Non-Classical Operations Hidden in Classical Logic

Vladimir Sotirov*

* *Institute of Mathematics and Informatics,*
Bulgarian Academy of Sciences,
8, Acad. G. Bonchev Str.,
1113 Sofia, Bulgaria

vlsot@math.bas.bg,
http://www.math.bas.bg/˜vlsot

ABSTRACT. *Objects of consideration are various non-classical connectives "hidden" in the classical logic in the form of $G \circ s$ with $\circ$ — a classical connective, and s — a propositional variable. One of them is negation, which is defined as $G \Rightarrow s$; another is necessity, which is defined as $G \wedge s$. The new operations are axiomatized and it is shown that they belong to the 4–valued logic of Łukasiewicz.A 2–point Kripke semantics is built leading directly to the 4-valued logical tables.*

KEYWORDS: *negation, necessity, 4–valued logic, Kripke semantics*

Dedication

In the issue of this Journal, which was dedicated to the memory of George Gargov, Johan van Benthem mentioned what he called "Sofia school of modal logic". Indeed, many Bulgarian logicians have been successfully working in the field of modal and non-classical logics for 40 years until now. Prof. Dimiter Vakarelov is at the foundations of this school. He taught and tutored dozens of Masters and Ph.D. students of mathematical logic. Some of his colleagues — as, for instance, the late George Gargov and myself, worked or have worked in close collaboration with him. Prof. Vakarelov is still generating logical ideas and is pleased to watch these ideas being realized by his disciples. At the same time, I would like to note his capability to make mathematics of everything, be it philosophical ideas or puzzles. During the most dogmatic period of Bulgaria, he published a logical theory of dogma. One of his books was devoted to the large variety of permutation toys like Rubik's cube; and the complete solution gave rise to new theorems of the theory of groups. Another aspect of his work that has always impressed me is the clarity and surprising naturalness of his ideas, even *when emerging from the most sophisticated demonstrations. Leaving aside the technicalities of the proofs, every logical invention of Prof. Vakarelov could be explained to the non-specialist. A popular presentation of his achievements in logic, algebra, topology, geometry, and philosophy, would be interesting and fascinating reading.*

Long life to you, Mitko, and to your logical school!

1. Preliminary and Historical Notes

"Hidden" are logical operations, which do not occur explicitly in a certain logical system but can be defined within it. For example, the disjunction is "hidden" in the pure implicational calculus because it can be expressed in it by $A \vee B = (A \Rightarrow B) \Rightarrow B$. We will denote the basic systems by listing the main elements of their signature. So $\langle\Rightarrow\rangle$ is the system of the classical implication, $\langle\Rightarrow, \wedge\rangle$ is that of implication plus conjunction, etc.

Maybe Ch. Peirce in the 1880-s was the first to observe that implication involves many properties of negation if any subformula of the form $X \Rightarrow s$, where s is a distinguished propositional letter, is interpreted as $\sim X$. Let us denote the negation so obtained by $\sim(\Rightarrow, s)$ (read "negation produced by implication and a propositional variable s"). When it is clear from the context, only the sign $\sim$ will be used. A few examples: one of the transitivity laws $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ produces one of the laws of contraposition $(A \Rightarrow B) \Rightarrow (\sim B \Rightarrow \sim A)$ after replacing C with s. In the same manner, the law of commutativity of the antecedents $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ gives another law of contraposition $(A \Rightarrow \sim B) \Rightarrow (B \Rightarrow \sim A)$. Frege's law of self-distrubutivity $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ gives a form of *reductio ad absurdum*: $(A \Rightarrow \sim B) \Rightarrow ((A \Rightarrow B) \Rightarrow \sim A)$. *Modus ponens* presented by the formula $A \Rightarrow ((A \Rightarrow C) \Rightarrow C)$ gives the weak law of double negation $A \Rightarrow \sim\sim A$. The law of reduction $((A \Rightarrow (A \Rightarrow C)) \Rightarrow (A \Rightarrow C)$ gives a variant of the Clavius law, $(A \Rightarrow \sim A) \Rightarrow \sim A$. Further on, the Peirce law $((A \Rightarrow C) \Rightarrow A) \Rightarrow A$ gives the other Clavius law, or *consequentia mirabilis* $(\sim A \Rightarrow A) \Rightarrow A$. Sometimes different laws can be derived: the tautology $((A \Rightarrow C) \Rightarrow B) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ gives both the law of alternative $(\sim A \Rightarrow B) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ and the law $\sim (A \Rightarrow C) \Rightarrow \sim\sim A$.

