Total progeny in a subcritical branching process with two types of immigration

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$$n(t) = \max\{n : \tau_n \leq t\}$$
  the number of renewal events in the sequence $\tau_n, n = 1, 2, \ldots$ during the time interval $[0, t]$.

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- life time θ of one particle with cdf $G(t),\ t \geq 0$, the offspring of one particle ξ with pgf $h(s)$,
- the pgf $f(s)$ of the random number ν_i of immigrants in the state zero and the cdf $K(t)$ of the duration X_i of the stay in the state zero.

- The construction is as follows (see e.g. Mitov and Yanev (1985)): Let σ_i be the life period of the process $Z_i(t)$.

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- Then the sequence $U_i = X_i + \sigma_i, \quad i = 1, 2, \ldots$ defines

$$S_0 = 0, \quad S_n = S_{n-1} + U_n, n = 1, 2, \ldots \tag{1}$$

and

$$N(t) = \max\{n : S_n \leq t\}. \tag{2}$$

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- The BHIO process $Y(t)$ is defined by

$$Y(t) = Z_{N(t)+1}(t - S_{N(t)} - X_{N(t)+1})\mathbb{I}_{\{S_{N(t)}+X_{N(t)+1}\leq t\}},$$

where $\mathbb{I}_A$ denotes the indicator of the event A.

- Now the process $X(t)$ can be defined as follows (taking into account that $\tau_0 \equiv 0$ is the first renewal event when the I_0 independent BHIO processes start)

$$X(t) = \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} Y^{(i,k)}(t - \tau_i), \quad t \geq 0,$$

where $Y^{(i,k)}(t), t \geq 0$ are independent copies of $Y(t)$.

- The process $X(t)$ is studied by Weiner(1991) in the critical case, and by Slavchova-Bojkova (1996) in the non-critical cases.

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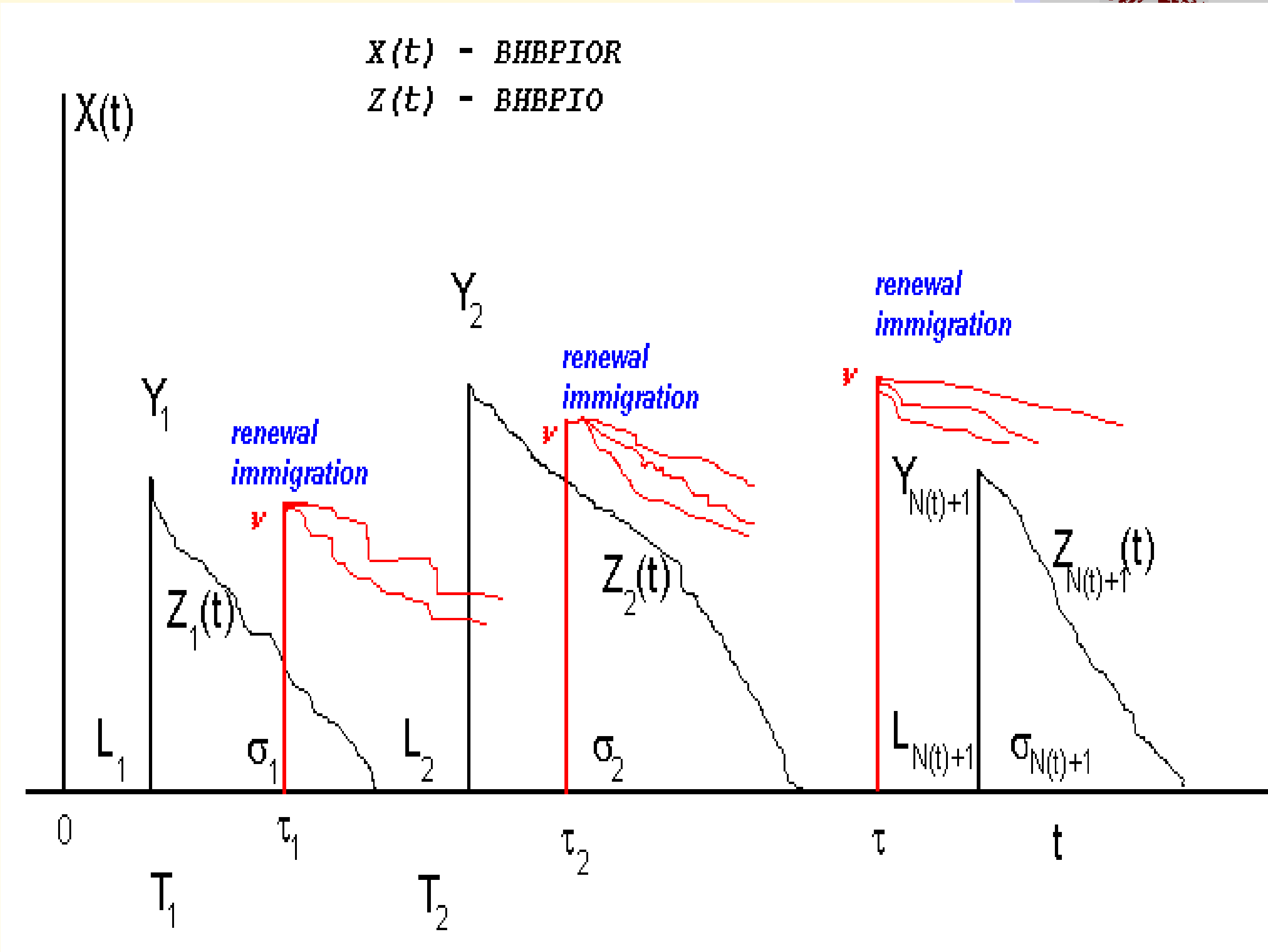

