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Dedicated to the 80th Anniversary of Professor Ivan Dimovski, Corr.-Member of Bulgarian Academy of Sceinces

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Nonclassical convolutions and their uses

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Keywords: convolution, operational calculus, boundary value problem, Duhamel principle

The following generalization of the classical Duhamel convolution

$$(f \overset{t}{*} g)(t) = \chi_\tau \left\{ \int_\tau^t f(t+\tau-\sigma) g(\sigma) d\sigma \right\} \tag{1}$$

with arbitrary linear functional χ, found independently by the author (1974) and by L. Berg (1976), has similar algebraic properties as these of (1). It can be used to build a generalization of Mikusinski's operational calculus, intended for nonlocal Cauchy problems.

As a next step, we proposed (1976) the operation

$$\begin{aligned}(f \overset{x}{*} g)(x) \quad = \quad & \Phi_\xi \left\{ \int_x^\zeta f(\zeta + x - \eta) g(\eta) d\eta \right. \\ & \left. - \int_{-x}^\zeta f(|\zeta - x - \eta|) g(|\eta|) \operatorname{sgn}(\eta(\zeta - x - \eta)) d\eta \right\},\end{aligned} \tag{2}$$

which happened to be useful for solving of nonlocal BVPs connected with the square of differentiation [1].

A detailed study of operations (1) and (2) and revealing their differential and functional properties is made in N. Bozhinov's book [2]. Recently, it became clear that both convolutions and their closest extensions are useful for practical applications too. Their multidimensional extension allow to develop multivariate operational calculi and to apply them to local and nonlocal BVPs of mathematical physics.

The generalizations of the classical Duhamel principle give explicit solutions of BVPs which solutions had been known till now only in series form.

References

[1] I.H. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht, 1990.

[2] N.S. Bozhinov, *Convolutional Representations of Commutants and Multipliers*, Publ. House of BAS, Sofia, 1988.

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On associativity of the convolution of ultradistributions

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Keywords: convolution of ultradistributions, associativity of convolution of ultradistributions

New results on the existence and associativity of the convolution in various spaces of ultradistributions are proved.

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References

[1] R.D. Carmichael, A. Kamiński, S. Pilipović, *Boundary Values and Convolution in Ultradistribution Spaces*, World Scientific, New Jersey, 2007.

[2] A. Kamiński, D. Perisić, S. Pilipović, Existence theorems for convolution of ultradistributions, *Dissertationes Math.* **340** (1995), 79–91.

[3] A. Kamiński, D. Perišić, S. Pilipović, On the convolution of tempered ultradistributions of Beurling type, *Integral Transforms Spec. Func.* **15** (2004), 323–329.

[4] S. Pilipović, B. Prangoski, Equivalence of several definitions of convolution of Roumieu ultradistributions, *Monatsh. Math.* **173** (2014), 83–105.

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Nonstandard quadratures of Gauss-Lobatto type and applications in the fractional calculus

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Keywords: Gauss-Lobatto quadratures, numerical methods, fractional derivatives, CAS Mathematica; 65D30, 33C45, 41A55, 65D32

In a joint paper with S. Esmaeili [2], a family of nonstandard Gauss-Lobatto quadratures for numerical calculating integrals of the form $\int_{-1}^{1} f'(x)(1-x)^\alpha \,\mathrm{d}x$, $\alpha > -1$, has been derived and applied to approximation of fractional derivatives of Riemann-Liouville and Caputo type.

In this lecture we start with a general weight function $w : (-1,1) \to \mathbb{R}$ for which all its moments $\mu_\nu = \int_{-1}^{1} x^\nu w(x)\,dx$, $\nu = 0,1,\ldots$, exist and are finite, and we consider nonstandard (algebraic) quadrature formulas of Gaussian or Gauss-Lobatto type of the form

$$I(f) = \int_{-1}^{1} (Lf)(x)w(x)\,\mathrm{d}x = A_0 f(-1) + \sum_{k=1}^{n} A_k f(x_k) + A_{n+1} f(1) + R_n(f),$$

