Vladimir Sotirov (Sofia)

Leibniz's Intensional Semantics of Syllogistics (a Reconstruction)*

1. Introductory Notes

"La maniere d'enoncer vulgaire regarde plustost les individus, mais celle d'Aristote a plus d'egard aux idées ou universaux. Car disant 'tout homme est animal', je veux dire que tous les hommes sont compris dans tous les animaux; mais j'entends en même temps que l'idée de l'animal est comprise dans l'idée de l'homme. L'animal comprend plus d'individus que l'homme, mais l'homme comprend plus d'idées ou plus de formalités; l'un a plus d'exemples, l'autre plus de degrés de realité; l'un a plus d'extension, l'autre plus d'intension" (**GP** V, 496).

These important words taken from *Nouveaux Essais sur l'Entendement humain* (1704) prove that Leibniz was acquainted with *two* semantics of the syllogistic propositions and thought them to be equipollent. Following his own expressions, the first semantics usually is called *extensional* and the second one is called *intensional*. For some hard reasons Leibniz preferred the intensional semantics and in practice, all his logical manuscripts contain it. There is one important exception: the geometric interpretation by circles consecutively follows the extensional semantics. In some places Leibniz notes that all his conclusions could be paraphrased also in the terms of the extensional semantics: "Using fitting characters, we could demonstrate all the rules of logic by another kind of calculus than the one developed here, merely by an inversion of our own calculus" (*in Latin*: **A** VI, II, 200). However I have to point out that no rules of the translation have been described explicitly.

Leibniz tried to build an adequate representation of the two basic syllogistic relations *sAp* ("all *s* are *p*") and *sIp* ("some *s* are *p*") using different tools: algebraic, geometric, set-theoretical, arithmetical. Here I will consider only the last one. The central idea of the arithmetical models was to interpret *notions* by *integers* (being their *characteristic numbers*) and to translate syllogistic relations *A* and *I* into arithmetical relations between integers, using mainly *divisibility*. In such a way Leibniz hoped to realize his programme of calculating "the truth" and to promote his popular motto '*Calculemus*!'. Leibniz's favourite example was: if the number of *animal* were 2 and that of *rational* were 3 then the number of *man* being by definition a *rational animal* should be obtained by multiplication 3·2. Then the answer to the question "Is every man a rational being?" could be reduced to the fact that 6 is divisible by 3.

The last example is a proof that Leibniz was clinging to the intensional semantics of the universal affirmative (UA) propositions (the predicate is contained in the subject). Unfortunately there is no translation of a concrete particular affirmative (PA) proposition in the manuscripts. What is even worse, all attempts to use single integers turned out to be incorrect. A detailed analysis of the faulty procedures, the causes of the faults, and two correct realizations of the primary Leibnizian

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idea have been already presented in my paper *Arithmetizations of syllogistic à la Leibniz*[1]. In fact, Leibniz mixed both semantics: the interpretation of UA-propositions followed the intensional semantics while the interpretation of PA-propositions followed the extensional one. Finally Leibniz constructed an adequate model using *pairs* of integers. However I will not review this model as a part of the scope of this report.

On the previous Leibniz Congress I presented a working arithmetical procedure in accordance with the extensional semantics[2]. Now I will continue the work by describing the intensional arithmetical semantics. It will be the extensional semantics on a concrete example. In such a way the model preferred by Leibniz will be completely rehabilitated and shown in action.

2. Extensional and intensional semantics: a comparison

Relevant to our task – an implement of a natural and working translation of syllogistic into arithmetic – are Leibniz's manuscripts of April 1679. The most important papers devoted to models with single numbers are *Elementa characteristicae universalis* (**A** VI, II, 181–194), *Elementa calculi* (**A** VI, II, 195–205), and *Calculi universalis elementa* (**A** VI, II, 205–216). The example of 'man = rational animal' gives a criterion for UA-propositions: *sAp* is true when *s* is divisible by *p*. (The letters used for terms are the same as the ones used for their characteristic numbers.) In Leibniz's general notation: $s = xp$.