It is reasonable to ask: which are the axioms of $\sim(\Rightarrow, s)$? In other words, what is the volume of the negation "hidden" within the classical implication? It includes a part of the intuitionistic negation, but not all because it does not satisfy $\sim A \Rightarrow (A \Rightarrow B)$. On the other hand, the new negation is not part of intuitionistic negation because $(\sim A \Rightarrow A) \Rightarrow A$ is not an intuitionistic law. If disjunction were present, even the law of excluded middle $A \vee \sim A$ would be on the list. Where is the exact place of $\sim(\Rightarrow, s)$ on the scale of known negations?

Of course, we have to define precisely the possible occurrences of the letter s in a tautology. Obviously we shall not interpret $s \Rightarrow s$ as $\sim s$. Perhaps it would be best not to touch such subformulas at all. But, what to do with the tautology $s \Rightarrow (A \Rightarrow s)$?

One way is to restrict the consideration to formulas not containing the letter s as an antecedent (except the case of $s \Rightarrow s$). A second way is to "incorporate" $s \Rightarrow$ into the paradigm of $\Rightarrow s$. This means to reduce any expression of the first kind to an expression of the second kind. Both variants will be regarded.

Let us shortly retrace the history of the "hidden" negation and the main related results. According to [Prior 62] (Part 1, Ch. 3, §1), the idea to use "maximum inference" and "minimum something else" for building up the whole propositional calculus was promoted by Peirce in 1885 and implemented by M. Wajsberg in 1937. The latter introduced the constant f ("falsehood"), defined $\sim X$ by $X \Rightarrow f$, added a new axiom $f \Rightarrow A$ to the axioms of implication, and obtained the full classical propositional calculus in the form $\langle \Rightarrow, f \Rightarrow A \rangle$. The alternative way for introducing negation was the axiomatic one. Then the full classical propositional calculus was obtained again but in the form $\langle \Rightarrow, \sim \rangle$. The two systems are not equivalent because the language of the first one is richer: it contains a *constant* while the second language contains *variables* only.

Nevertheless they are *equipollent*. A. Church devoted to this question three pages of his eminent book [Church 56] (§23). That is why we will not discuss this topic in detail. Given a formula B of $\langle \Rightarrow, \sim \rangle$, after replacing all its subformulas of the kind $\sim C$ with $C \Rightarrow f$, its representative B_f in $\langle \Rightarrow, f \Rightarrow A \rangle$ will be obtained. Then it can be proved that B is a theorem of $\langle \Rightarrow, \sim \rangle$ iff B_f is a theorem of $\langle \Rightarrow, f \Rightarrow A \rangle$. In this sense we may say that the negation $\sim (\Rightarrow, f)$ is *axiomatizable* by the axioms of $\langle \Rightarrow, \sim \rangle$.

Earlier on, A. Kolmogorov [Kolmogorov 25] and I. Johansson [Johansson 37] defined "minimal" negation in two ways. The first one was by adding a propositional constant f to the syntax of the positive implication $\Rightarrow^+$ without special axioms for it. The negation $\sim X$ was an abbreviation of $X \Rightarrow^+ f$. We will denote this form of negation by $\sim (\Rightarrow^+, f)$. The second way was axiomatic and used the single axiom Ax: $(A \Rightarrow \sim B) \Rightarrow (B \Rightarrow \sim A)$. Denote this form of negation by $\sim (\Rightarrow^+, Ax)$. The two systems are equipollent again.