where L is a linear operator acting between certain functional spaces, and R_n is the remainder term, which is equal zero for all algebraic polynomials of degree at most $2n+1$. A special attention is devoted to an important case when $(Lf)(x) = \alpha f(x) + (1+x)f'(x)$, $\alpha > 0$. Under some conditions on the moment sequence $\{\mu_\nu\}_{\nu\ge 0}$ we prove that such kind of quadratures exist for each $n \in \mathbb{N}$. The nodes x_k are real, mutually different and lie in $(-1,1)$. The weights A_k can be expressed in terms of the corresponding Christoffel numbers of an equivalent Gauss-Christoffel quadrature formula (cf. [3]). We also analyze some special weight functions, including weights of Jacobi type, and give some applications of such quadrature rules in the fractional calculus.

A software implementation of these quadratures was done by the recent MATHEMATICA package `OrthogonalPolynomials` (cf. [1] and [4]), which is downloadable from the Web Site: `http://www.mi.sanu.ac.rs/~gvm/`. Several numerical examples are presented and they show the effectiveness of the proposed approach.

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References

[1] A.S. Cvetković, G.V. Milovanović, The Mathematica Package "OrthogonalPolynomials", *Facta Univ. Ser. Math. Inform.* **9** (2014), 17–36.

[2] S. Esmaeili, G.V. Milovanović, Nonstandard Gauss–Lobatto quadrature approximation to fractional derivatives, *Manuscript* prepared for publication (2014).

[3] G. Mastroianni, G.V. Milovanović, *Interpolation Processes: Basic Theory and Applications*, Springer–Verlag, Berlin (2008).

[4] G.V. Milovanović, A.S. Cvetković, Special classes of orthogonal polynomials and corresponding quadratures of Gaussian type, *Math. Balkanica* **26**, No 1-2 (2012), 169–184.

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Stronger Rolle's Theorem for Complex Polynomials

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A domain Θ_n is called *Rolle's domain* if for every complex polynomial $p(z)$ of degree $n \geq 2$ and $p(-i) = p(i)$ there exists at least one $\zeta \in \Theta_n$, such that $p'(\zeta) = 0$.

A Theorem X is called Rolle's theorem for complex polynomials if it states that a given domain Θ_n^X is a Rolle's domain.

A Rolle's Theorem X is *stronger* than the Rolle's Theorem Y, if $\Theta_n^X \subset \Theta_n^Y$ and $\Theta_n^X \neq \Theta_n^Y$.

A Rolle's Theorem X is *sharp*, if from $\Theta_n^Y \subset \Theta_n^X$ follows that $\Theta_n^Y = \Theta_n^X$.

There are several Rolle's theorems for complex polynomials. The most famous one is, see [1, p. 126]:

Theorem 1 (Grace-Heawood). *The disk*

$$\Theta_n^{GH} = D\left[0; \cot\frac{\pi}{n}\right] = \left\{z:\ |z| \leq \cot\frac{\pi}{n}\right\} \tag{1}$$

is a Rolle's domain.

Another complex Rolle's theorem, see [1, Theorem 4.3.4, p. 128], is the following:

Theorem 2. *The double disk* $\Theta_n^F = DD[c;r] = D[-c;r] \bigcup D[c;r]$*, where*

$$c = \cot\frac{\pi}{n-1}, \quad r = \sin^{-1}\frac{\pi}{n-1}; \quad n \geq 3,$$

is a Rolle's domain.

Neither one of the above two theorems is stronger than the other.

The main goal of this lecture is to prove the following:

Theorem 3. *The double disk* $\Theta_n^{SS} = DD[c; r]$*, where*

$$c = \cot \frac{2\pi}{n}, \quad r = \sin^{-1} \frac{2\pi}{n}; \quad n \geq 3,$$

is a Rolle's domain.

It is easy to see that Theorem 3 is stronger than Theorem 1 and Theorem 2. In Figure 1, the Rolle's domains of Theorems 1, 2 and 3, for $n = 20$, are presented.

