Following a Sangaku

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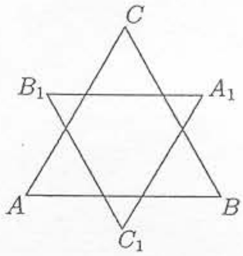
... there is that maketh himself poor, yet hath great reaches.
Proverbs 13:7

One can see wooden tablets hung under the roof of ancient Japanese temples. On these tablets, named sangaku, geometrical configurations were represented. Their properties, in antiquity, had excited Japanese mathematicians and at the present time still remain intriguing. At first sight some sangaku configurations seem too simple and ordinary, but upon a closer look are revealed to be rather challenging. This is the case with the problem given in [1].

Problem 1. Consider a unit equilateral triangle ABC with center O. Centered at O, equilateral triangle $A_1B_1C_1$ of side x is positioned so that its sides are parallel to the respective sides of $\triangle ABC$. Find the value of x, minimizing the combined area of the parts, which belongs to exactly one of the triangles ABC and $A_1B_1C_1$.



Perhaps the shortest way to solve this extreme value problem is by the use of parametrization and the properties of the quadratic function. Since such a solution hides some geometric properties of the configuration, we choose a more "geometrical" approach.

Solution of Problem 1. Parts belonging to exactly one of the triangles ABC and $A_1B_1C_1$ will be called *non-overlapping.* By symmetry it follows that there are three non-overlapping congruent equilateral triangles inside each of $\triangle ABC$ and $\triangle A_1B_1C_1$. Therefore the desired minimum is attained when the sum of the squares of respective altitudes CE and C_1E_1 of $\triangle CDF$ and $\triangle C_1D_1F_1$ (Fig. 1) reaches its minimum possible value.

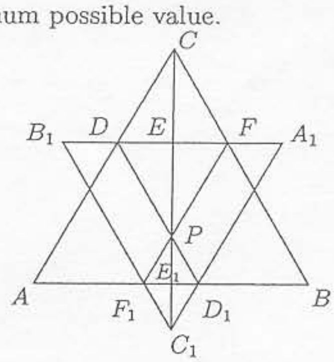


Figure 1

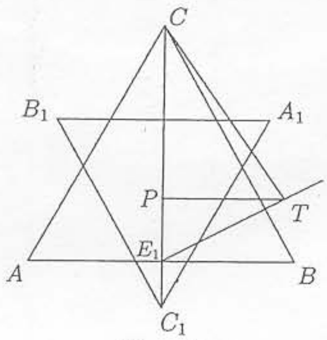


Figure 2

Note that the segments DD_1 and FF_1 are parallel, respectively, to the sides BC and AC. The intersection of these segments forms equilateral triangles DFP and D_1F_1P, congruent to DFC and $D_1F_1C_1$, respectively. It is easy to see that P lies on CC_1, and since $CP = 2 \cdot CE$ and $C_1P = 2 \cdot C_1E_1$, we have to find the value of x which minimizes the sum $CP^2 + C_1P^2$. Constructing segment PT, equal to C_1P and perpendicular to CC_1 (as shown in Fig. 2), we obtain $CP^2 + C_1P^2 = CT^2$. So the problem is to minimize CT.

On the other hand, we have $\tan \sphericalangle CE_1T = \dfrac{PT}{PE_1} = 2$. This means that point T lies on a line t through the midpoint E_1 of AB, which does not depend on $\triangle A_1B_1C_1$. Consequently CT is a minimum when it is perpendicular to t. Then $\tan \sphericalangle PTC = 2$, e.g. $CP = 2 \cdot PT$.

The latter equality implies that the side of $\triangle DFC$ is twice the side of $\triangle D_1F_1C_1$. Since the side of $\triangle ABC$ equals 1 it follows that the sides of triangles $D_1F_1C_1$, DFC and $A_1B_1C_1$ equal $\frac{1}{5}, \frac{2}{5}$ and $x = \frac{4}{5}$, respectively. ◇

Now a remarkable result follows: the common area of non-overlapping parts is minimized when $A_1B_1 = 2 \cdot DF$. In other words, the desired minimum is attained when half the perimeter of $\triangle A_1B_1C_1$ is inside the given triangle.