The interpretation of PA-propositions in Leibniz's manuscripts is less clear and more problematic. In the last variant it means that *sIp* is true when *s*, being multiplied by another integer, is divisible by *p*. This is an arithmetical expression of Leibniz's reduction of PA-propositions to UA-propositions: *sIp* is true when *s* enhanced with an additional requisite *x* is *p*. In Leibniz's notation: $sx = yp$. In the paper quoted in the footnote 1 I have remarked that if we take this rule literally, it will become trivially true because *any* integer *s* becomes divisible by *any other* integer *p* after multiplying it by a suitable integer, e.g., by *p* itself. That is why it is natural to complete the criterion for PA-propositions by the condition that the multiplier must be *less* than the number of the predicate. Then it is easy to prove that both criteria proposed by Leibniz can be formulated in a uniform manner: *sAp* is true when each divisor of *p* is also a divisor of *s*; *sIp* is true when *s* and *p* have a common divisor greater than 1, or, $\gcd(s, p) > 1$.

However, some syllogisms cease to be true given this interpretation. For example, let us check the syllogism *Darii*: all *m* are *p*, some *s* are *m*, ergo some *s* are *p*. Take the notions *mammal*, *biped*, and *rational* and let *a*, *b*, and *c* be their characteristic numbers. The composition *ca* gives the notion *man* and *ba* gives the notion *biped mammal*. Now, the UA-proposition "All men are rational" is true because *ca* is divisible by *c*. The PA-proposition "Some biped mammals are men" is also true because (*ba*)*c* is divisible by *ca*. Or if translated in words, all *biped mammals* with the additional requisite *rational* are *men*. To prove that the conclusion "Some biped mammals are rational" is true we must check whether (*ba*)*x* is divisible by *c* for some additional requisite *x*. We see however that *x* may be only *c* – a trivial situation that is excluded by the rule. Therefore this arithmetical model is not adequate.

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[1] Journal of Applied Non-Classical Logics, 9(1999), pp. 387–405.

[2] V. Sotirov: "Leibniz's 'Calculemus!' at Work", in: *Akten des VIII. Internationalen Leibniz-Kongresses. Hannover, 24.-29. Juli 1972.*

The equivalent formulation of the criteria using divisors of the characteristic numbers reveals the confusion of the two semantics: *sAp* is true when the set of divisors of *s* includes the set of *p* (the intensional semantics) while *sIp* is true when *s* and *p* have a common part (the extensional semantics). Both criteria can take their own places by obtaining in effect two adequate semantics. In the extensional arithmetical interpretation terms are evaluated by integers greater than 1; *sAp* is replaced with '*s* is a divisor of *b*', and *sIp* with 'gcd(*s*, *p*) > 1'. If empty terms are admitted, they are evaluated by 1. For the intensional interpretation, an arbitrary integer $u > 1$ (a "universe") must be introduced and terms are evaluated by *u*'s arbitrary proper divisors (i. e., the divisors which are less than *u*); *sAp* is replaced by '*s* is divisible by *p*', and *sIp* by 'lcm(*s*, *p*) < *u*' (lcm denotes the least common multiple). If empty terms are admitted, they are evaluated by *u*. In both semantics the characteristic numbers do not admit multiple factors.

Considering integers *s* and *p* as sets of their (prime) factors and applying the usual symbols of set theory, *sAp* and *sIp* are true in the *extensional semantics* when $s \subseteq p$ and $s \cap p \neq \varnothing$ respectively ($\varnothing$ is the empty set, the second relation corresponds to Leibniz's criterion for PA-propositions). For Aristotelian syllogistics all sets are supposed to be non-empty (and the characteristic numbers are different from 1). In the *intensional semantics*, *sAp* and *sIp* are true when $s \supseteq p$ and $s \cup p \neq u$ respectively (*u* is the universe, the first relation corresponds to Leibniz's criterion for UA-propositions). For Aristotelian syllogistics no set (number) is equal to *u*. The difference between both semantics is not big from a mathematical point of view because they are *mutually dual* (Leibniz calls them *inverse*). Namely, when the case is the one of sets, the intensional semantics will be obtained from the extensional one after replacing each set by its complement to *u*, $\cap$ by $\cup$, $\subseteq$ by $\supseteq$, $\varnothing$ by *u*, and vice versa. When numbers are used, each number *k* has to be replaced by u/k, the expression 'is a divisor of' by the expression 'is divisible by', gcd by lcm, 1 by *u*, and vice versa.

The situation fundamentally changes when a configuration of the representing sets is drawn. Passing from extensional to intensional semantics all plausibility of the Leibniz circles (known as well as Euler circles and sometimes incorrectly named Venn diagrams) disappears. Our intuition loses the transparency and clarity of the overlaying and overlapping circles and leads us astray into the jungle of strange and artificial curves. Only an extremely inventive mind is able to draw a syllogism containing two particular propositions! In the next figure the rather elementary diagrams of *sIp* in both semantics should be compared: the black segment on the left side denotes that *s* and *p* have a common part and the black triangle on the right denotes that there is a part out of both of *s* and *p*.

