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Nonclassical convolutions and their uses

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Keywords: convolution, operational calculus, boundary value problem, Duhamel principle

The following generalization of the classical Duhamel convolution

$$(f \overset{t}{*} g)(t) = \chi_\tau \left\{ \int_\tau^t f(t+\tau-\sigma) g(\sigma) d\sigma \right\} \tag{1}$$

with arbitrary linear functional χ, found independently by the author (1974) and by L. Berg (1976), has similar algebraic properties as these of (1). It can be used to build a generalization of Mikusinski's operational calculus, intended for nonlocal Cauchy problems.

As a next step, we proposed (1976) the operation

$$\begin{aligned} (f \overset{x}{*} g)(x) \quad = \quad & \Phi_\xi \left\{ \int_x^\zeta f(\zeta + x - \eta) g(\eta) d\eta \right. \\ & \left. - \int_{-x}^\zeta f(|\zeta - x - \eta|) g(|\eta|) \mathrm{sgn}(\eta(\zeta - x - \eta)) d\eta \right\}, \end{aligned} \tag{2}$$

which happened to be useful for solving of nonlocal BVPs connected with the square of differentiation [1].

A detailed study of operations (1) and (2) and revealing their differential and functional properties is made in N. Bozhinov's book [2]. Recently, it became clear that both convolutions and their closest extensions are useful for practical applications too. Their multidimensional extension allow to develop multivariate operational calculi and to apply them to local and nonlocal BVPs of mathematical physics.

The generalizations of the classical Duhamel principle give explicit solutions of BVPs which solutions had been known till now only in series form.

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On associativity of the convolution of ultradistributions

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Keywords: convolution of ultradistributions, associativity of convolution of ultradistributions

New results on the existence and associativity of the convolution in various spaces of ultradistributions are proved.

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Nonstandard quadratures of Gauss-Lobatto type and applications in the fractional calculus

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Keywords: Gauss-Lobatto quadratures, numerical methods, fractional derivatives, CAS Mathematica; 65D30, 33C45, 41A55, 65D32

In a joint paper with S. Esmaeili [2], a family of nonstandard Gauss-Lobatto quadratures for numerical calculating integrals of the form $\int_{-1}^{1} f'(x)(1-x)^{\alpha}\,\mathrm{d}x$, $\alpha > -1$, has been derived and applied to approximation of fractional derivatives of Riemann-Liouville and Caputo type.

In this lecture we start with a general weight function $w : (-1,1) \to \mathbb{R}$ for which all its moments $\mu_\nu = \int_{-1}^{1} x^\nu w(x)\,dx$, $\nu = 0, 1, \ldots$, exist and are finite, and we consider nonstandard (algebraic) quadrature formulas of Gaussian or Gauss-Lobatto type of the form

$$I(f) = \int_{-1}^{1} (Lf)(x)w(x)\,\mathrm{d}x = A_0 f(-1) + \sum_{k=1}^{n} A_k f(x_k) + A_{n+1} f(1) + R_n(f),$$

where L is a linear operator acting between certain functional spaces, and R_n is the remainder term, which is equal zero for all algebraic polynomials of degree at most $2n+1$. A special attention is devoted to an important case when $(Lf)(x) = \alpha f(x) + (1+x)f'(x)$, $\alpha > 0$. Under some conditions on the moment sequence $\{\mu_\nu\}_{\nu\geq 0}$ we prove that such kind of quadratures exist for each $n \in \mathbb{N}$. The nodes x_k are real, mutually different and lie in $(-1,1)$. The weights A_k can be expressed in terms of the corresponding Christoffel numbers of an equivalent Gauss-Christoffel quadrature formula (cf. [3]). We also analyze some special weight functions, including weights of Jacobi type, and give some applications of such quadrature rules in the fractional calculus.

A software implementation of these quadratures was done by the recent MATHEMATICA package `OrthogonalPolynomials` (cf. [1] and [4]), which is downloadable from the Web Site: `http://www.mi.sanu.ac.rs/~gvm/`. Several numerical examples are presented and they show the effectiveness of the proposed approach.

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Stronger Rolle's Theorem for Complex Polynomials

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A domain Θ_n is called *Rolle's domain* if for every complex polynomial $p(z)$ of degree $n \geq 2$ and $p(-i) = p(i)$ there exists at least one $\zeta \in \Theta_n$, such that $p'(\zeta) = 0$.

A Theorem X is called Rolle's theorem for complex polynomials if it states that a given domain Θ_n^X is a Rolle's domain.

A Rolle's Theorem X is *stronger* than the Rolle's Theorem Y, if $\Theta_n^X \subset \Theta_n^Y$ and $\Theta_n^X \neq \Theta_n^Y$.

