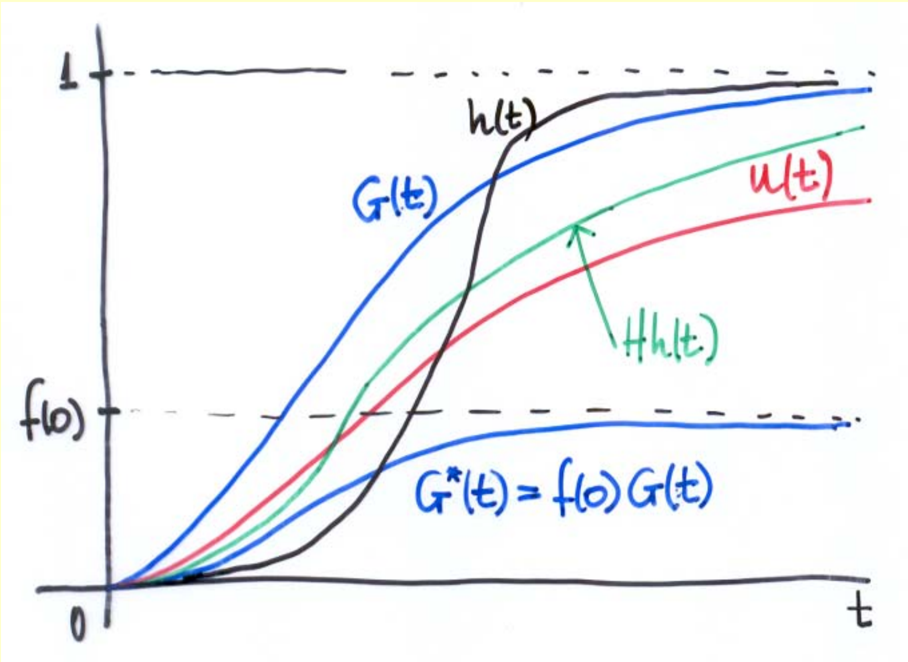

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$$v(t) = v^*(t) = u(t)\,, \quad t \ge 0$$

Let $\{Z_f(t)\}_{t\geq 0}$ be an Age-Dependent Branching Process defined by a reproduction law given by the p.g.f. f $(m_f = f'(1))$ and a life-length distribution $G(t)$

For each of these processes we consider its extinction time, T_f, denoting by $u_f(t)$ its distribution function and by H_f its associated integral operator

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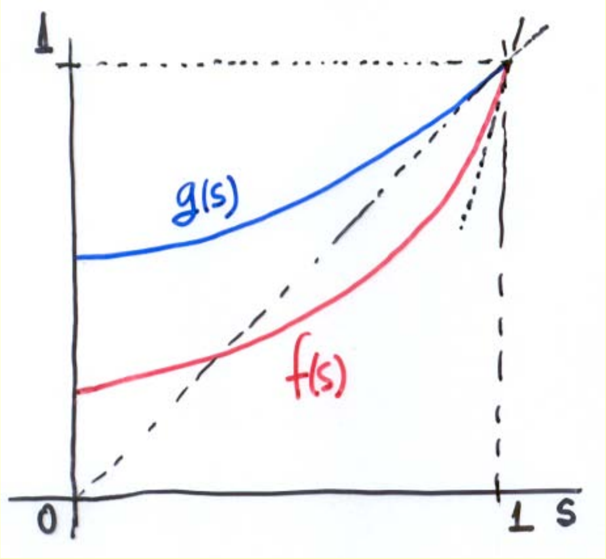

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- $u_f(t) \leq \lim_{n\to\infty} H_g^n u_f(t) = u_g(t)$, $t \geq 0$.

Theorem 2. Let f be a p.g.f. such that $m_f = f'(1) < 1$. For each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, f) > 0$ such that, if g is a p.g.f. with $\sup_{0 \leq s \leq 1} |f(s) - g(s)| \leq \delta$, then $\sup_{t \geq 0} |u_f(t) - u_g(t)| \leq \varepsilon$

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Proof.

For ε, f, let $\delta = \varepsilon(1 - m_f)$.

For each p.g.f. g such that $\sup_{0\leq s\leq 1} |f(s) - g(s)| \leq \delta$, it is verified

$$\sup_{t\geq 0} |H_f^n G(t) - H_g^n G(t)| \leq \varepsilon(1 - m_f^n)$$

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- Different types of stochastic models have been used to model the evolution of an infectious disease into a population: branching processes.
- Branching processes approach is appropriate when the number of infected individuals is small in relation to the total population size (see Ball (1997)).
- We shall use age-dependent branching processes because allow us to control the extinction time more accurately than discrete-time processes.

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- The disease is spreading when an infected individual is in contact with susceptible individuals.
- We denote by p_k the probability that one infected individual contacts k healthy individuals, $k \geq 0$, and by α $(0 \leq \alpha \leq 1)$ the proportion of immune individuals in the population.