H. Curry [Curry 63] (Ch. 6, Sec. C, §6) noticed that Johansson had introduced also a system denoted by HD which was an extension of $\langle \Rightarrow^+, Ax \rangle$ by the additional axiom $(\sim A \Rightarrow A) \Rightarrow A$. Curry judged that "no applications are known for HD, and the system has been little studied. Johansson [...] suggested that it formed a natural system of strict implication, but this has not been worked out". Adding the Peirce law to HD, Johansson obtained the system HE and proved that the axiom $(\sim A \Rightarrow A) \Rightarrow A$ is superfluous in it. As we see, $HE = \langle \Rightarrow, Ax \rangle$ and is equipollent with $\langle \Rightarrow, f \rangle$.

S. Kanger in [Kanger 55] also considered the system $\langle \Rightarrow, Ax \rangle$ and noted that it was "a weakened classical calculus in the same sense as the minimal calculus is a weakened intuitionistic calculus". He obviously had in mind the axiom $\sim A \Rightarrow (A \Rightarrow B)$ which, when added to $\langle \Rightarrow, Ax \rangle$, produces the classical calculus, and, when added to the minimal calculus, produces the intuitionistic one. Kanger proved that each formula of $\langle \Rightarrow, Ax \rangle$ has a representative A^* in $\langle \Rightarrow \rangle$ obtained by replacing all subformulas of the form $\sim B$ with $B \Rightarrow s$ (s is a variable occurring neither in B nor in the axioms of $\langle \Rightarrow, Ax \rangle$). Then, A is a theorem of $\langle \Rightarrow, Ax \rangle$ $(\mathrm{resp.}\langle \Rightarrow^+, \mathrm{Ax}\rangle)$ iff A^* is a theorem of the classical (intuitionistic) implicational calculus $\langle \Rightarrow \rangle$ $(\mathrm{resp.}\langle \Rightarrow^+ \rangle)$.

It will be shown below that the answer of our question about $\sim$ $(\Rightarrow, s)$ is: *its adequate axiom is* Ax. This means that the two systems, $\langle \Rightarrow, Ax \rangle$ and $\langle \Rightarrow \rangle$, are equipollent. However, this fact cannot be immediately derived from the equipollency of $\langle \Rightarrow, Ax \rangle$ and $\langle \Rightarrow, f \rangle$. It is not correct to transfer the operations with a *constant* to operations with a *variable*, even with an "arbitrary but fixed" one. The syntax of $\langle \Rightarrow \rangle$ is weaker than the syntax of $\langle \Rightarrow, f \rangle$ because the constant f is not definable in $\langle \Rightarrow \rangle$. Furthermore, the system $\langle \Rightarrow, f \rangle$ contains theorems such as $f \Rightarrow\sim A$, which is obtained from $B \Rightarrow (A \Rightarrow B)$ but has no analogue in $\langle \Rightarrow \rangle$. Therefore a special proof for $\sim$ $(\Rightarrow, s)$ is needed and it cannot be the proof of Kanger. He interprets $\langle \Rightarrow, Ax \rangle$ into $\langle \Rightarrow \rangle$ but we need an interpretation in the opposite direction.

A short exposition of some results concerning "hidden" negation were presented in [Sotirov 01].

2. The "hidden" negation

In a two-part paper published in 1967–1968 [Vakarelov 65], [Vakarelov 66], D. Vakarelov investigated various aspects of some kinds of modalities and negations added to the classical propositional logic, some of them "hidden" according to our terminology. He axiomatically introduced two unary operators: D ("doubtful") and L ("verisimilar"). The axiom schemes (beside those of the classical implication) were the following five:

$L(LA \Rightarrow A)$;

$(A \Rightarrow B) \Rightarrow (LA \Rightarrow LB)$;

$(A \Rightarrow B) \Rightarrow (DB \Rightarrow DA)$;

$(LA \Rightarrow B) \Rightarrow DDB$;

$DA \Rightarrow (A \Rightarrow (LB \Rightarrow B))$.

Let us denote this system by $\langle \Rightarrow, L, D \rangle$. Afterwards Vakarelov built a translation φ_s of formulas of $\langle \Rightarrow, L, D \rangle$ into formulas of $\langle \Rightarrow \rangle$ using a propositional letter s, which does not occur in the given formulas, by induction on the construction of formulas:

$\varphi_s(X) = X$ when X is a propositional letter;

$\varphi_s(LX) = s \Rightarrow \varphi_s(X)$ and $\varphi_s(DX) = \varphi_s(X) \Rightarrow s$;

$\varphi_s(X \Rightarrow Y) = \varphi_s(X) \Rightarrow \varphi_s(Y)$.