A Rolle's Theorem X is *sharp*, if from $\Theta_n^Y \subset \Theta_n^X$ follows that $\Theta_n^Y = \Theta_n^X$.

There are several Rolle's theorems for complex polynomials. The most famous one is, see [1, p. 126]:

Theorem 1 (Grace-Heawood). *The disk*

$$\Theta_n^{GH} = D\left[0; \cot\frac{\pi}{n}\right] = \left\{z : \ |z| \leq \cot\frac{\pi}{n}\right\} \tag{1}$$

is a Rolle's domain.

Another complex Rolle's theorem, see [1, Theorem 4.3.4, p. 128], is the following:

Theorem 2. *The double disk* $\Theta_n^F = DD[c; r] = D[-c; r] \bigcup D[c; r]$, *where*

$$c = \cot\frac{\pi}{n-1}, \quad r = \sin^{-1}\frac{\pi}{n-1}; \quad n \geq 3,$$

is a Rolle's domain.

Neither one of the above two theorems is stronger than the other.

The main goal of this lecture is to prove the following:

Theorem 3. *The double disk* $\Theta_n^{SS} = DD[c; r]$, *where*

$$c = \cot\frac{2\pi}{n}, \quad r = \sin^{-1}\frac{2\pi}{n}; \quad n \geq 3,$$

is a Rolle's domain.

It is easy to see that Theorem 3 is stronger than Theorem 1 and Theorem 2. In Figure 1, the Rolle's domains of Theorems 1, 2 and 3, for $n = 20$, are presented.

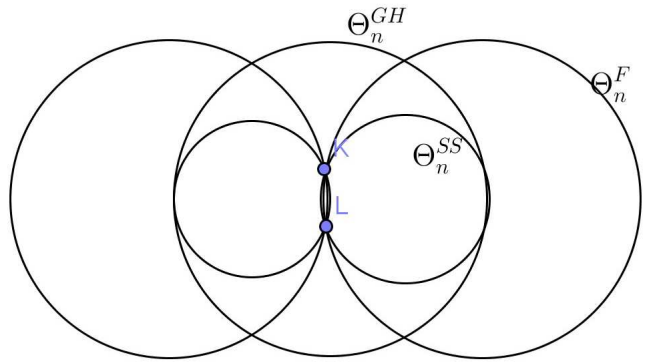

Fig. 1, K=i, L=-i.

The prove of Theorem 3 is based on the notion *locus holder*, on an analogue of the Grace-Walsh-Szegő coincidence theorem, called Argument coincidence theorem and on the Sector theorem, see [2].

To make the lecture selfcontent, we present the needed facts for the notion locus holder, see [3] and the formulation of the Argument coincidence theorem. We also emphasize on the Sector theorem, which is an analogue of the Gauss-Lucas theorem for the algebraic polynomials with real end non negative coefficients.

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Buschman-Erdélyi transmutations: classification, analytical properties and applications to differential equations and integral transforms

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This work is dedicated to Professor Ivan Dimovski,
on the remarkable occasion of his 80th *birthday*

Keywords: transmutations, Sonine-type, Poisson-type, Buschman–Erdélyi transmutations

The methods of transmutation theory form now an important part of modern mathematics, cf. [1]– [7]. They have many applications to theoretical and applied problems.

Let us just itemize some problems in the modern transmutation theory:

1. Theory of Buschman–Erdélyi transmutations [8]– [9]. This class of operators have many applications to partial differential equations, Radon transform theory and many other problems.

2. Theory of operator convolutions and commuting operators [2]. The transmutation operators are closely connected with the commutants. And if the commutants in different spaces of analytic functions are completely described by the convolutional calculus theory of I. Dimovski, the commutants in standard spaces like C^k are much more difficult to characterize, it has been done only recently.

3. Sonine–Dimovski and Poisson–Dimovski transmutations for the hyper–Bessel operators, equations nd functions [2], [3], [5], [7].

4. Sonine and Poisson type transmutations for difference–differential operators of Dunkl type.

5. Applications of transmutations to generalized analytic function theory, cf. [6].

6. Methods of fractional integro-differentiation and integral transforms with special function kernels [5]. In this field let us mention a composition method to derive many classes of transmutations in the unified way [10].