- We assume that the population size is fixed and large enough so that α and the contact distribution law, $\{p_k\}_{k\geq 0}$ (with p.g.f. f and mean m), can be considered stable along time.

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$$p_{\alpha,k} = \sum_{j=k}^{\infty} \binom{j}{k} \alpha^{j-k} (1-\alpha)^k p_j.$$

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$$p_{\alpha,k} = \sum_{j=k}^{\infty} \binom{j}{k} \alpha^{j-k} (1-\alpha)^k p_j.$$

- We call $\{p_{\alpha,k}\}_{k\geq 0}$ the infection distribution law when the proportion of immune individuals in the population is α. Its p.g.f. is $f_\alpha(s) = f(\alpha + (1-\alpha)s)$.

Following this spreading scheme along time, infected individuals pass on the disease to other susceptible individuals and so on. We model the number of infected individuals in the population by an age-dependent branching process: $\{Z_\alpha(t)\}_{t\geq 0}$

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- Reproduction law: $f_\alpha(s) = f(\alpha + (1-\alpha)s)$, $m_\alpha = (1-\alpha)m$.

- Life-length: $G(t)$

  Intuitively: By life-length we mean the period (measured in real time) till one infected individual infects susceptible individuals or the disease disappears in this individual

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1) To investigate the distribution of the extinction time of the infection depending on the proportion of immune individuals into the population.

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2) From the previous study, to suggest vaccination policies based on the quantiles on the infection extinction time.

Extinction Time: T_α $\qquad u_\alpha(t) = P(T_\alpha \leq t)$, $t \geq 0$

Intuitively: T_α is the maximal time that the infection survives into the population when the proportion of immune individuals is α

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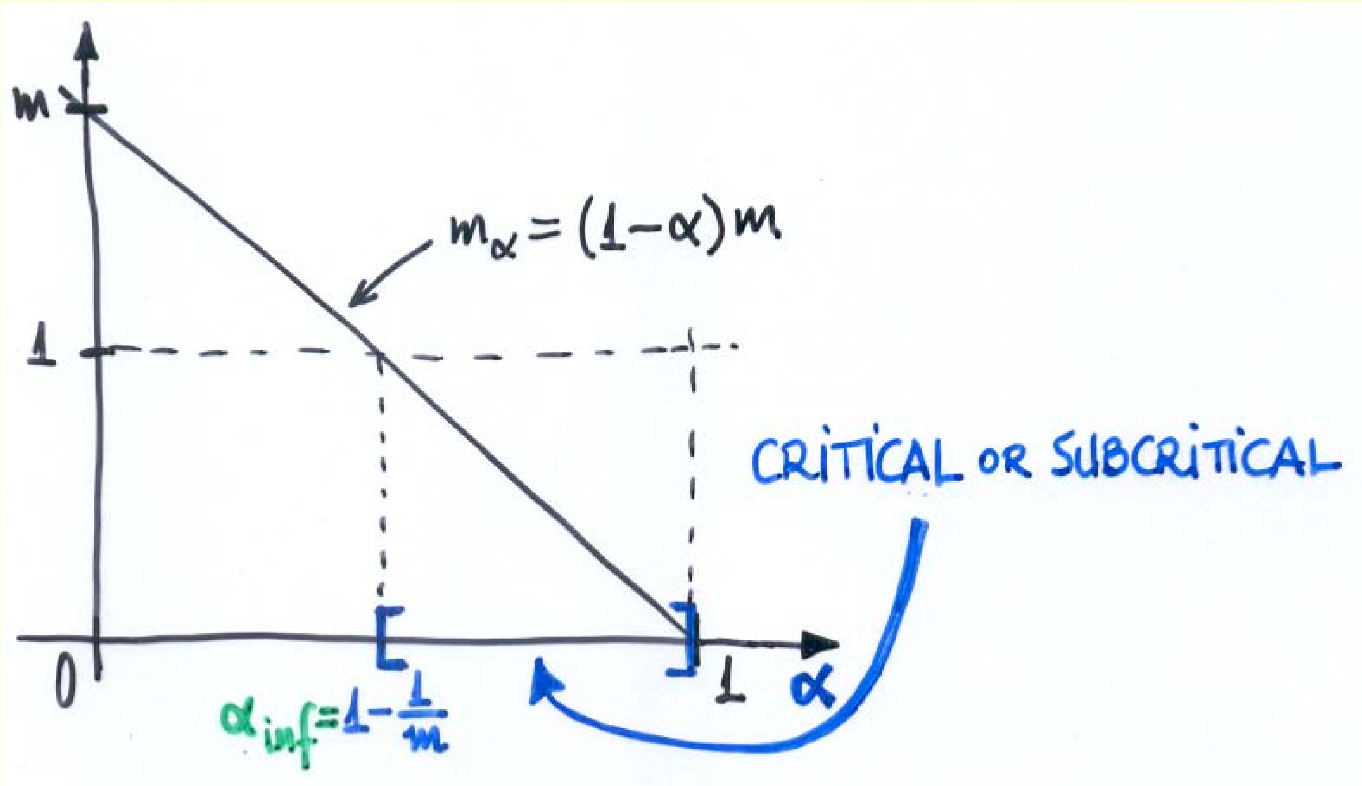