X(t) - BHBPIOR
Z(t) - BHBPIO
X(t)
Y1
Y2
renewal
immigration
renewal
immigration
renewal
immigration
Z1(t)
Z2(t)
YN(t)+1
ZN(t)+1(t)
L1
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0
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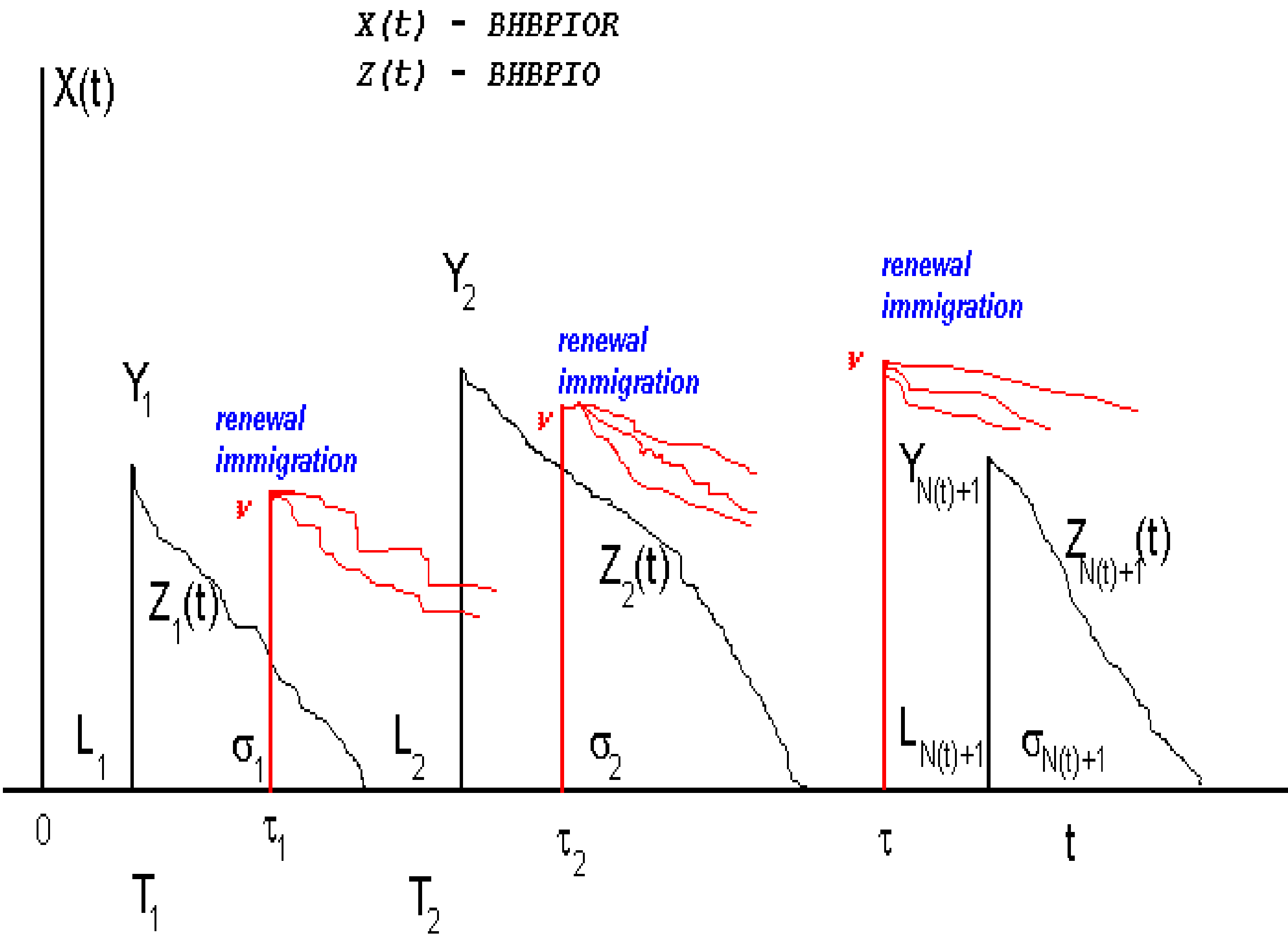