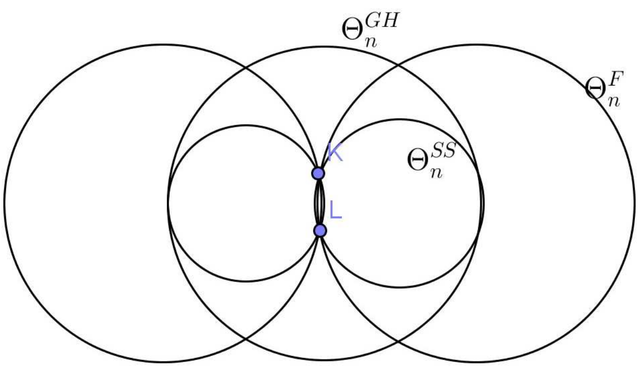

Fig. 1, K=i, L=-i.

The prove of Theorem 3 is based on the notion *locus holder*, on an analogue of the Grace-Walsh-Szegő coincidence theorem, called Argument coincidence theorem and on the Sector theorem, see [2].

To make the lecture selfcontent, we present the needed facts for the notion locus holder, see [3] and the formulation of the Argument coincidence theorem. We also emphasize on the Sector theorem, which is an analogue of the Gauss-Lucas theorem for the algebraic polynomials with real end non negative coefficients.

References

[1] Q.I. Rahman, G. Schmeisser, *Analytic Theory of Polynomials*, Oxford Univ. Press Inc., New York, 2002.

[2] Bl. Sendov, Analogue of Gauss-Lucas theorem for non convex set on the complex plane, *C. R. Acad. Sci. Bulgare*, to appear.

[3] Bl. Sendov, H. Sendov, Loci of complex polynomials, Part I, *Trans. Amer. Math. Soc.*, to appear.

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Buschman-Erdélyi transmutations: classification, analytical properties and applications to differential equations and integral transforms

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This work is dedicated to Professor Ivan Dimovski,
on the remarkable occasion of his 80^{th} *birthday*

Keywords: transmutations, Sonine-type, Poisson-type, Buschman–Erdélyi transmutations

The methods of transmutation theory form now an important part of modern mathematics, cf. [1]– [7]. They have many applications to theoretical and applied problems.

Let us just itemize some problems in the modern transmutation theory:

1. Theory of Buschman–Erdélyi transmutations [8]– [9]. This class of operators have many applications to partial differential equations, Radon transform theory and many other problems.

2. Theory of operator convolutions and commuting operators [2]. The transmutation operators are closely connected with the commutants. And if the commutants in different spaces of analytic functions are completely described by the convolutional calculus theory of I. Dimovski, the commutants in standard spaces like C^k are much more difficult to characterize, it has been done only recently.

3. Sonine–Dimovski and Poisson–Dimovski transmutations for the hyper–Bessel operators, equations nd functions [2], [3], [5], [7].

4. Sonine and Poisson type transmutations for difference–differential operators of Dunkl type.

5. Applications of transmutations to generalized analytic function theory, cf. [6].

6. Methods of fractional integro-differentiation and integral transforms with special function kernels [5]. In this field let us mention a composition method to derive many classes of transmutations in the unified way [10].

7. Unitary Sonine–Katrakhov and Poisson–Katrakhov transmutations [8]–[10].

8. Applications to partial differential equations with singularities [4], [7]– [10].

References

[1] R.W. Carroll, *Transmutation Theory and Applications*, North Holland, 1986.

[2] I. Dimovski, *Convolutional Calculus*, Kluwer Acad. Publ., Dordrecht, 1990.

[3] R. Gilbert, H. Begehr, *Transmutations and Kernel Functions*, Vol. 1-2, Longman, Pitman, 1992.

[4] V.V. Katrakhov, S.M. Sitnik, Composition method for constructing B–elliptic, B–hyperbolic, and B–parabolic transformation operators, *Russ. Acad. Sci. Dokl. Math.* **50**, No 1 (1995), 70–77.

[5] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman Sci. & J. Wiley, Harlow - N. York, 1994.