A lemma follows: if A is a theorem of $\langle\Rightarrow, L, D\rangle$ then $\varphi_s(A)$ is a theorem of $\langle\Rightarrow\rangle$. For the converse proposition Vakarelov defined a translation ψ_s from $\langle\Rightarrow\rangle$ into $\langle\Rightarrow, L, D\rangle$ by induction on the construction of formulas:

$\psi_s(X) = X$ for X — a propositional variable or the formula $s \Rightarrow s$;

$\psi_s(s \Rightarrow X) = L\psi_s(X)$ and $\psi_s(X \Rightarrow s) = D\psi_s(X)$ for $X \neq s$;

$\psi_s(X \Rightarrow Y) = \psi_s(X) \Rightarrow \psi_s(Y)$ for $X, Y \neq s$.

A next lemma follows: if A is a theorem of $\langle\Rightarrow\rangle$ then $\psi_s(A)$ is a theorem of $\langle\Rightarrow, L, D\rangle$. Combining both lemmas and using $\psi_s(\varphi_s(A)) = A$, a theorem is obtained: a formula A is provable in $\langle\Rightarrow, L, D\rangle$ iff $\varphi_s(A)$ is provable in $\langle\Rightarrow\rangle$. (This theorem provides the decidability of $\langle\Rightarrow, L, D\rangle$.) Therefore $\langle\Rightarrow, L, D\rangle$ and $\langle\Rightarrow\rangle$ are equipollent.

One can count the result of Vakarelov as close to the answer of our question because his D is our $\sim (\Rightarrow, s)$. Two peculiarities of his exposition however prevented the direct answer. The first one is the "direction" of the view: Vakarelov transforms the theorems of $\langle\Rightarrow, L, D\rangle$ into $\langle\Rightarrow\rangle$ while we want to transform the theorems of $\langle\Rightarrow\rangle$ into…And here is the second peculiarity: the system of Vakarelov includes L besides $D = \sim$. The two operations interact in axioms and it is impossible to separate them. To reverse the direction of the view an inverse equality is needed: $\varphi_s(\psi_s(A)) = A$. Then the definition of φ_s must be modified with the permission for s to occur in A. The demonstration is almost the same. As a result, the following theorem is obtained:

THEOREM 1. — *A formula A is a theorem of $\langle\Rightarrow\rangle$ iff $\psi_s(A)$ is a theorem of $\langle\Rightarrow, L, D\rangle$.*

Roughly speaking, the system of L and D axiomatizes $s \Rightarrow$ and $\Rightarrow s$. However, the inconvenience with the appearance of L remains. Indeed, we could benefit from this inconvenience because it gives an answer to the question: what are the axioms characterizing the two operations generated by the two positions of the letter s in the implication? The axioms are those of L and D. However, if we insist on the unique operation generated by $\Rightarrow s$, the second part of Vakarelov's paper provides a solution. Following the spirit of the paper, we shall change some denotations and shorten the proof.

LX can be represented as $DT \Rightarrow X$ with T — a fixed theorem. This is the mentioned above incorporating of the expression $s \Rightarrow$ into an expression containing only $\Rightarrow s$. In addition, replace D with $\sim$. Denote this transformation by λ. It is easy to show that the axioms of $\langle\Rightarrow, L, D\rangle$ so transformed can be deduced from Ax. On the other hand, commute $\sim$ to D in $\langle\Rightarrow, Ax\rangle$. Then Ax so transformed can be deduced in $\langle\Rightarrow, L, D\rangle$ (actually Vakarelov worked with the equivalent pair of axioms for negation $(A \Rightarrow B) \Rightarrow (\sim B \Rightarrow \sim A)$ and $A \Rightarrow \sim\sim A$ which are closer to his axioms of L and D). Hence

THEOREM 2. — *A formula A is a theorem of $\langle\Rightarrow, L, D\rangle$ iff $\lambda(A)$ is a theorem of $\langle\Rightarrow, Ax\rangle$.*