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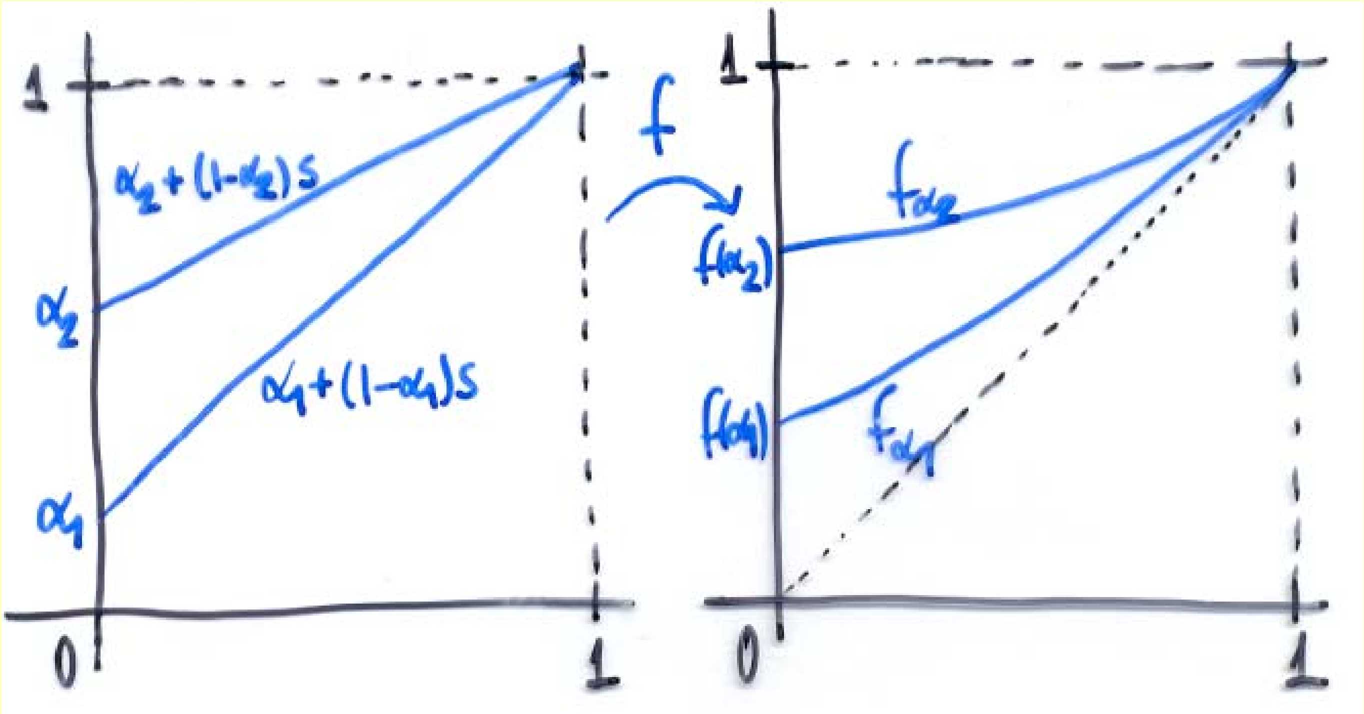