[6] V.V. Kravchenko, *Pseudoanalytic Function Theory*, Birkhäuser Verlag, 2009.

[7] S.M. Sitnik, Transmutations and applications: A survey, *arXiv*: 1012.3741 (2012), 141 p.

[8] S.M. Sitnik, Buschman–Erdélyi transmutations, classification and applications, In: *Analytic Methods of Analysis And Differential Equations: AMADE 2012* (Proc. Conf., Ed. by M.V. Dubatovskaya, S.V. Rogosin), Cambridge Scientific Publishers (2013), 31 p.

[9] S.M. Sitnik, Buschman–Erdélyi transmutations, classification and applications, *arXiv*: 1304.2114 (2013), 67 p.

[10] S.M. Sitnik, Factorization and estimates of the norms of Buschman–Erdélyi operators in weighted Lebesgue spaces, *Soviet Mathematics Doklades* **44**, No 2 (1992), 641–646.

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Application of the operational calculus approach of Dimovski for solving the backward heat problem

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Keywords: operational calculus, non-classical convolution, Duhamel principle, ill-posed problem, quasireversibility

The final value problem for the heat equation is known to be ill-posed. To deal with this, in the method of quasireversibility, the equation or the final value condition is perturbed to form an approximate well-posed problem, depending on a small parameter ε. In this work, several known quasireversibility techniques for the backward heat problem are considered and the obtained new problems are treated using the operational calculus approach developed by Dimovski [1]. For every approximate problem, applying an appropriate bivariate operational calculus, a Duhamel-type representation of the solution is obtained. It is in the form of a convolution product of a special solution of the problem and the given final value function. The idea for such Duhamel-type representations for ill-posed problems is originally proposed in [2]. Here it is further developed for different regularizations of the backward heat problem and its application for calculating the numerical solution is illustrated on some test problems.

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References

[1] I.H. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht (1990).

[2] I.H. Dimovski, S.L. Kalla and I. Ali, Operational calculus approach to PDE arising in QR-regularisation of ill-posed problems, *Math. Comput. Model.* **35** (2002), 835–848.

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Simulation of viscoelastic flows with fractional derivative models: an approach via the operational calculus of Dimovski

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Keywords: operational calculus, non-classical convolution, Riemann-Liouville fractional derivative, generalized Oldroyd-B fluid, finite difference scheme

The boundary value problem for the velocity distribution of a viscoelastic flow with generalized fractional Oldroyd-B constitutive model is studied. The model contains two Riemann-Liouville fractional derivatives in time of different orders. Based on the eigenfunction expansion, the unique existence of the solution is established and some regularity results and qualitative properties are obtained. Further, applying the operational calculus approach proposed by Dimovski [1], a Duhamel-type representation of the solution with respect to the space variables is found. This is a compact representation, containing a non-classical convolution product of a special solution and the given initial function. It is appropriate for numerical computation of the solution. To illustrate this, a finite difference scheme is also constructed and the solutions of some test problems are calculated numerically in different ways: using the finite difference approximation, using the Duhamel-type representation, or combining both of them. Numerical results for one- and two-dimensional examples are presented and the different techniques are compared in terms of efficiency, accuracy, and CPU time.

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References

[1] I.H. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht (1990).

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Existence of solutions to boundary value problem for impulsive fractional equations

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Keywords: fractional differential equations, impulsive conditions, weak solution, classical solution, three critical point theorem

In this paper we study the existence and the multiplicity of solutions for an impulsive boundary value problem for fractional differential equations. The notions of classical and weak solutions are introduced. Then, the existence of at least one and three solutions are proved. An example is given.

For related studies, see references below.

References

[1] F. Jiao, Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory. *Comput. Math. Appl.* **62** (2011), 1181–1199.

[2] J. Nieto, D. O'Regan, Variational approach to impulsive differential equations. *Nonlinear Anal. RWA* **10**, (2009), 680–690.

