Teaching of Scientific Concepts' Integration – an Example

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Abstract. *One of the aims of the mathematical education is to enable students to understand and use connections between mathematical topics and to apply mathematics in context outside it. Purposely we present an educational module based on example of a discrete Dirichlet problem connected with random walk on a discrete lattice. Problems discussed in this article integrate ideas from Physics, Numerical Methods, Probability Theory and Mathematical Modeling within the capacity of high school students.*

Key Words. *Mathematics education, Mathematical modeling, applications of mathematics, Computer science applications.*

The dynamics of the contemporary society provokes the change of the way of the education. This change is connected not just with the question *how much* to teach but rather with the question *how* to teach. Daniel Pink comments this in his book *A Whole New Mind: Moving from the Information Age to the Conceptual Age* ([4]):

> *In a new quickly changing world one needs in a compelling way ability to apprehend the large-scale picture rather than analyzing from single point of view.*

The contemporary teaching is oriented towards the conception of combining the elements horizontally instead the narrow specialization. Many colleges and universities offer various courses from different areas of science that comprehend in a flexible and intuitive way. This kind of education is aimed to develop different form of thinking, qualities like inventiveness and meaning-predominate and to create meaning makers and big picture thinkers in result.

Of course, these are not new ideas. At the beginning of 20th century Felix Klein debated this "more vital direction of advance" of mathematics and teaching it which "stresses on the organic connection between distinct areas ... and correspondingly prefers those methods giving simultaneous understanding of many topics from one and the same point of view" ([8]).

The proposed essay is an attempt to connect the physical model of thermal equilibrium and random walk problem from the point of view of grid modeling.

1 A Lattice Model of the Temperature Distribution in Thermal Equilibrium

Consider a square plate with fixed temperature distribution on its boundary. Let the temperature inside the plate be in equilibrium. To find the equilibrium temperature distribution at different points on the plate is an interesting and important problem. Theoretically, it is given by the solution to the corresponding Dirichlet problem[1] involving partial differential equations. In practice we can grid the plate and choosing a fine lattice, we approximate the temperature at *X* with this at the closest to *X* lattice point. So we will be interested in the temperatures at lattice points in the plate. Following this way, we face a new problem - *discrete Dirichlet problem*.

Both the intuition and experiment prompt that the equilibrium temperature at the midpoint of a segment is the average of the temperatures fixed at its ends. More generally, in one-dimensional case we have linear temperature distribution. That way, taking into consideration symmetry and for other physical reasons ([5]), we establish that if we have two conductors of equal length with common mid-point *O*, then the equilibrium temperature at *O* is the average of temperatures at the four conductors' ends. This experimentally verified conjecture (Mean-Value Property) serves as basis of modeling the temperature equilibrium.

Let *D* be the set of all lattice points and *I*(*D*) be the set of the interior lattice points of *D* (i.e. the points with four neighbors in *D*). The points of $\Gamma(D) = D \setminus I(D)$ are called boundary points. Denote by $[X]$ the set of the neighbor lattice points of any $X \in D$. The statement of the discrete Dirichlet problem is:
Given a temperature distribution $f(X),\ X \in \Gamma(D)$ on the boundary, find a function $T(X)$, satisfying the conditions:

$$\left| \begin{array}{ll} T(X) = \dfrac{1}{4} \sum\limits_{Y \in [X]} T(Y), & X \in I(D) \\ T(X) = f(X), & X \in \Gamma(D) \end{array} \right. \tag{1}$$

The above means that if the plate is in thermal equilibrium and *X* is an interior lattice point, then the temperature *T*(*X*) is the average of the temperatures at the four closest to *X* lattice points.

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[1] The Dirichlet problem for Laplace's equation can be formulated in the following way: find a function *T*(*x*) in some domain $D \subset \mathbb{R}^2$ such that

$$\left| \begin{array}{l} \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0 \\ T(X) = f(X) \quad \text{for} \quad X \in \Gamma(D) \end{array} \right. ,$$

where $\Gamma(D)$ is the boundary of *D* ([2]).

From physical point of view, the existence of a solution of (1) follows from the existence of thermal equilibrium. Further, for the solutions of (1) the Maximum Value principle holds; it claims that the temperature distribution reaches its maximal and minimal values on the boundary of *D*. Thus, if $f(X) \equiv 0,\ X \in \Gamma(D)$ it follows that $T(X) \equiv 0,\ X \in I(D)$ is the only solution of (1). Hence the homogeneous system corresponding to (1) has an unique solution and therefore (1) also has an unique solution.

Note that it is convenient to choose a system of coordinates in such a way that the coordinates of the lattice points are integers.

Problem 1. Let us start with a square grid of nine interior points, and let $t(i; j)$ be the temperatures at points $(i; j)$; $i; j = 1,..., 5$ (Fig. 1). Using the boundary conditions in the figure, determine the equilibrium values at the lattice points of the plate.