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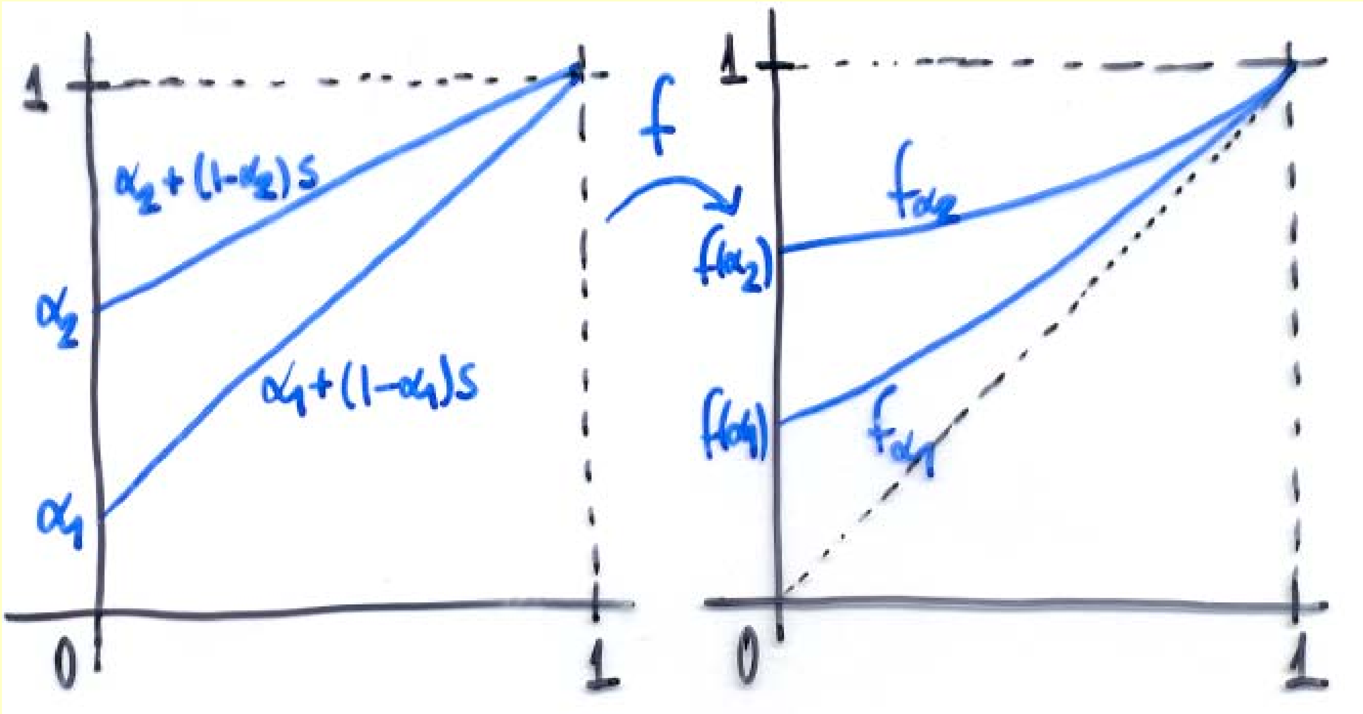

Intuitively, it is clear that the greater is the proportion of the immune