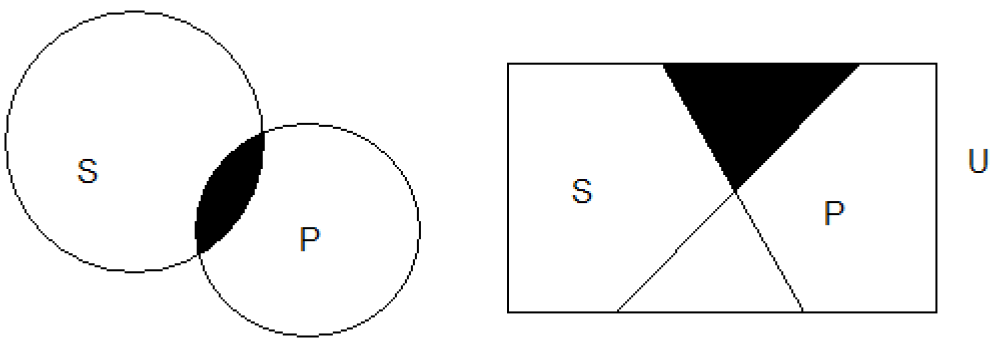

Fig. 1. *sIp* in extensional semantics (*left*) and in intensional semantics (*right*)

The representation by circles in extensional semantics has the advantage to avoid the empty sets, so to say, automatically: it is not possible to draw a "null-circle". Respectively, the number corresponding to the empty set is 1 and it does not need to be introduced separately. It is constant for all models. The universe becomes necessary only when term negation is introduced into syllogistic. On the contrary, the universe u occurs in the interpretation of sIp in intensional semantics even when negation is not present. Moreover, the universe has to be changed if new elements are added to the model. Beside the intuitive and graphical obstacles intensional semantics represents some linguistic obscurities. Therefore it is not surprising that Leibniz invented a few geometric extensional representations and no intensional one. The most popular of them is described in *De formae logicae comprobatione per linearum ductus* (**C**, 292–321). All syllogisms are presented there by circles in a perfect form following the extensional semantics. It is notable that on many places of the manuscripts sAp is thought also as $s \sqsupseteq p$ following the intensional semantics. However not the slightest idea of a diagram of sIp in that semantics can be found. We may conclude that Leibniz had correct geometric but wrong arithmetical intuition concerning syllogistics. The 20 year span between them is a possible explanation of this contradiction. Obviously the intensional semantics is sophisticated if it represents a problem even to many contemporary authors.

3. The intensional arithmetical interpretation in action

In what fallows we shall test the intensional arithmetical semantics on an extremely simple model, which nevertheless should be sufficient for the purposes of this work because it contains all Boolean term operations. The example for our test is rather illustrative and does not claim adequacy to established taxonomy. Here it should be noted that Leibniz himself composed detailed classifications with definitions filling lots of pages (e. g., in *Table de definitions*, **C**, 437–510). Furthermore I believe it would be a better illustration of his idea to apply the arithmetical models to certain real biological classifications.

In order to become comparable, the frame of notions is the same as that in the report quoted in footnote 2. In the previous interpretation the characteristic numbers followed the extensional semantics (Fig. 2) and now they will follow the intensional one (Fig. 3). Then the notions were treated as classes of *objects* and now they are classes of *properties*. The notion on the top is *animal*. From the extensional view it is the wider class of objects but according to the intensional view it has a minimal bundle of properties and none is specific. The class of *animals* is supposed to consist of *reptiles*, *mammals*, *birds*, and *insects* only. *Mammals* are subdivided into *dogs*, *mankind*, and *bats*. The easiest way to obtain the new characteristic numbers is to replace each number in Fig. 2 by its reciprocal with respect to the "universe" ($u = 2{\cdot}3{\cdot}\ldots{\cdot}13$). The results are represented on Fig. 3.