7. Unitary Sonine–Katrakhov and Poisson–Katrakhov transmutations [8]–[10].

8. Applications to partial differential equations with singularities [4], [7]– [10].

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Application of the operational calculus approach of Dimovski for solving the backward heat problem

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Keywords: operational calculus, non-classical convolution, Duhamel principle, ill-posed problem, quasireversibility

The final value problem for the heat equation is known to be ill-posed. To deal with this, in the method of quasireversibility, the equation or the final value condition is perturbed to form an approximate well-posed problem, depending on a small parameter ε. In this work, several known quasireversibility techniques for the backward heat problem are considered and the obtained new problems are treated using the operational calculus approach developed by Dimovski [1]. For every approximate problem, applying an appropriate bivariate operational calculus, a Duhamel-type representation of the solution is obtained. It is in the form of a convolution product of a special solution of the problem and the given final value function. The idea for such Duhamel-type representations for ill-posed problems is originally proposed in [2]. Here it is further developed for different regularizations of the backward heat problem and its application for calculating the numerical solution is illustrated on some test problems.

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Simulation of viscoelastic flows with fractional derivative models: an approach via the operational calculus of Dimovski

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Keywords: operational calculus, non-classical convolution, Riemann-Liouville fractional derivative, generalized Oldroyd-B fluid, finite difference scheme

The boundary value problem for the velocity distribution of a viscoelastic flow with generalized fractional Oldroyd-B constitutive model is studied. The model contains two Riemann-Liouville fractional derivatives in time of different orders. Based on the eigenfunction expansion, the unique existence of the solution is established and some regularity results and qualitative properties are obtained. Further, applying the operational calculus approach proposed by Dimovski [1], a Duhamel-type representation of the solution with respect to the space variables is found. This is a compact representation, containing a non-classical convolution product of a special solution and the given initial function. It is appropriate for numerical computation of the solution. To illustrate this, a finite difference scheme is also constructed and the solutions of some test problems are calculated numerically in different ways: using the finite difference approximation, using the Duhamel-type representation, or combining both of them. Numerical results for one- and two-dimensional examples are presented and the different techniques are compared in terms of efficiency, accuracy, and CPU time.

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Existence of solutions to boundary value problem for impulsive fractional equations

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Keywords: fractional differential equations, impulsive conditions, weak solution, classical solution, three critical point theorem

In this paper we study the existence and the multiplicity of solutions for an impulsive boundary value problem for fractional differential equations. The notions of classical and weak solutions are introduced. Then, the existence of at least one and three solutions are proved. An example is given.

For related studies, see references below.

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Commutant of Sturm-Liouville operator in an invariant subspace

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Keywords: commutant, invariant subspace, Sturm-Liouville operator, convolution

We characterize the continuous linear operators $M : C \to C$, with $C = C[0,\infty)$, commuting with the Sturm-Liouville operator $D = \dfrac{d^2}{dx^2} - q(x)$ in the invariant subspace $C_{h,\Phi} = \{f \in C, f'(0) - hf(0) = 0, \Phi\{f\} = 0\}$, where Φ is an arbitrary nonzero continuous linear functional. Additionally, we assume $M : C^k \to C^k$, $k = 1, 2$.

Using a convolution $f * g$ found by the authors (see [1]) in 1976, we found explicitly the commutant as consisting of all operators of the form

$$Mf(x) = \mu f(x) + m * f,$$

where $\mu = const$ and $m \in C$.

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Explicit solutions of BVPs for multidimensional heat equation

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Keywords: convolution, operational calculus, nonlocal boundary value problem, Duhamel principle

We consider a general nonlocal BVP (boundary value problem) of the form:

$$\begin{aligned}
&u_t = u_{x_1x_1} + ... + u_{x_nx_n} + F(x_1, ..., x_n, t),\\
&u(x_1, ..., x_n, 0) = 0,\\
&u(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n, t) = 0,\\
&\Phi_{j,\xi}\left\{u(x_1, ..., x_{j-1}, \xi, x_{j+1}, ..., x_n, t)\right\} = 0, \quad j = 1, 2, ..., n,
\end{aligned}$$

with given linear functionals Φ_1, ...,Φ_n.

It is shown that using a multidimensional operational calculus, the problem could be reduced to n one-dimensional BVPs of the form

$$\begin{aligned}
&v_t = v_{x_kx_k}, \quad v(x_k, 0) = x_k,\\
&v(0, t) = 0, \quad \Phi_{j,\xi}\{v(\xi, t)\} = 0, \quad k = 1, ..., n,
\end{aligned}$$

with corresponding solutions $\Omega_k(x_k, t)$.

Then a general solution is obtained as an extension of the Duhamel principle for the space variables in the form

$$u(x_1, ..., x_n, t) = \frac{\partial^{2n}}{\partial x_1^2 ... \partial x_n^2}\left[(\Omega_1 ... \Omega_n) * F\right],$$

where $*$ is a multidimensional convolution.