individuals, the more probable is that the infectious disease disappears faster

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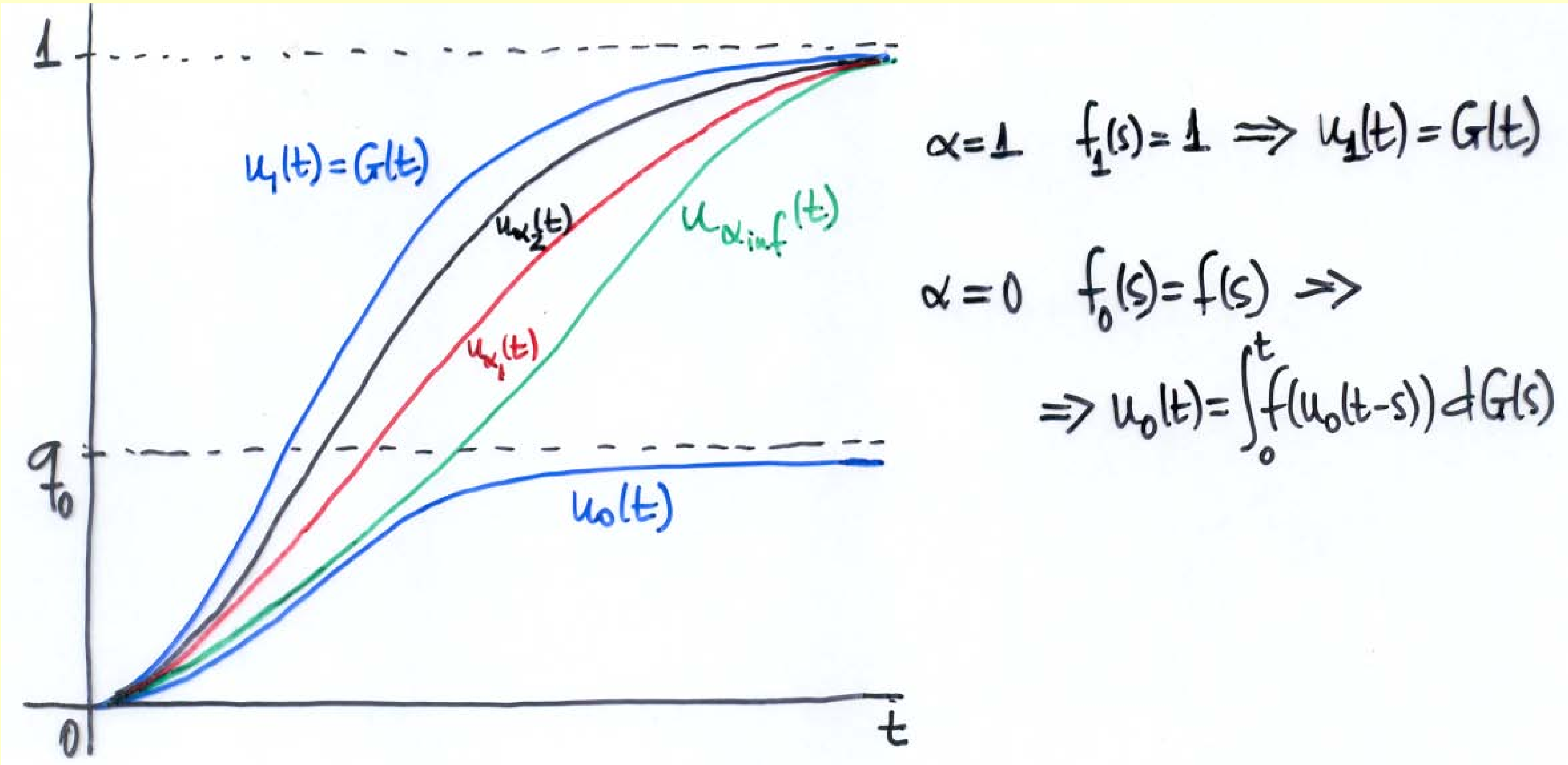