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Commutant of Sturm-Liouville operator in an invariant subspace

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Keywords: commutant, invariant subspace, Sturm-Liouville operator, convolution

We characterize the continuous linear operators $M : C \to C$, with $C = C[0,\infty)$, commuting with the Sturm-Liouville operator $D = \dfrac{d^2}{dx^2} - q(x)$ in the invariant subspace $C_{h,\Phi} = \{f \in C, f'(0) - hf(0) = 0, \Phi\{f\} = 0\}$, where Φ is an arbitrary nonzero continuous linear functional. Additionally, we assume $M : C^k \to C^k$, $k = 1, 2$.

Using a convolution $f * g$ found by the authors (see [1]) in 1976, we found explicitly the commutant as consisting of all operators of the form

$$Mf(x) = \mu f(x) + m * f,$$

where $\mu = const$ and $m \in C$.

References

[1] N.S. Bozhinov, I.H. Dimovski, Boundary value operational calculi for linear differential operators of second order, *C.R. Acad. Bulg. Sci.*, **29** (1976), 1201-1264.

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Explicit solutions of BVPs for multidimensional heat equation

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Keywords: convolution, operational calculus, nonlocal boundary value problem, Duhamel principle

We consider a general nonlocal BVP (boundary value problem) of the form:

$$\begin{aligned}
&u_t = u_{x_1x_1} + ... + u_{x_nx_n} + F(x_1, ..., x_n, t),\\
&u(x_1, ..., x_n, 0) = 0,\\
&u(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n, t) = 0,\\
&\Phi_{j,\xi}\left\{u(x_1, ..., x_{j-1}, \xi, x_{j+1}, ..., x_n, t)\right\} = 0, \quad j = 1, 2, ..., n,
\end{aligned}$$

with given linear functionals Φ_1, ...,Φ_n.

It is shown that using a multidimensional operational calculus, the problem could be reduced to n one-dimensional BVPs of the form

$$\begin{aligned}
&v_t = v_{x_kx_k}, \quad v(x_k, 0) = x_k,\\
&v(0, t) = 0, \quad \Phi_{j,\xi}\{v(\xi, t)\} = 0, \quad k = 1, ..., n,
\end{aligned}$$

with corresponding solutions $\Omega_k(x_k, t)$.

Then a general solution is obtained as an extension of the Duhamel principle for the space variables in the form

$$u(x_1, ..., x_n, t) = \frac{\partial^{2n}}{\partial x_1^2 ... \partial x_n^2}\left[(\Omega_1 ... \Omega_n) * F\right],$$

where $*$ is a multidimensional convolution.

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On the convolutors in the D_{Lp}-type spaces associated with a singular second order differential operator

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Keywords: D_{Lp}-type spaces, convolution product, differential operator

We consider the D_{Lp}-type spaces associated with a singular second order differential operator $\triangle_A$. Some results are established. Next, using the convolution associated with $\triangle_A$, we study the convolutors and the surjective convolution operators acting on spaces of distributions of L_p^A-growth.

For more details, see References.

References

[1] S. Abdullah, On convolution operator and multipliers of distributions of Lp-growth. *J. Math. Anal. Appl.* 183 (1994), 196-207.

[2] S. Abdullah, Solvability of convolution equations in D'_{Lp}. *Hokkaido Math. J.* 29 (2000), 247-254.

[3] J.J. Betancor and B.J. Gonzalez, Spaces of D_{Lp}-type and Hankel convolution. *Proc. Amer. Math. Soc.* 129, No 1 (2001), 219-228.

[4] W.R. Blom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, De Gruyter Studies in Mathematics, Walter de Gruyter, Berlin, New York, 1995.

[5] H. Brezis, *Analyse fonctionnelle théorie et applications*, Masson, Paris, 3eme tirage, 1992.

[6] M. Dziri, M. Jelassi and L.T. Rachdi, Spaces of DLp type and a convolution product associated with a singular second order differential operator, *J. of Concrete and Applicable Mathematics* 10, No 3-4 (2012), 207-232.

[7] M. Jelassi and H. Mejjaoli, Fractional Sobolev type spaces associated with a singular second order differential operator and applications. *Fract. Calc. Appl. Anal.* 17, No 2 (2014), 401–423; DOI: 10.2478/s13540-014-0177-1.