Now it would be straightforward to verify the UA-propositions by using simple division. For example, "Every man is a mammal" is true because the number $3{\cdot}5{\cdot}7{\cdot}11{\cdot}13$ is divisible by $7{\cdot}11{\cdot}13$. "Every bat is winged" is also true: $2{\cdot}3{\cdot}7{\cdot}11{\cdot}13$ is divisible by $2{\cdot}3{\cdot}7$. Let us check some PA-propositions. For example, let us answer the question "Are there winged mammals?". We have to verify if there is a prime number between 2 and 13 that does not divide neither the number of *mammals* ($7{\cdot}11{\cdot}13$) nor the number of winged ($2{\cdot}3{\cdot}7$). Such a number is 5 and $2{\cdot}3{\cdot}7{\cdot}11{\cdot}13$ obviously corresponds to the class of *bats*, which appears below *mammals* and *winged* on the diagram. According to our scheme, the answer to the question "Are there winged reptiles?" is "no", because $2{\cdot}3{\cdot}5{\cdot}11{\cdot}13$ together with $2{\cdot}3{\cdot}7$ contain all factors of u. Indeed, nothing is placed below *reptiles* and *winged*. Here it is important to note that Leibniz treated existence like a *logical consistency*. As a result of this we have to assume that *flying mammals* should exist even if nobody has them seen.

Their existence is logically possible because no property of *mammals* contradicts a property of *winged* according to our data. On the contrary, *winged reptiles* do not exist because they cannot logically exist as all *reptiles* are *wingless*.