Continuity property: Let α be such that $m_\alpha < m_{\alpha_{inf}}$. For each $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, \alpha) > 0$ such that if $|\alpha - \alpha^*| \leq \delta$, then $\sup_{t \geq 0} |u_\alpha(t) - u_{\alpha^*}(t)| \leq \varepsilon$.

- f is uniformly continuous
- $|\alpha + (1-\alpha)s - \alpha^* + (1-\alpha^*)s| \leq |\alpha - \alpha^*|$

Quantiles of the infection extinction time T_α

$$0 < p < 1 \qquad t_p^\alpha = \inf\{t : u_\alpha(t) \geq p\}$$

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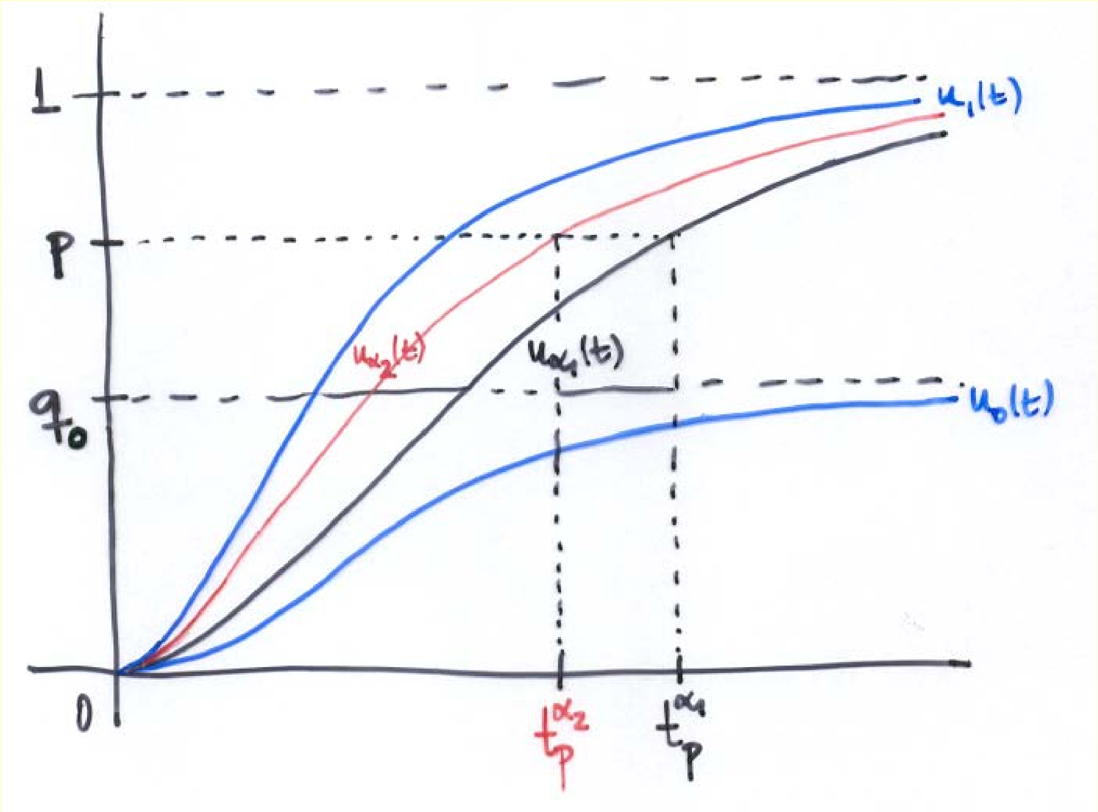
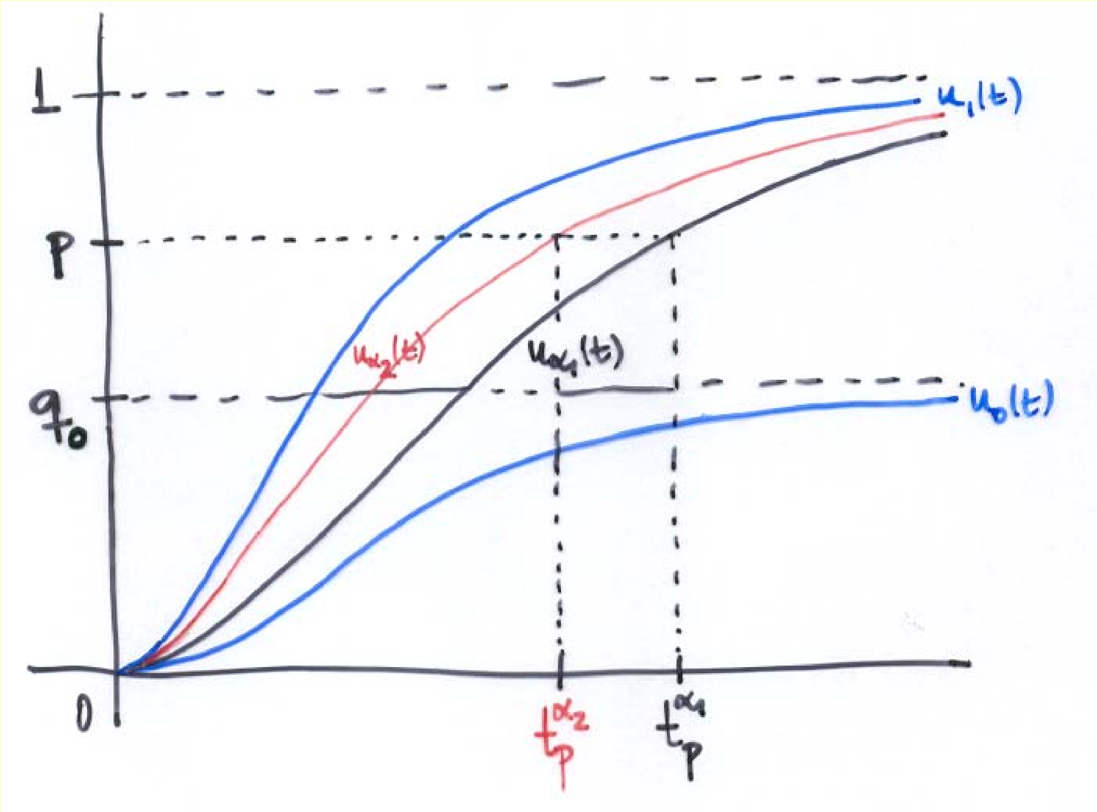