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On the convolutors in the D_{Lp}-type spaces associated with a singular second order differential operator

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Keywords: D_{Lp}-type spaces, convolution product, differential operator

We consider the D_{Lp}-type spaces associated with a singular second order differential operator $\triangle_A$. Some results are established. Next, using the convolution associated with $\triangle_A$, we study the convolutors and the surjective convolution operators acting on spaces of distributions of L_p^A-growth.

For more details, see References.

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From the hyper-Bessel operators of Dimovski to the generalized fractional calculus

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Keywords: hyper-Bessel operators and functions, integrals and derivatives of fractional order, special functions and integral transforms related to fractional calculus

In several works since 1966, Dimovski [1] introduced and studied a very wide class of singular differential operators with variable coefficients of arbitrary integer order $m \geq 1$, known nowadays as *hyper-Bessel differential operators* and appearing in alternative forms in many problems of mathematical physics:

$$B = x^{\alpha_0} D x^{\alpha_1} D x^{\alpha_2} \cdots D x^{\alpha_m} = x^{-\beta} \prod_{i=1}^{m} (xD + \beta\gamma_i)$$
$$= x^{-\beta} \left(x^m D^m + a_1 x^{m-1} D^{m-1} + \cdots + a_{m-1} x D + a_m \right), \quad \beta > 0, \quad m \geq 2, \qquad (1)$$

with $D := d/dx$, and real parameters α_k, γ_i, a_j. The best known example, giving rise to their name, is the second order differential operator of Bessel $B_\nu = x^{-2}(xD+\nu)(xD-\nu) = x^{\nu-1} D x^{2\nu+1} D x^{\nu} = D^2 + x^{-1}D - x^{-2}\nu^2$, related to the Bessel function $y(x) = J_\nu(x)$ as a solution of the equation $B_\nu y(x) = -y(x)$. Another simple representative of higher order is the operator of m-fold differentiation $D^m = (d/dx)^m$.

Dimovski constructed a very general operational calculus for the operators (1) following the algebraic approach of Mikusinski, and giving rise to the new notion of *convolutional calculus*, [2]. Besides, he discovered that an integral transform introduced by another Bulgarian mathematician in 1958 – the Obrechkoff transform – can serve as a transform approach to the same operational calculus for (1). Many well-known mathematicians (to mention Ditkin, Prudnikov, Meller, Botashev, Krätzel, Rodriguez, etc) have studied later very particular cases of the hyper-Bessel operators (1) and rediscovered operational calculi and Laplace-Meijer or Hankel type integral transforms related to them, years after Dimovski's most general works.

The present author has started her studies on Dimovski's hyper-Bessel operators and on the Obrechkoff transform since 1975, and finally benefitted of them in developing a theory of the *generalized fractional calculus* (GFC) [3] and in introducing new classes of integral transforms [4,5] and special functions [6]. The GFC deals with generalized integrals and derivatives of *fractional multi-orders* $(\delta_1, \delta_2, ..., \delta_m)$ as analogues of the Riemann-Liouville (R-L) and Erdélyi-Kober integrals (E-K) $I^{\delta}, I_{\beta}^{\gamma,\delta}$ and derivatives $D^{\delta}, D_{\beta}^{\gamma,\delta}$ of arbitrary order $\delta > 0$ in the classical fractional calculus. Having the structure of commuting compositions of E-K operators $I_{(\beta_k)_1^m,m}^{(\gamma_k)_1^m,(\delta_k)_1^m} = \prod_1^m I_{\beta_k}^{\gamma_k,\delta_k}$, the generalized fractional integrals of our GFC are represented by means of integral operators involving special functions:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = \int_0^1 \Phi(\sigma) f(x\sigma) d\sigma, \text{ where } \Phi(\sigma) = H_{m,m}^{m,0}\left[\sigma \middle| \begin{matrix} (\gamma_i + \delta_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \\ (\gamma_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \end{matrix}\right] \tag{2}$$

is the Fox H-function, a generalized hypergeometric function of very general nature. The corresponding R-L and Caputo-type generalized fractional derivatives $D_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ are defined by means of suitable differ-integral expressions, see [3, 7].

The other operators of fractional calculus and many generalized integrations and differentiations used in applied analysis are shown to be special cases of (2). But the worthy fact to emphasize is that *the hint to introduce the GFC came from the hyper-Bessel operators* (1) of integer order m that appear also to be generalized "fractional derivatives" of multiorder $(1, 1, ..., 1)$: $B = x^{-\beta} D_{(\beta,...,\beta),m}^{(\gamma_1,...,\gamma_m),(1,...,1)}$.