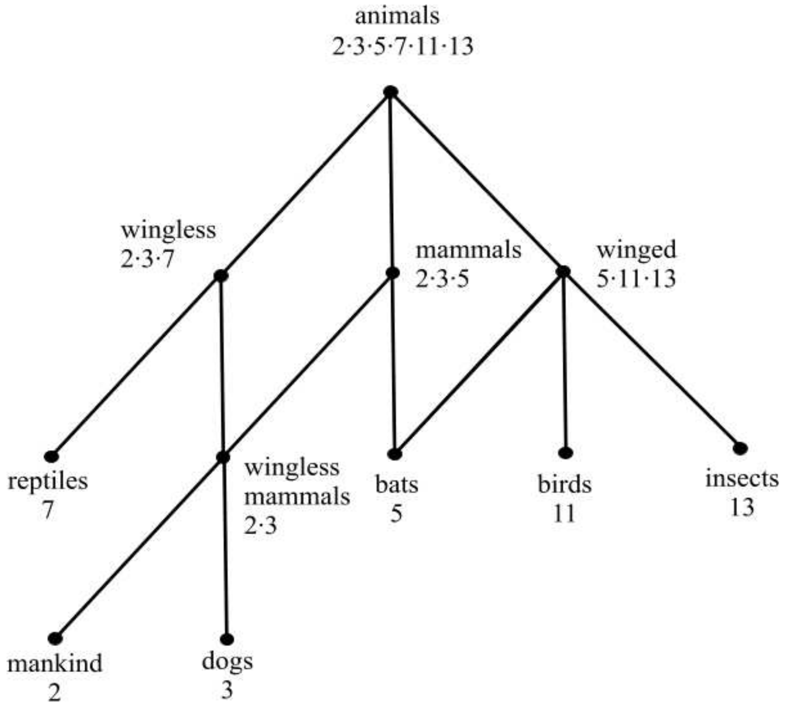

Fig. 2. Extensional semantics

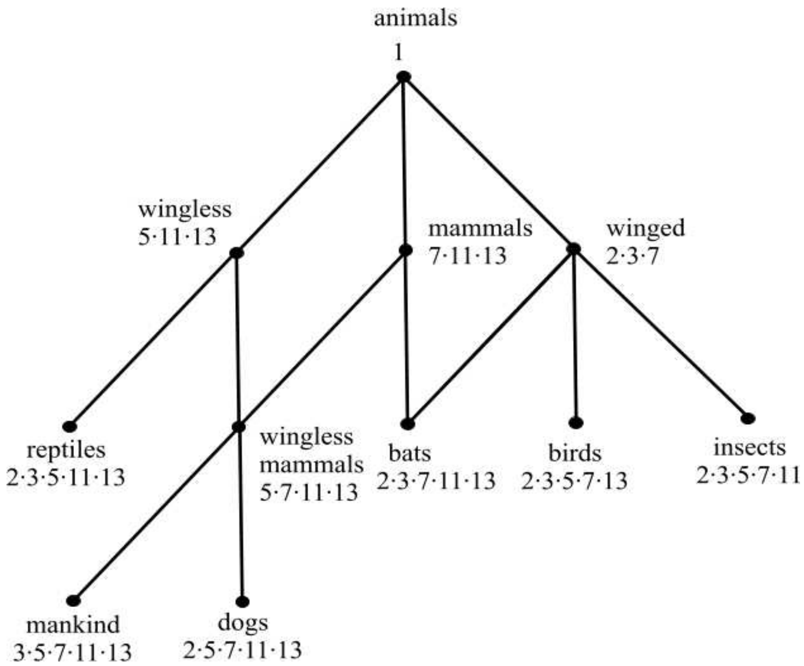

Fig. 3. Intensional semantics

Based on our example we can demonstrate the arithmetical translation of all Boolean operations with notions. If the question is disjunctive, e. g., "Which is the class of *men* and *dogs*?", or in other words "What does unite *men* and *dogs*?", the answer is: the class of the *wingless mammals* unites them because the gcd(3·5·7·11·13, 2·5·7·11·13) is 5·7·11·13 and the corresponding class is the nearest one placed above the *dogs* and *mankind*. Furthermore, the negation of *winged* is *wingless* (*apterouse*) and its number will be obtained by dividing u by 2·3·7 (for *winged*). Of course the result is 5·11·13 – the number of *wingless*.

Unfortunately, the full Boolean diagram contains $2^6 = 64$ vertices and is difficult to fit on a page. Therefore for the next consideration a simple algebra of $2^3 = 8$ elements (Fig. 4) would be useful. Take the three basic painting colours (as the *atoms* of the algebra): *yellow*, *red*, and *blue* and allocate the prime numbers 2, 3, and 5 to them respectively. Then the *pairs* of properties will represent the composed colours: *orange* (2·3), *green* (2·5), and *purple* (3·5). The "empty" colour *white* does not contain any other colour however it is included into each of the others. The combination of all three colours is *black* – the "universe", and its number is 2·3·5. If a question "Does *orange* colour contain *red*?" is asked then the answer will be "Yes, because 2·3 is divisible by 3". The answer to "Does *green* contain *red*?" will be "No, because 2·5 is not divisible by 3". "Are *orange* and *purple* contained in any colour?" – "Yes, but only in *black*, because 2·3 and 3·5 together contain all factors." "Does *orange* and *purple* contain a common colour?" – "Yes, because 2·3 and 3·5 have a common divisor 3 and this colour is *red*." "Which is the 'negative' colour of *blue*?" – 2·3·5 divided by 5 gives 2·3 and the answer is "*orange*". &c.

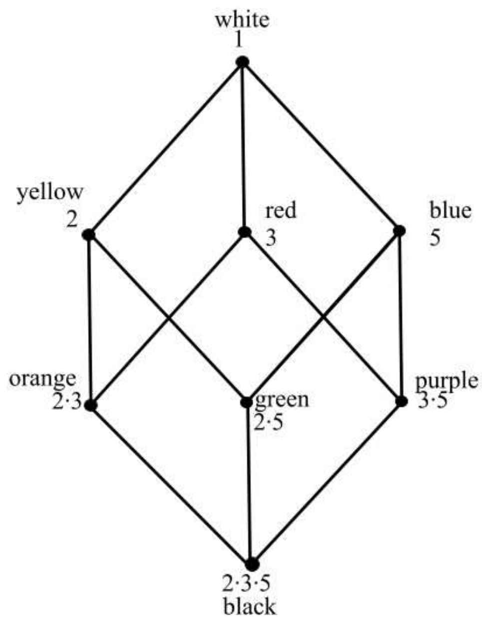

Fig. 4

Finally, let us return to the initial Leibniz criterion for *sIp*: there exist x and y such that $sx = py$. If $x < p$ and $y < s$ this requirement is equivalent to gcd(s, p) > 1. The latter is equivalent to lcm(s, p) < sp. However we already know that the correct rule is lcm(s, p) < u. Indeed, the last example clearly shows the difference between trivial and appropriate calculations. For example, there is a colour containing both *red* (3) and *blue* (5) and this is *purple* (3·5). However we cannot consider this fact as a triviality because lcm(3, 5) < 2·3·5. The *composition* of two notions a and b (the conjunction of properties such as *rational animal*) always is ab in the intensional semantics (with the obvious limitation $cc = c$ for any factor c). The problem is in the existence of ab, i. e., in the truthfulness of *aIb*. The multiplication 3·2 = 6 will not guarantee by itself that *rational animal* exists. Otherwise the fact that 6 is divisible by 3 with the same success would lead to the conclusion that "All flying men are flying" and "All square triangles are square" as well. In order to distinguish between the logically possible objects (e. g. *angels*) and the self-contradictory ones, a universe u is necessary to distinguish the case $ab < u$ from $ab = u$. This distinction has not been pursued by Leibniz.