The next sangaku (see [1]) is a generalization of Problem 1.

Problem 2. Solve the preceding problem if the center of $\triangle A_1B_1C_1$ is not necessarily the same as the center of the given triangle ABC.

Solution of Problem 2. Let a, b, c be the sides of non-overlapping equilateral triangles with respective vertices A_1, B_1, C_1 as shown in Fig. 3. Then the sides of the equilateral triangles with respective vertices A, B, C are $x-b-c$, $x-c-a$, $x-a-b$. Since the side of $\triangle ABC$ is of length 1, we conclude that $a + b + c = 2x - 1$.

Denoting by S the common area of non-overlapping parts, we have

$$\begin{aligned} \frac{4}{\sqrt{3}}S &= a^2 + b^2 + c^2 + (x-a-b)^2 + (x-b-c)^2 + (x-c-a)^2 \\ &\geq \frac{(a+b+c)^2}{3} + \frac{(x-a-b+x-b-c+x-c-a)^2}{3} \\ &= 3\left(\frac{2x-1}{3}\right)^2 + 3\left(\frac{2-x}{3}\right)^2. \end{aligned}$$

The above inequality means that when x is fixed, the minimum value of S is attained when $a = b = c = \dfrac{2x-1}{3}$, e.g. when triangles ABC and $A_1B_1C_1$ have a common

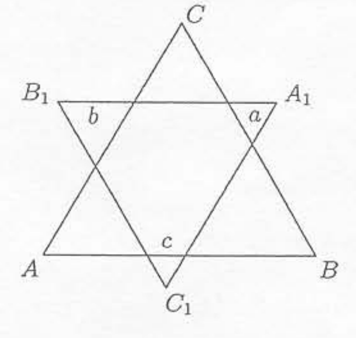


Figure 3

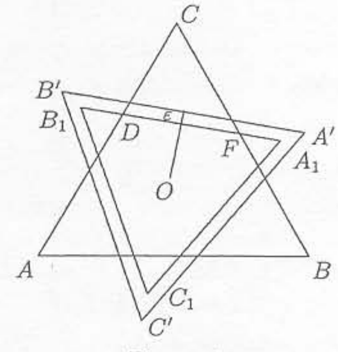


Figure 4

We can interpret these old Japanese problems in terms of contemporary mathematics. The common area of parts belonging to exactly one of the polygons A and B (e.g. parts of $(A \cup B) \setminus (A \cap B)$) is defined in [2] as the symmetric distance $\Delta(A, B)$ between the polygons. Considering this distance as a special case is the idea of the above sangaku. Given polygon A and the set of polygons M, the polygon $B \in M$ of minimum distance to A will be called the *closest* to A in M. The set of equilateral triangles with sides parallel to the respective sides of $\triangle ABC$ defined in problem 1 was widened in problem 2 without changing the answer. Let us consider other sets of equilateral triangles in which we seek the invariant property of being the closest to $\triangle ABC$.

[1] Some of these results hold in general without the equilateral assumption. Consider triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ with parallel corresponding sides and coefficient of similarity x. The non-overlapping triangles inside $\triangle A_1B_1C_1$ are similar to $\triangle ABC$ and let a, b, c be the respective coefficients of similarity. One can repeat the exact solution of problem 2 to prove that $a = b = c$.

Problem 3. Let ABC be a unit equilateral triangle with center O. Consider the set M_φ of equilateral triangles $A_1B_1C_1$ with the same center having sides which form a fixed angle $\varphi \in (0°, 60°)$ with the respective sides of $\triangle ABC$. Prove that half the perimeter of the closest to $\triangle ABC$ in M_φ is inside $\triangle ABC$.