- If $\alpha_1 < \alpha_2$, then $t_p^{\alpha_2} \leq t_p^{\alpha_1}$
- If $u_\alpha(t)$ is an increasing and absolutely continuous function, then $\lim_{\alpha^* \to \alpha} t_p^{\alpha^*} = t_p^{\alpha}$.



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- This proportion of susceptible individuals to be vaccinated depends on the time that we allow the infectious disease to survive after vaccination.

Assume:

- Before vaccination, every healthy individual which is in contact with an infected individual is not immune, i.e. the contact always produces the infection. Then, with probability p_k an infected individual passes the disease on k susceptible individuals.

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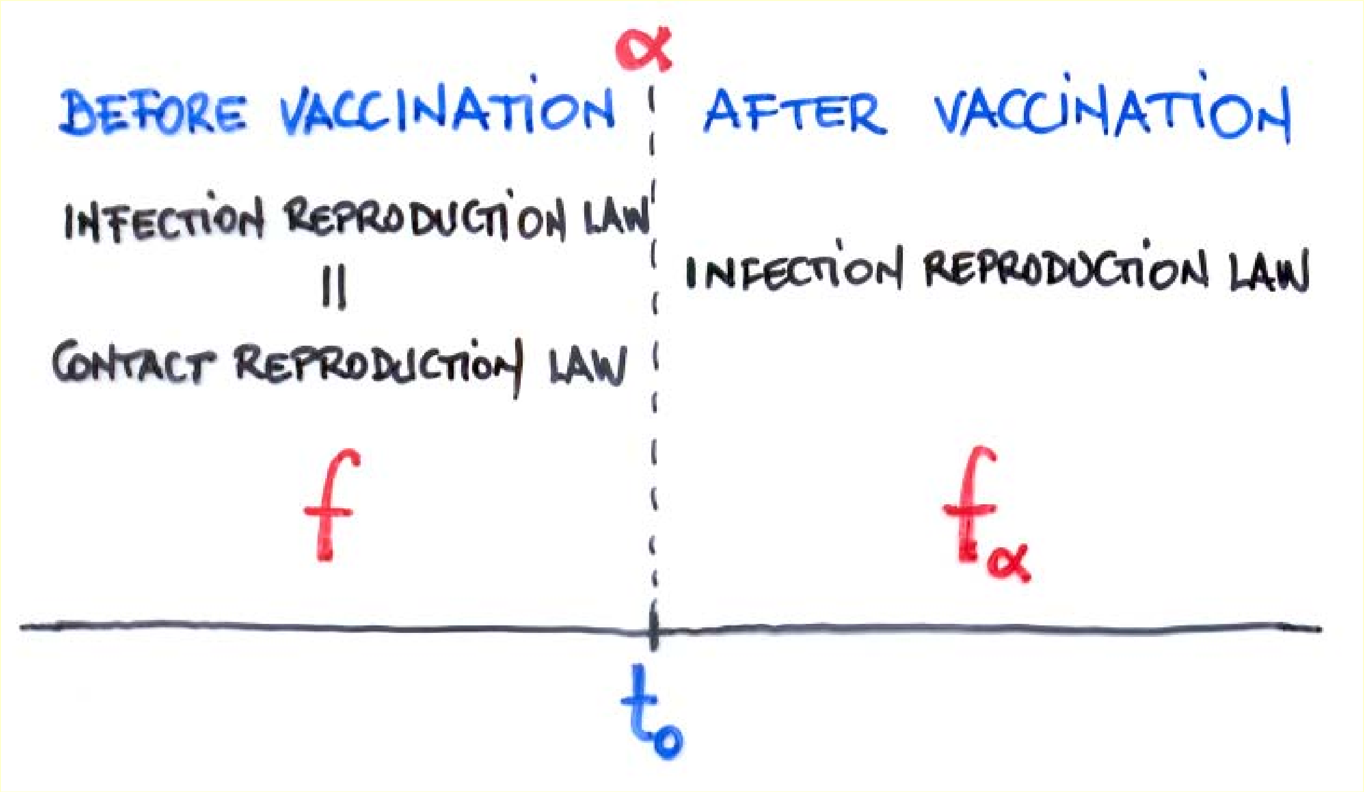
- At an arbitrary time t_0 after the infection occurred, we vaccinate a proportion α of susceptible individuals. We suppose that the vaccination process is instantaneous and every vaccinated individual is immune to the infectious disease from this time on.
- After vaccination, with probability $p_{\alpha,k}$ an infected individual transmits the disease to k susceptible individuals.


α
BEFORE VACCINATION
AFTER VACCINATION
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||
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f
f_α
t_0

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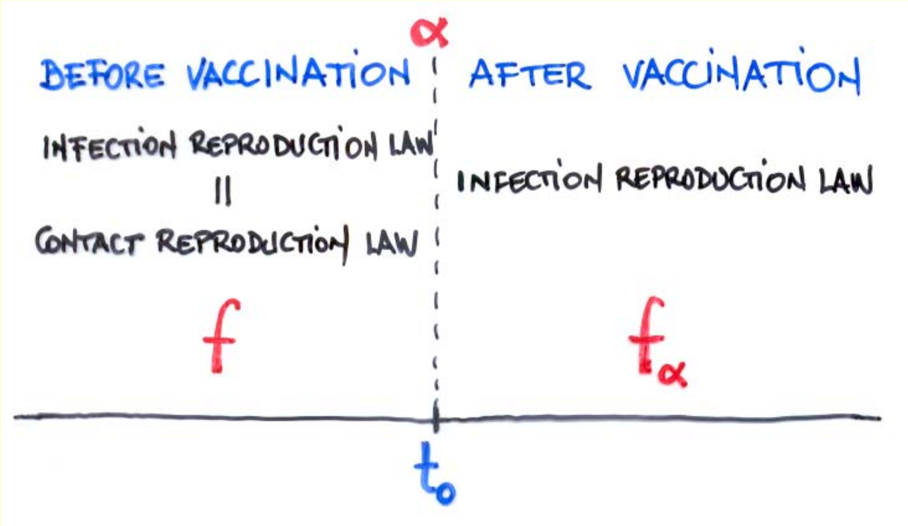
- Optimal proportion of vaccinated individuals:

  Fixed p, $0 < p < 1$, and $t > 0$, we are looking for vaccination policies, i.e. α-values, such that it can be guaranteed the extinction of the disease, with probability greater than or equal to p, no later than time t after vaccination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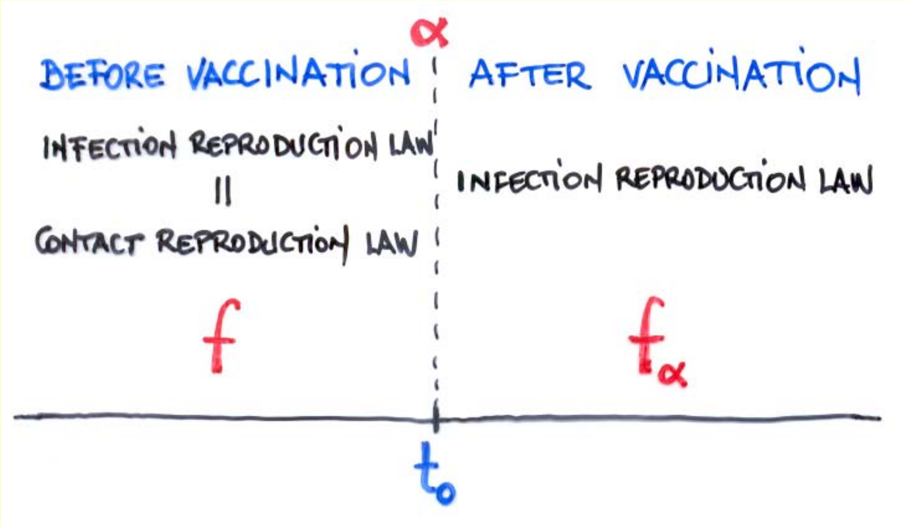
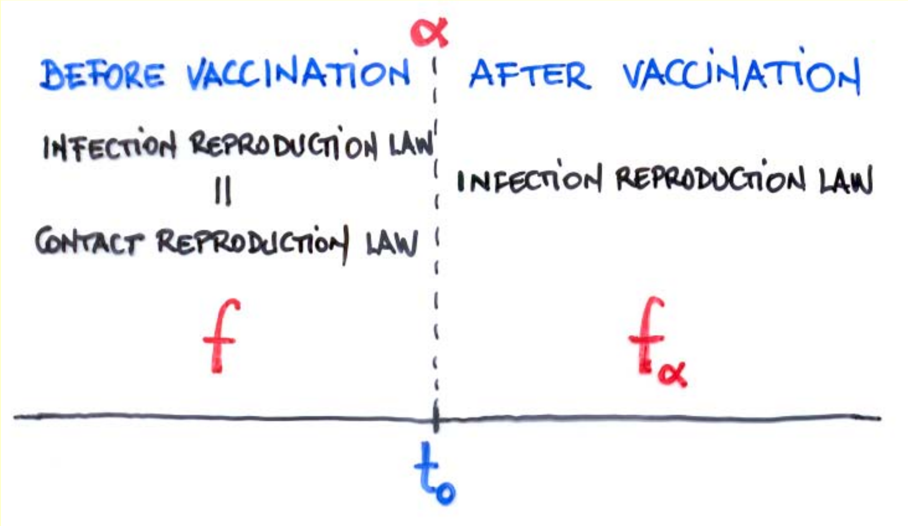


$$P(Z_\alpha(t) = 0 | Z_\alpha(0) = z) = P(T_\alpha \leq t)^z = u_\alpha(t)^z$$



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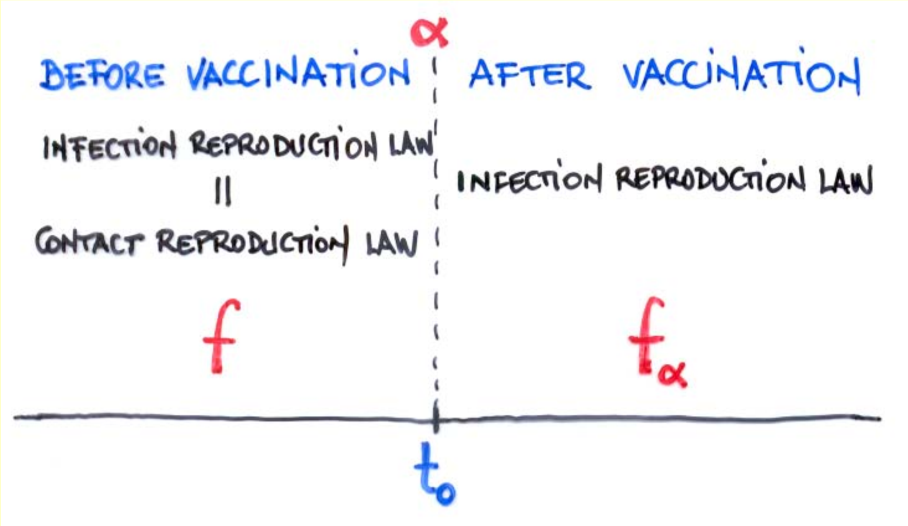

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$$\begin{aligned} \alpha_q = \alpha_q(p, t, z) &= \inf\{\alpha : \alpha_{inf} \leq \alpha \leq 1, u_\alpha(t) \geq p^{(z)}\} \\ &= \inf\{\alpha : \alpha_{inf} \leq \alpha \leq 1, t^\alpha_{p^{(z)}} \leq t\} \end{aligned}$$

Epidemic Modelling: Open questions for further research

1) How it can be chosen t_0?

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2) How to model the vaccination period?

   By age-dependent branching processes in varying environment.

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