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Fractional order linear autonomous system with distributed delay

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Keywords: fractional order differential equations, Riemann-Liouville fractional derivative, Caputo derivative, distributed delay

Asymptotic properties of the solutions of fractional order linear autonomous system with distributed delay are studied.

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Evolution equations for the Stefan problem

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Keywords: Stefan problem, boundary value problem, abstract parabolic evolution equations

The Stefan problem is a particular kind of a free boundary value problem which models phase transition phenomena, for example melting of ice and freezing of water.

We study a quasi-steady variant and propose in our model a boundary condition with surface tension and kinetic undercooling that reflects the relaxation dynamics. In our approach to the problem we use the theory of abstract quasilinear parabolic evolution equations. The obtained results are in Sobolev spaces.

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Mean value theorems for analytic functions

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Keywords: mean value theorems, real and complex analytic functions, Rolle's theorem in the complex domain

The problem of extending Rolle's theorem to the complex domain, as well as other related questions, have been of enduring interest (see for example [2]). In this talk, we will examine some interesting and little-known mean value theorems concerning real and complex analytic functions, focusing on the complex case. A sharper Evard-Jafari theorem (see [1]) will be proved. The remarkable contributions of the Bulgarian school of Mathematics to this field will be emphasized throughout the presentation, and the paper will be dedicated to the 118th anniversary of the birth of Academician Nikola Obrechkoff.

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Fractional differential equations involving impulses

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Keywords: fractional differential equation, Riemann–Liouville fractional derivative, impulse, Dirac delta function, Laplace transform

Fractional calculus and fractional differential equations (FDEs) have become important in recent decades as mathematical models of processes that exhibit such properties as long-term memory and self-similarity. FDEs appear naturally in a number of fields such as rheology, seismology, biophysics, blood flow phenomena, aerodynamics, fluid flow in porous media, viscoelasticity, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. There exists an extensive literature on this topic including monographs [3, 7]. In particular, the problems of control of FDEs and fractional variational problems are addressed in the papers [1, 2, 4].

Impulsive differential equations have recently received considerable attention as mathematical models of processes where some parameters can change instantly in a jump-like manner.

The monographs [5, 8] are devoted to the impulse differential equations and related issues, i.e. stability, control etc. To describe the impulsive character of the processes special types of differential equations allowing discontinuous solutions [5] can be employed.

A common approach to modeling the impulsive behavior is the use of difference equations to describe the impulse impact.

Other approaches employed to deal with the impulsive behavior are based on the technique of generalized functions (distributions) such as Dirac delta function. All these approaches have their advantages and drawbacks.

Both FDEs and impulsive differential equations have drawn intense attention from researchers in the last decades due to the numerous applications. The idea that combining these two classes of differential equations may yield an interesting and promising object of investigation, viz., impulsive FDEs, prompted significant interest.

In the paper [6] impulsive FDEs were treated, where the jump-like discontinuities at certain time instants were described using difference equations. Here we adopt the other approach and investigate linear fractional differential equations whose right hand side contains additive Dirac distributions. Analytical solutions to these equations are obtained on the basis of the Laplace transform method.

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On various existence conditions for the convolution of Beurling ultradistributions

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Keywords: ultradistribution of Beurling type, convolution of ultradistributions of Beurling type

Theorems on the existence of the convolution as well as on the convergence of convolutions in some spaces of ultradistributions of Beurling type are proved under certain general conditions.

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Extending the Stieltjes Transform

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Keywords: abelian theorem, Boehmians, generalized function, iterated Laplace transform, Stieltjes transform

Several authors have extended the classical Stieltjes transform onto spaces of generalized functions. Many have investigated the Stieltjes transform on the space $J'(r)$, which consists of distributions of the form $T = D^n f$ (for some $n \in \mathbb{N}$), where f is a locally integrable function supported on the interval $[0, \infty)$ and satisfies a growth condition at infinity.

The space of generalized functions known as Boehmians, whose construction is algebraic, has been used to extend integral transforms such as Fourier, Laplace, Hilbert, and Hankel. Roopkumar [2] has extended the Stieltjes transform onto a space of Boehmians. However, the transform is a Boehmian, not a function.

In this note, by using iteration of the Laplace transform, we extend the Stieltjes transform onto a subspace of Boehmians which contains a proper subspace that can be identified with $J'(r)$. In this case, the transform is an analytic function in the half-plane $\operatorname{Re} z > 0$. This allows, in a natural way, to establish an abelian theorem of the final type.