Solution of Problem 3. Assuming the notation in Fig. 4 it suffices to prove that $A_1B_1 = 2 \cdot EF$. Suppose $A_1B_1 < 2 \cdot DF$ for $\triangle A_1B_1C_1 \in M_\varphi$. We will find that $\triangle A'B'C' \in M_\varphi$ is "closer" to $\triangle ABC$. Let the sides of $\triangle A'B'C'$ be ε-more distanced to the center O than are the sides of $\triangle A_1B_1C_1$ (Fig. 4). Then

$$\Delta(ABC, A'B'C') = \Delta(ABC, A_1B_1C_1) - 3\varepsilon(2 \cdot DF - A_1B_1) + 3\sqrt{3}\varepsilon^2 k_\varphi,$$

where $k_\varphi = \dfrac{3 + 2\cos 2\varphi}{1 + 2\cos 2\varphi}$ is a positive number for the given values of φ. Now choosing $\varepsilon \in \left(0, \frac{\sqrt{3}(2 \cdot DF - A_1B_1)}{3k_\varphi}\right)$ we obtain that $\triangle A'B'C'$ is "closer" to $\triangle ABC$ than $\triangle A_1B_1C_1$.

Similar considerations hold in the case $A_1B_1 > 2 \cdot DF$. Consequently, for the $\triangle A_1B_1C_1$ in M_φ closest to $\triangle ABC$, the equality $A_1B_1 = 2 \cdot DF$ holds.[2]$\diamond$

One can prove in the same way as in Problem 2 that translating triangles of M_φ does not change the situation. Further generalizations are available in the case of arbitrary triangles. We will note that similar results may be obtained for regular n-gons.

Problem 4. Let $A = A_1A_2 \ldots A_n$ be a unit regular n-gon with its center at O. Consider the set M_φ of regular n-gons $B = B_1B_2 \ldots B_n$ with side x and center O, whose sides form the fixed angle φ with respective sides of A. Prove that half the perimeter of the closest regular n-gon to A in the set of n-gons M_φ is inside A.

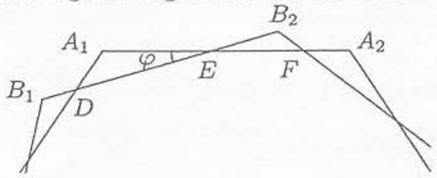


Figure 5

Solution of Problem 4. Denote the angle of the regular n-gon by $2\varphi_n$ and assume the notation in figure 5. If $DE = y$ by the Law of Sines for $\triangle A_1DE$ we get $A_1D + A_1E = yt_n$, where $t_n = \dfrac{\sin(\varphi + \varphi_n)}{\sin \varphi_n}$. By symmetry it is obvious that $A_1D = A_2F$ and therefore $EF = 1 - yt_n$. By the Law of Sines for $\triangle B_2EF$ we obtain $B_2E + B_2F = EF \cdot t_n = (1 - yt_n)t_n$. Since $B_2F = B_1D$ it implies that $y + (1 - yt_n)t_n = x$. The symmetric distance $\Delta(A, B)$ is a minimum if and only if the sum of areas of similar triangles A_1DE and B_2EF is a minimum. The angles of this triangles do not depend of x, so we have to find $\min\{DE^2 + EF^2\} = \min\{y^2 + (1 - yt_n)^2\}$. The desired minimum is attained when $y = \dfrac{t_n}{1 + t_n^2}$ and $x = \dfrac{2t_n}{1 + t_n^2} = 2y$.$\diamond$

Ancient Japanese mathematics can be characterized by its shrewd but brief descriptions of the properties of geometric figures. Understanding these terse descriptions and revealing their meanings presents a challenge. However, such research is exciting every step of the way toward understanding.

[2]The existence of such a triangle follows by continuity.

References

[1] Traditional Japanese Mathematics Problems of the 18th & 19th Centuries, H. Fukagawa, J. Rigby, SCT Publishing, Singapore, 2002

[2] D. Shklqrskii, N. Chencov, I. Yaglom, Combinatorial Geometry Problems, Moskow, 1974 (in Russian)