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For continuous time branching processes, which is our case ($Z(t)$, $X(t)$ or $Y(t)$), one can count the total number of particles up to the instant t or consider the following continuous time characteristic of the process, (e.g. for $Z(t)$),

$$\int_0^t Z(u)du, t \geq 0,$$

which is analogous to the total number of particles up to the instant t. More comments and discussions on this characteristic can be found in Pakes (1972) or in Jagers (1975).

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Let us denote by $V(t)$ the total number of particles up to the moment t in the process $Y(t)$.

Then

$$V(t) = \sum_{i=1}^{N(t)} \zeta_i + \zeta_{N(t)+1}(t - S_{N(t)} - X_{N(t)+1})\mathbb{I}_{\{S_{N(t)}+X_{N(t)+1}\le t\}}.$$

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Comment 2 *Kulkarni and Pakes (1983) have studied the corresponding quantity to $V(t)$ for Galton-Watson branching processes. In the recent paper of Glynn and Whitt (2001) the problem is solved in a more general setting. They have obtained necessary and sufficient conditions for LLN and CLT for an integral of a delayed regenerative process, i.e. $\int_0^t Y(u)du$ in our notations.*

Finally, denote by $W(t)$ the total number of particles in the process $X(t)$, i.e.

$$W(t) = \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} V^{(i,k)}(t-\tau_i) \tag{3}$$

$$= \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \sum_{l=0}^{N^{(i,k)}(t-\tau_i)} \zeta_l^{(i,k)}$$

$$+ \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \zeta^{(i,k)}_{N^{(i,k)}(t-\tau_i)+1} (t - S^{(i,k)}_{N^{(i,k)}(t-\tau_i)} - X^{(i,k)}_{N^{(i,k)}(t-\tau_i)+1})$$

$$\times \ \mathbb{I}_{\{S^{(i,k)}_{N^{(i,k)}(t-\tau_i)} + X^{(i,k)}_{N^{(i,k)}(t-\tau_i)+1} \le t\}},$$

where $V^{(i,k)}(t), t \ge 0$ are independent copies of $V(t), t \ge 0$.

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&+ \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \zeta^{(i,k)}_{N^{(i,k)}(t-\tau_i)+1} (t - S^{(i,k)}_{N^{(i,k)}(t-\tau_i)} - X^{(i,k)}_{N^{(i,k)}(t-\tau_i)+1}) \\
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where $V^{(i,k)}(t), t \geq 0$ are independent copies of $V(t), t \geq 0$.

Comment 3 *The process $W(t)$ is partially investigated in Weiner (1991) in the critical case.*

We will investigate the limiting behaviour of the process $W(t)$, $t \geq 0$, assuming the following basic conditions:

1. For the processes $Z(t)$:

$$0 < A = E\xi = h'(1) < 1, \quad 0 < B = Var\xi < \infty, \tag{4}$$

$$r_1 = E\theta = \int_0^\infty x dG(x) < \infty, \quad r_2 = Var\theta < \infty; \tag{5}$$

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2. For the processes $Y(t)$:

$$m_1 = E\nu = f'(1) < \infty, \quad 0 < m_2 = Var\nu < \infty, \tag{6}$$

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3. For the characteristics of the sequence $\{\tau_n, n = 0, 1, 2, \ldots\}$, we assume

$$c_1 = EI_i = f_0'(1) < \infty, \quad c_2 = f_0''(1) < \infty, \quad c_3 = VarI_i < \infty, \tag{8}$$

$$\mu_0 = E\tau_1 = \int_0^\infty x dG_0(x) < \infty, \quad \beta_0 = Var\tau_1 < \infty. \tag{9}$$