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Application of fractional integro-differentiation in telecommunications

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Keywords: fractal, fractal signal, fractal modulation, wavelet modulation, fractional calculus

Currently exists a large number of powerful sources of electromagnetic noise having a destructive impact on transmission of useful signals and thus significantly distorting the information transmitted. Therefore, the primary objective in the development of new radio systems is to achieve the required noise robustness of the transmitted signals. One of the possible ways of solving this problem is the use of fractal signals in communication systems. Phenomena and objects of fractal nature are extremely common. In many applications modeling of natural fractal behavior is necessary for certain signal processing problems. In particular, there are many problems of identification, classification, smoothing, filtering and prediction related to fractal signals. On the other hand, the ubiquity of fractal behavior in nature suggests that the fractal structure is in a sense optimal or efficient. This causes growing interest in development communication, telemetry and other engineering systems based on the use of fractal signals. The problems of building communication systems based on the use of fractals are discussed in the papers [1, 4]. Generally speaking, fractal signal is a function that has a structure at all scales. However, of the greatest interest are fractals whose structure at all scales is similar. In this case it is said that fractal is self-similar or scale-invariant, to emphasize the fact that the fractal has no absolute scale reference. Fractal signals are divided into two broad categories: those whose self-similarity is statistical in nature, and those, which are deterministically self-similar. Statistically self-similar fractal signal or stochastic fractal signal has the same statistical properties at all scales, while the structure of the deterministic fractal signal is the same at all scales.

The fractal modulation method described in [1, 4] involves the generation of fractal signals, where a value of "0" and "1" stream of binary data correspond to two different values of the Hausdorff–Besicovitch fractal dimension.

In the practical implementation of this approach it is necessary to solve the problem of generating a signal of a given fractal dimension. The existing approaches to solving this problem are usually associated with using fractional integro-differentiation operators. The theory of Fractional Calculus was first systematically outlined in the fundamental monograph [2] and its further development, the theory of fractional differential equations – in the book [3]. In particular, to obtain stochastic signal of a given fractal dimension, where the parameter H is the Hurst exponent, $1 < D < 2$, one can pass white noise $w(t)$ through a linear time invariant filter having impulse response of the form $\frac{1}{\Gamma(H+1/2)} t^{H-1/2} u(t)$, where $\Gamma(\cdot)$ is the Euler Gamma function that satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$ and $u(t)$ is the Heaviside unit step function. This filter corresponds to the fractional integration of order $H + \frac{1}{2}$ in the sense of Weyl:

$$I_+^{H+1/2} w(t) = \frac{1}{\Gamma(H+1/2)} \int\limits_{-\infty}^{t} (t-\tau)^{H-1/2} w(\tau) d\tau.$$

The physical meaning of the parameter H is that it determines the exponent of statistical self-similarity of the signal, that is, for any $a > 0$, the following equalities hold true for the expectation and covariance:

$$E\left[x(t)\right] = a^{-H} E\left[x(at)\right], \; E\left[x(t)x(s)\right] = a^{-2H} E\left[x(at)x(as)\right].$$

Hence, the problem of constructing integrators and differentiators of fractional order is important, and in addition can be used for digital processing of images and medical diagnostic data. Due to Fourier transform for fractional integral:

$$\mathcal{F}(I_+^{\alpha} w) = \frac{\hat{w}(\omega)}{(-i\omega)^{\alpha}}, \;\; 0 < \alpha < 1,$$

one can propose the following algorithm to generate discrete noise signal v_k of given fractal dimension $D = 2 - H$: 1. Generate pseudo-random array w_k, $k = 0, 1, ..., N-1$, with the white noise (flat) spectrum; 2. Calculate the discrete Fourier transform of w_k using Fast Fourier Transform (FFT); 3. Filter w_k with $(-i\omega)^{-(H+1/2)}$; 4. Apply the Inverse Fast Fourier Transform (IFFT) to the resulting sequence.

The above method of fractal modulation can be used for concealed transmission of digital data, since the signal that is transmitted can be hardly distinguished from the natural noise present in the communication channel. This makes appropriate its use in communications systems with low probability of interception (LPI). **References:** [1] J.M. Blackledge, B. Foxon, S. Mikhailov, A fractal modulation technique for digital communications systems, *Proc. Military Communications Conf. MILCOM 98*, IEEE, Vol. **1** (1998), 140–144; [2] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives*, Gordon & Breach, 1993; [3] N.O. Virchenko, V.Ya. Rybak, *Foundations of Fractional Integro-Differentiation*, Kiev, 2007 (in Ukrainian); [4] G.W. Wornell, *Signal Processing with Fractals: A Wavelet-Based Approach*, Prentice Hall, Boston, 1996.