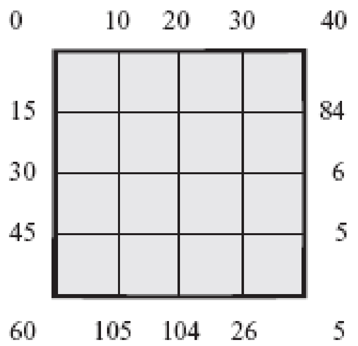

Figure 1

By the Mean-Value Property we obtain the following system of linear equations:

$$
\left|\begin{array}{l}
t(2,2) = \frac{1}{4}\left(45 + t(2,3) + t(3,2) + 105\right) \\
t(2,3) = \frac{1}{4}\left(30 + t(2,4) + t(3,3) + t(2,2)\right) \\
\vdots \\
t(4,4) = \frac{1}{4}\left(t(3,4) + 30 + 84 + t(4,3)\right)
\end{array}\right.
$$

(this is in fact the first row of (1)). Solving the above system is a standard technique for school students but requires a lot of computations. In practice many problems lead to similar systems with more equations because the temperature approximation is more accurate if we consider a grid with more interior points.

This requires using numerical methods; for example, the method of iteration. Assuming an arbitrary temperature (say 0) at each interior point, $T^{(0)}(X) \equiv 0,\ X \in I(D)$, we construct a sequence $T^{(k)}(X),\ X \in I(D),\ k = 1, 2, \ldots,$

called successive approximations, as average of the previous temperatures at the neighbors of each point:

$$\left| \begin{array}{ll} T^{(k)}(X) = \dfrac{1}{4} \sum\limits_{Y \in [X]} T^{(k-1)}(Y), & X \in I(D) \\ T^{(k)}(X) = f(X), & X \in \Gamma(D). \end{array} \right.$$

The iterations could be interpreted as stages of reaching the heat equilibrium. We stop computing when the difference between two successive values of temperature $\left| T^{(k)}(X) - T^{(k-1)}(X) \right|$ at each point $X \in I(D)$ becomes small enough. This process is convergent ([3]). School students could easily model it using Excel and so find the solution (Fig. 2).

0	**10**	**20**	**30**	**40**
15	24,00	32,00	44,00	**84**
30	39,00	40,00	30,00	**6**
45	62,00	59,00	30,00	**15**
60	**105**	**104**	**16**	**15**

Figure 2

Note that the lattice model has an important advantage - it does not require using of partial derivatives. Moreover, its application in more complicated domains is intuitively clear ([6]). The combination of the lattice model with modern computer tools makes it suitable for school teaching.

Now we shall show that the solution of the discrete Dirichlet problem seems similar to that of a random walk problem although they come from different topics.

2 Equal Probability Random Walk on a Discrete Lattice

Let us consider 4 x 4 lattice (Fig. 3). A point walks randomly on the lattice so that no direction is more probable than another. This means that if the point occupies a point with k neighbor lattice points, then the probability of visiting any of them at the next step of its walk equals $\frac{1}{k}$.

Problem 2. A random walk starts at lattice point (x; y); x; y = 1,..., 5 (Fig. 3). Find the probability $p(x; y)$ to reach the lattice boundary for the first time at A(3; 1)[2].

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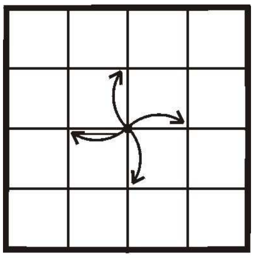
[2] Similar problem is discussed in [1].



$B \quad A$

Figure 3

The symmetry of the lattice implies that

$$p(2,i) = p(4,i), \quad i = 2..4. \tag{2}$$

Obviously, on the boundary we have

$$p(3,1) = 1, \;\; p(2,1) = p(4,1) = p(1,i) = p(5,i) = p(i,5) = 0, \quad i = 1..5. \tag{3}$$

Let the point land at interior point (x, y) with 4 neighbors. With probability $\frac{1}{4}$ it moves to the neighbor (x', y') at the next stage. At the point (x', y') the probability of touching the boundary firstly at A is $p(x', y')$. So the conditional probability equals $\frac{1}{4} \cdot p(x', y')$. Since the point (x, y) has 4 neighbors it follows that

$$p(x, y) = \sum_{(x',y') \in [(x,y)]} \frac{1}{4} \cdot p(x', y'). \tag{4}$$

From (2), (3) and (4) we obtain a system of six variables:

$$\left| \begin{array}{l} p(2,2) = \frac{1}{4}(0 + p(2,3) + p(3,2) + 0) \\ p(3,2) = \frac{1}{4}(1 + p(2,2) + p(3,3) + p(2,2)) \\ \vdots \\ p(3,4) = \frac{1}{4}(p(3,3) + p(2,4) + 0 + p(2,4)) \end{array} \right. .$$

Applying a linear system solving program (as Maple) gives:

$p(2,2)$	$p(3,2)$	$p(2,3)$	$p(3,3)$	$p(2,4)$	$p(3,4)$
$\frac{11}{112}$	$\frac{37}{112}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{112}$	$\frac{5}{112}$

The systems of equations giving the solutions for heat equilibrium and random walk problems seem similar. It is interesting that problems from completely different areas lead to almost equivalent lattice models. There is one more connection between the heat equilibrium problem and random walk which will be revealed in the next section.