For the moments of the total number of particles ζ in the process $(Z(t), 0 \le t \le \sigma)$,

$$v_1 = E\zeta = \frac{E\nu}{1 - E\xi} = \frac{m_1}{1 - A}, \tag{10}$$

$$v_2 = Var\zeta = \frac{E(\nu)B}{(1-A)^3} + \frac{Var(\nu)}{(1-A)^2} = \frac{m_1 B}{(1-A)^3} + \frac{m_2}{(1-A)^2}. \tag{11}$$

Under the conditions (4) and (5),

$$P(\sigma > t) = P(Z(t) > 0) \sim C \exp(\alpha t)$$

where C is a positive constant and α is a Malthusian parameter defined by

$$A \int_0^\infty e^{-\alpha t} dG(t) = 1.$$

We always assume that a Malthusian parameter exists. Hence σ has finite moments of all orders. Therefore, for the moments of $U_i = X_i + \sigma_i$ we get

$$\mu_1 = EU_i = EX_i + E\sigma_i < \infty, \quad \beta_1 = VarU_i < \infty. \tag{12}$$

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2. Basic equations and inequalities

The following inequalities are fulfilled almost surely:

$$\sum_{i=1}^{N(t)} \zeta_i \leq V(t) \leq \sum_{i=1}^{N(t)+1} \zeta_i, \tag{13}$$

and

$$\sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \sum_{j=1}^{N^{(i,k)}(t-\tau_i)} \zeta_j^{(i,k)} \leq W(t) \leq \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \sum_{j=1}^{N^{(i,k)}(t-\tau_i)+1} \zeta_j^{(i,k)}. \tag{14}$$

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$$n^*(t) = \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} N^{(i,k)}(t - \tau_i) \qquad (15)$$

the number of the cycles in all the renewal processes S_n governing the processes $Z^{i,k}(t - \tau_i), \quad t \geq 0$ which are completely finished up to the moment t and

$$n^{**}(t) = \sum_{i=0}^{n(t)} I_i \qquad (16)$$

the number of BHIO processes starting at the moments $\tau_0, \tau_1, \ldots, \tau_{n(t)}$ during the interval $[0, t]$. In other words, $n^{**}(t)$ is the number of the cycles unfinished at the instant t.

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If we enumerate the iid r.v.'s $\zeta_j^{(i,k)}$ by one index (in some order) then by (15), (16) and (14) we can write:

$$\sum_{l=1}^{n^*(t)} \zeta_l \le W(t) \le \sum_{l=1}^{n^*(t)+n^{**}(t)} \zeta_l. \tag{17}$$

We will use these inequalities together with the definition (3) to investigate the limiting behaviour of $W(t)$.

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3. Moments of $n^*(t)$ and $n^{**}(t)$

$$M_1^*(t) = En^*(t) \quad M_2^*(t) = En^*(t)[n^*(t) - 1], \quad D^*(t) = Var(n^*(t)),$$

and

$$M_1^{**}(t) = En^{**}(t) \quad M_2^{**}(t) = En^{**}(t)[n^{**}(t) - 1], \quad D^{**}(t) = Var(n^{**}(t)).$$

Lemma 1 *The moments of $n^*(t)$ satisfy:*

$$M_1^*(t) \sim \frac{c_1 t^2}{2\mu_0\mu_1}, \quad t \to \infty, \tag{18}$$

$$M_2^*(t) \sim \frac{c_1^2 t^4}{4\mu_0^2\mu_1^2}, \quad t \to \infty, \tag{19}$$

$$D^*(t) = o(t^4), \quad t \to \infty. \tag{20}$$

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Lemma 2 *The moments of $n^{**}(t)$ satisfy:*

$$M_1^{**}(t) = \frac{c_1}{\mu_0} t + o(t), \quad t \to \infty, \tag{21}$$

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Lemma 3 *The following limits take place:*

$$\frac{n^*(t)}{M_1^*(t)} \xrightarrow{p} 1, \quad t \to \infty \tag{24}$$

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$$\frac{n^{**}(t)}{M_1^{**}(t)} \xrightarrow{p} 1, \quad t \to \infty. \tag{25}$$