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A family of hyper-Bessel functions and convergent series in them

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Keywords: hyper-Bessel function, hyper-Bessel differential operator, series in hyper-Bessel functions, convergence of series in complex plane

In 1953, Delerue introduced generalizations $J^{(m)}_{\nu_1,\ldots,\nu_m}(z)$ of the Bessel function of the first type $J_\nu(z)$ with vector indices $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$. Later these functions were studied and explored also by other authors, for example Marichev, Kljuchantcev, Dimovski, Dimovski and Kiryakova (for details see e.g. [1]– [4]), etc. The Delerue hyper-Bessel functions are closely related to the hyper-Bessel differential operators of arbitrary order $m > 1$, introduced by Dimovski [1], see [3].

In this work we consider an enumerated family of hyper-Bessel functions and study the convergence of series in such kind of functions. The obtained results are analogues to the ones in the classical theory of widely used power series.

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A note on a subclass of close-to-convex functions

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Keywords: univalent functions, close-to-convex functions

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic and univalent in the open unit disk $E = \{z : |z| < 1\}$.

Let C denote the class of convex functions [1]:

$$f(z) \in C \text{ if and only if for } z \in E,\ \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

Let S^* denote the class of starlike functions [2]:

$$f(z) \in S^* \text{ if and only if for } z \in E,\ \Re\frac{zf'(z)}{f(z)} > 0.$$

A function $f(z)$ analytic in E is said to be close-to-convex in E, if there exists a function $g(z) \in S^*$ such that for $z \in E$

$$\Re\frac{zf'(z)}{g(z)} > 0.$$

The class of such functions is denoted by K, [3].

The classes S, K, S^* and C are related by the proper inclusions

$$C \subset S^* \subset K \subset S.$$

Now we will consider a class $\widetilde{K}$ defined as follows:

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E. Then $f(z) \in \widetilde{K}$ if and only if there exists a function $g(z) \in C$ such that for $z \in E$

$$\Re\frac{zf'(z)}{g(z)} > 0.$$

Since $C \subset S^*$, it follows that $\widetilde{K} \subset K$ and so, functions in $\widetilde{K}$ are univalent.

Theorem 1. *Let $f(z) \in \widetilde{K}$. Then for $z = re^{i\theta} \in E$:*

$$\frac{1-r}{(1+r)^2} \leq |f'(z)| \leq \frac{1+r}{(1-r)^2},$$

$$-\ln(1+r) + \frac{2r}{1+r} \leq |f(z)| \leq \ln(1-r) + \frac{2r}{1-r}.$$

Each inequality is sharp for $f_0(z)$ defined by

$$f_0(z) = \overline{x}\log(1-zx) + \frac{2z}{1-xz} \quad \textit{with} \quad |x| = 1.$$

Theorem 2. *Let $f(z) \in \widetilde{K}$, with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then for $z \in E$:*

$$|a_n| \leq 1 - \frac{1}{n}$$

for $n \geq 2$. Equality is attained for $f_0(z)$.

Theorem 3. *Let $f(z) \in \widetilde{K}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{5}{3} - \frac{9}{4}\mu, \ \textit{if} \ \ \mu \leq \frac{2}{9} \\ \frac{2}{3} + \frac{1}{9}\mu, \ \textit{if} \ \ \frac{2}{9} \leq \mu \leq \frac{2}{3} \\ \frac{5}{6}, \ \textit{if} \ \ \frac{2}{3} \leq \mu \leq 1. \end{cases}$$

For each μ there is a function in $\widetilde{K}$ such that equality holds.

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Somos-4 property of Hankel determinants derived from number sequences in an elliptic integral form

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Keywords: special numbers, determinants, polynomials, recurrence relations

In continuation of our paper [1], we will consider the special number sequences in the next integral form

$$g_n^{(p)} = \frac{1}{2\pi}\int_a^b x^{n-\delta_{1,p}}(b-x)^{\nu-1}(x-a)^{\mu-1}(b_1-x)^{\nu_1-1}(x-a_1)^{\mu_1-1}\,dx,$$

where $a_1 < b_1 < 0 < a < b$; $p \in \{0,1\}$; $\mu, \nu, \mu_1, \nu_1 > 0$; $n \in \mathbb{N}$ and their *Hankel transform* $H = \{h_n\}$ given by $h_n = |g_{i+j-2}|_{i,j=1}^n$.