[8] D.H. Pahak, On the convolution equations in the spaces of distributions of *Lp*-growth, *Proc. Amer. Math. Soc.* 94 (1985) 81-86.

[9] L. Schwartz, *Théories des distribution*s, Hermann, Paris, 1966.

[10] K. Trimèche, Hypoelliptic distribution on Chébli-Trimèche hypergroups, *Global J. of Pure and Applied Mathematics* 1, No 3 (2005), 251-271.

[11] K. Trimèche, Inversion of the Lions transmutation operators using generalized wavelets, *Appl. and Compu. Harm. Anal.* 4 (1997), 97-112.

[12] K. Trimèche, Transformation intégrale de Weyl et théorème de Paley-Wiener associés a un opérateur différentiel singulier sur $(0;+\infty)$, *J. Math. Pure et Appl.* 60 (1981) 51-98.

[13] Z. Xu, H*armonic Analysis on Chébli-Trimèche Hypergroups*, PhD Thesis, Murdock Uni, Australia, 1994.

The present author has started her studies on Dimovski's hyper-Bessel operators and on the Obrechkoff transform since 1975, and finally benefitted of them in developing a theory of the *generalized fractional calculus* (GFC) [3] and in introducing new classes of integral transforms [4,5] and special functions [6]. The GFC deals with generalized integrals and derivatives of *fractional multi-orders* $(\delta_1, \delta_2, ..., \delta_m)$ as analogues of the Riemann-Liouville (R-L) and Erdélyi-Kober integrals (E-K) $I^\delta, I_\beta^{\gamma,\delta}$ and derivatives $D^\delta, D_\beta^{\gamma,\delta}$ of arbitrary order $\delta > 0$ in the classical fractional calculus. Having the structure of commuting compositions of E-K operators $I_{(\beta_k)_1^m,m}^{(\gamma_k)_1^m,(\delta_k)_1^m} = \prod_1^m I_{\beta_k}^{\gamma_k,\delta_k}$, the generalized fractional integrals of our GFC are represented by means of integral operators involving special functions:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = \int_0^1 \Phi(\sigma) f(x\sigma) d\sigma, \text{ where } \Phi(\sigma) = H_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_i + \delta_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \\ (\gamma_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \end{matrix} \right. \right] \quad (2)$$

is the Fox H-function, a generalized hypergeometric function of very general nature. The corresponding R-L and Caputo-type generalized fractional derivatives $D_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ are defined by means of suitable differ-integral expressions, see [3,7].

The other operators of fractional calculus and many generalized integrations and differentiations used in applied analysis are shown to be special cases of (2). But the worthy fact to emphasize is that *the hint to introduce the GFC came from the hyper-Bessel operators* (1) of integer order m that appear also to be generalized "fractional derivatives" of multiorder $(1,1,...,1)$: $B = x^{-\beta} D_{(\beta,...,\beta),m}^{(\gamma_1,...,\gamma_m),(1,...,1)}$.

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References

[1] I.H. Dimovski, Operational calculus for a classs of differential operators, *C. R. Acad. Bulg. Sci.* **19** (1966), 1111-1114.

[2] I. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht (1990).

[3] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman & J. Wiley, Harlow - N. York, 1994.

[4] I. Dimovski, V. Kiryakova, The Obrechkoff integral transform: Properties and relation to a generalized fractional calculus, *Numer. Funct. Anal. and Optimization* **21** (2000), 121-144.

[5] F. Al-Musallam, V. Kiryakova, Vu Kim Tuan, A multi-index Borel-Dzrbashjan transform, *Rocky Mountain J. Math.* **32** (2002), 409-428.

[6] V. Kiryakova, The multi-index Mittag-Leffler functions as important class of special functions of fractional calculus, *Computers Math. with Appl.* 59 (2010), 1885-1895.

[7] V. Kiryakova, Yu. Luchko, Riemann-Liouville and Caputo type multiple Erdelyi-Kober operators, *Central European J. of Phys.* **11** (2013), 1314-1336.