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Lemma 4 *Under the conditions above*

$$\frac{n^{**}(t)}{n^{*}(t)} \xrightarrow{p} 0, \quad t \to \infty \tag{26}$$

and

$$\frac{n^{**}(t)}{\sqrt{n^{*}(t)}} \xrightarrow{p} \sqrt{\frac{2c_1\mu_1}{\mu_0}}, \quad t \to \infty. \tag{27}$$

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Lemma 5 *We even have a stronger convergence in (26):*

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4. Results

Theorem 1 *Under the conditions (4)-(9), as* $t \to \infty$,

$$\frac{W(t)}{n^*(t)} \xrightarrow{p} v_1, \tag{29}$$

$$\frac{W(t)}{n^*(t) + n^{**}(t)} \xrightarrow{p} v_1, \tag{30}$$

$$\frac{W(t)}{t^2} \xrightarrow{p} \frac{c_1 v_1}{2\mu_0\mu_1}, \tag{31}$$

$$EW(t) \sim \frac{v_1 c_1 t^2}{2\mu_0\mu_1}. \tag{32}$$

4. Results

Theorem 1 *Under the conditions (4)-(9), as* $t \to \infty$,

$$\frac{W(t)}{n^*(t)} \xrightarrow{p} v_1, \tag{29}$$

$$\frac{W(t)}{n^*(t) + n^{**}(t)} \xrightarrow{p} v_1, \tag{30}$$

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Theorem 2 *Under the conditions (4)-(9),*

$$\frac{\sum_{i=1}^{n^*(t)} \zeta_i - v_1 n^*(t)}{\sqrt{v_2 n^*(t)}} \xrightarrow{d} N(0,1), \quad t \to \infty, \tag{33}$$

and

$$\frac{\sum_{i=1}^{n^*(t)+n^{**}(t)} \zeta_i - v_1[n^*(t) + n^{**}(t)]}{\sqrt{v_2 n^*(t)}} \xrightarrow{d} N(0,1), \quad t \to \infty. \tag{34}$$

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Theorem 3 *Under the conditions (4)-(9),*

$$\limsup_{t\to\infty} P\left(\frac{W(t) - v_1 n^*(t)}{\sqrt{v_2 n^*(t)}} \le x\right) \le \Phi(x), \qquad (35)$$

$$\liminf_{t\to\infty} P\left(\frac{W(t) - v_1 [n^*(t) + n^{**}(t)]}{\sqrt{v_2 n^*(t)}} \le x\right) \ge \Phi(x). \qquad (36)$$

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Comment 4 *It is evident from the last theorem that the random elements used for centering are not appropriate to obtain a CLT. Let us consider the second sum in the definition (3):*

$$S(t) = \sum_{i=0}^{n(t)} \sum_{k=1}^{I_i} \zeta^{(i,k)}_{N^{(i,k)}(t-\tau_i)+1} (t - S^{(i,k)}_{N^{(i,k)}(t-\tau_i)} - X^{(i,k)}_{N^{(i,k)}(t-\tau_i)+1})$$

$$\times \quad \mathbb{I}_{\{S^{(i,k)}_{N^{(i,k)}(t-\tau_i)} + X^{(i,k)}_{N^{(i,k)}(t-\tau_i)+1} \le t\}}.$$

We can enumerate the summands in some order by one index to write

$$S(t) = \sum_{p=1}^{n^{**}(t)} \bar{\zeta}_p(t).$$

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$$\times \quad \mathbb{I}_{\{S^{(i,k)}_{N^{(i,k)}(t-\tau_i)}+X^{(i,k)}_{N^{(i,k)}(t-\tau_i)+1}\le t\}}.$$

We can enumerate the summands in some order by one index to write

$$S(t) = \sum_{p=1}^{n^{**}(t)} \bar{\zeta}_p(t).$$

Note that the random variables $\bar{\zeta}_l(t)$ are neither independent nor identically distributed. Clearly

$$W(t) = \sum_{l=1}^{n^*(t)} \zeta_l + \sum_{p=1}^{n^{**}(t)} \bar{\zeta}_p(t),$$

It seems that the right centering for $W(t)$ must be $v_1 n^*(t) + \sum_{p=1}^{n^{**}(t)} E\bar{\zeta}_p(t)$ and the following CLT must be true

$$\frac{W(t) - (v_1 n^*(t) + \sum_{p=1}^{n^{**}(t)} E\bar{\zeta}_p(t))}{\sqrt{v_2 n^*(t)}} \xrightarrow{d} N(0,1),$$

but we have not proved it by now.

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