We are interested in the special cases of $\{g_n\}$ which satisfy the *generalized convolution property*

$$g_n = \sum_{k=1}^{r} \alpha_k g_{n-k} + \beta \sum_{k=0}^{n-r} g_k g_{n-r-k},$$

and the Hankel determinants have the *generalized Somos-4 property*

$$h_n h_{n-4} = r\, h_{n-1}h_{n-3} + s\, h_{n-2}^2 \quad (n = 4, 5, \ldots) \quad (r, s \in \mathbb{N}).$$

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Representation of holomorphic functions by Schlömilch's series

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Keywords: Schlömilch's series, Bessel functions, representations of holomorphic functions, Erdélyi-Kober operators

A necessary and sufficient condition is given for a holomorphic function to be represented by a series of the kind

$$\sum_{n=0}^{\infty} a_n J_0(nz), \quad z,\, a_n \in \mathbb{C},$$

where J_0 is the Bessel function of first kind with zero index.

To derive the result, we use an Erdélyi-Kober operator of fractional order known as Uspensky transform, and the Poisson integral representation for $J_0(z)$ via $\cos(z)$.

Some of these results have been published in [1].

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On various polynomials of Mittag–Leffler type

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Keywords: hypergeometric function, generating function, polynomial sequence, recurrence, orthogonality

The Mittag–Leffler polynomials $\{g_n(y)\}$ can be represented by the following hypergeometric function or by its generating function, as follows:

$$g_n(y) = 2y\ {}_2F_1\Big(\begin{matrix}1-n,\ 1-y\\ 2\end{matrix}\ \Big|\ 2\Big): \quad \Big(\frac{1+x}{1-x}\Big)^y = \sum_{n=0}^{\infty} g_n(y)x^n \quad (|x|<1).$$

They were introduced by Mittag-Leffler in a study on the integral representations. Their main properties were found by H. Bateman [1]. They can be considered as a special case of the Meixner polynomials $M_n(x;\beta,c)$ for $\beta=2$ and $c=-1$ (in spite of the fact that the Meixner polynomials require the constraint $0<c<1$) or the Pidduck polynomials by the expression $P_n(y) = ((e^D+1)/2)g_n(y)$, where we use series for the exponential function and D is differentiation.

A few new papers considering these polynomials, appeared recently, see [2].

Based on the generalized integer powers of real numbers

$$z^{(0,h)} = z^{[0,h]} = 1, \quad z^{(n,h)} = \prod_{k=0}^{n-1}(z-kh), \quad z^{[n,h]} = \prod_{k=0}^{n-1}(z+kh) \quad (n\in\mathbb{N}),$$

and on the deformed exponential function

$$e_h(x,y) = (1+hx)^{y/h} \qquad (x\in\mathbb{C}\setminus\{-1/h\},\ y\in\mathbb{R}),$$

we introduce *the deformed Mittag–Leffler polynomials* as the coefficients in the expansion

$$G_h(x,y) = e_h(x,y)\ e_{-h}(x,y) = \sum_{n=0}^{\infty} g_n^{(h)}(y) x^n \ .$$

We investigate their recurrence relations, hypergeometric representation and orthogonality. Since they have all zeros on the imaginary axis, we consider the associated real polynomials.

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An operational calculus approach for periodic and anti-periodic solutions in the environment of a computer algebra system

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Dedicated to the 80th anniversary of Professor Ivan Dimovski, Corresponding member of the Bulgarian Academy of Sciences

Keywords: convolution, operational calculus, Mikusiński calculus, linear ordinary differential equation, periodic solution, boundary value problem

An operational calculus approach for obtaining periodic and anti–periodic solutions of Linear Ordinary Differential Equations (LODE) with constant coefficients in the environment of a Computer Algebra System (CAS) is considered.

A Miikusiński's type operational calculus, based on the non–classical convolution of Dimovski

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f(t+\tau-\sigma) g(\sigma) d\sigma \right\},$$

where Φ is a linear functional in $C(\mathbb{R})$, is used (see [1]). In the framework of this operational calculus an extension of the classical Heaviside algorithm is proposed. It is intended for solving nonlocal boundary value problems for LODEs with constant coefficients. The problems of obtaining periodic and anti–periodic solutions of LODEs with constant coefficients are reduced to such problems.

The algorithms based on the considered approach are implemented in the environment of the CAS *Mathematica*. Illustrative examples are presented.

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On the asymptotic behavior of generalized functions

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Keywords: Stieltjes transform, asymptotic behavior, distributions

In this paper we present some new versions of the distributional Stieltjes transform, and apply it to the asymptotic behaviour and asymptotic expansion.

These integral transforms allow corresponding Abelian and Tauberian type results.

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Some operational solutions of higher order fuzzy differential equations

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Keywords: fuzzy calculus, operational calculus, fuzzy differential equations, Mikusiński calculus and operators

In this paper fuzzy differential equations of higher order with fuzzy coefficients are studied within the frames of the Mikusiński calculus. Some preliminaries on the subject can be seen in the references.

The exact and the approximate solutions of the considered problem are constructed and their characters are analyzed.

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