3 Probability and the Dirichlet Problem

We will illustrate a solution of the discrete version of the Dirichlet problem (1) in terms of the random walk ([7]).

For any point $X \in I(D)$ consider random walks starting at *X*. Let X_Γ be the first point on the boundary that the walk gets to. We claim that the solution of (1) is

$$T(X) = \sum_{X_\Gamma \in \Gamma(D)} P_{X_\Gamma}(X) \cdot f(X_\Gamma) \qquad (5)$$

where $P_{X_\Gamma}(X)$ is the probability to reach the boundary at first time in X_Γ.

Before proving the above formula, note that the multiplier $P_{X_\Gamma}(X)$ represents the influence of the temperature at X_Γ on establishing the equilibrium temperature at *X*. And as the answer of problem 2 shows, the smaller the distance XX_Γ, the greater the probability $P_{X_\Gamma}(X)$.

To prove (5) we have two steps to complete. First that $T(X) = \frac{1}{4} \sum_{Y \in [X]} T(Y)$ for $X \in I(D)$. As in (4) we can write

$$P_{X_\Gamma}(X) = \sum_{Y \in [X]} \frac{1}{4} \cdot P_{X_\Gamma}(Y)$$

Therefore

$$T(X) = \sum_{X_\Gamma \in \Gamma(D)} P_{X_\Gamma}(X) \cdot f(X_\Gamma) = \sum_{X_\Gamma \in \Gamma(D)} \sum_{Y \in [X]} \frac{1}{4} \cdot P_{X_\Gamma}(Y) \cdot f(X_\Gamma)$$
$$= \frac{1}{4} \sum_{Y \in [X]} \sum_{X_\Gamma \in \Gamma(D)} P_{X_\Gamma}(Y) \cdot f(X_\Gamma) = \frac{1}{4} \sum_{Y \in [X]} T(Y)$$

The second step that $T(X) = f(X)$ on $\Gamma(D)$ is obvious.

Following this method we can use the probabilities computed in problem 2 to find the solution of problem 1. Because of symmetry, estimating all the probabilities $P_{X_\Gamma}(X)$, $X \in I(D)$, $X_\Gamma \in \Gamma(D)$ needs only to find $P_B(X)$, $X \in I(D)$, where the point *B* has coordinates (2; 1) (Fig. 3). As in problem 2, we find $P_B(X) = q(X)$:

$q(2,2)$	$q(3,2) = q(2,3)$	$q(4,2) = q(2,4)$	$q(3,3)$	$q(4,3) = q(3,4)$	$q(4,4)$
$\frac{67}{224}$	$\frac{11}{112}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{3}{112}$	$\frac{3}{224}$

Example. Compute the equilibrium temperature $t(2;\ 2)$.

Using (5), we obtain

$$\begin{aligned} t(2,2) = {} & p(2,2)\big(t(3,1)+t(1,3)\big)+p(2,4)\big(t(3,5)+t(5,3)\big) \\ & + q(2,2)\big(t(2,1)+t(1,2)\big)+q(4,2)\big(t(4,1)+t(1,4)\big) \\ & + q(2,4)\big(t(2,5)+t(5,2)\big)+q(4,4)\big(t(4,5)+t(5,4)\big)=62 \end{aligned}$$

As expected, the result is the same as in the iterative solution.

Conclusion

Our attempt is to find such models of teaching process, that develops aptitude to combine seemingly unrelated ideas into something new and ability to stretch beyond the quotidian in pursuit of meaning. As the experiment in National High School for Natural Sciences and Mathematics shows, mathematical models considered in a simple way enable students to touch and connect concepts from Applied Mathematics, Physics, Probability Theory and Numerical Methods.

The computational part of the problems could be carried out by one of the available computer packages (Excel and Maple) but also could be put as a separate programming task for the students. That's why in the described topic there are ample opportunities for research in the classroom.

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Един опит за интегриране на физични и математически идеи в обучението

Резюме. *Една от целите на математическото образование е да развива умение за съставяне на математически модел на реална ситуация. Това умение се базира на разбиране на математическите идеи и връзките им с идеи от други научни области. В тази насока представяме модул, който включва дискретен модел на задача на Дирихле за разпределение на температурата при топлинно равновесие и задача за блуждаене в крайна мрежа. На тази основа задачата на Дирихле се интерпретира от гледна точка на вероятностните процеси. Решенията интегрират знания по физика, числени методи, теория на вероятностите и математическо моделиране по достъпен за ученици начин.*