Combining the two theorems gives rise to the following

COROLLARY 3. — *A formula A is a theorem of $\langle\Rightarrow\rangle$ iff $\lambda(\psi_s(A))$ is a theorem of $\langle\Rightarrow, Ax\rangle$.*

The corollary is an expression of the fact that although $\langle\Rightarrow\rangle$ and $\langle\Rightarrow, Ax\rangle$ possess different syntactic capacity, they have equal semantic capacity having isomorphic sets of theorems. This is the precise sense in which we say that Ax *axiomatizes* the negation $\sim(\Rightarrow, s)$ "hidden" in $\langle\Rightarrow\rangle$.

The translation $\lambda\psi_s$ transforms a formula containing both $s\Rightarrow$ and $\Rightarrow s$ into a formula containing $\sim$. As we mentioned, the class of transformed formulas might be restricted to formulas without "left s", i. e., without subformulas of the form $s\Rightarrow X$ (except $s\Rightarrow s$). The corollary and the axiom of $\sim(\Rightarrow, s)$ would then be the same. However the proofs become much simpler.

LEMMA 4. — *If A is a theorem of $\langle\Rightarrow\rangle$ not containing left s, there exists a proof of A not containing left s.*

PROOF. — If $A_1, \ldots, A_n = A$ is the proof of A, replace each left s in the inference with $T\Rightarrow s$ (T is a fixed theorem). $A_1^*, \ldots, A_n^* = A^*$ is obtained. To be it an inference of A^*, it has to be filled out by the proofs of T, $(T\Rightarrow s)\Rightarrow s$, and $s\Rightarrow(T\Rightarrow s)$ together with the members providing the substitutivity of equivalent formulas. In result an inference of A^* will be obtained not containing left s. But it will be an inference of A as well because $A^* = A$. ∎

We shall modify the transformation φ_s of $\langle\Rightarrow, Ax\rangle$ into $\langle\Rightarrow\rangle$:

$\varphi_s(X) = X$ when X is a propositional letter;

$\varphi_s(\sim X) = \varphi_s(X)\Rightarrow s$;

$\varphi_s(X\Rightarrow Y) = \varphi_s(X)\Rightarrow\varphi_s(Y)$.

To prove that $\varphi_s(A)$ is a theorem of $\langle\Rightarrow\rangle$ when A is a theorem of $\langle\Rightarrow, Ax\rangle$ it is enough to prove $\varphi_s(Ax) = (A\Rightarrow(B\Rightarrow s))\Rightarrow(B\Rightarrow(A\Rightarrow s))$ but it is obvious.

Modify also the converse transformation for formulas not containing left s (except $s\Rightarrow s$):

$\psi_s(X) = X$ for X — a propositional variable or the formula $s\Rightarrow s$;

$\psi_s(X\Rightarrow s) = \sim\psi_s(X)$ for $X\neq s$;

$\psi_s(X\Rightarrow Y) = \psi_s(X)\Rightarrow\psi_s(Y)$ for $X, Y\neq s$.

To prove that $\psi_s(A)$ is a theorem of $\langle\Rightarrow, Ax\rangle$ when A is a theorem of $\langle\Rightarrow\rangle$, it is enough to prove all "s-readings" of the implicative axioms. We can choose them to be, say, $A\Rightarrow(B\Rightarrow A)$, $(A\Rightarrow B)\Rightarrow((B\Rightarrow C)\Rightarrow(A\Rightarrow C))$, and $((A\Rightarrow B)\Rightarrow A)\Rightarrow A$. Only the last two axioms have such readings and these readings are unique: $(A\Rightarrow B)\Rightarrow(\sim B\Rightarrow\sim A)$ and $(\sim A\Rightarrow A)\Rightarrow A$. It is not difficult to deduce them from Ax. Again $\varphi_s(\psi_s(A)) = A$. In such a way the following theorem is obtained:

THEOREM 5. — *A formula A containing right s only is a theorem of $\langle\Rightarrow\rangle$ iff $\psi_s(A)$ is a theorem of $\langle\Rightarrow, Ax\rangle$.*

Summarizing the results, we see that the weak law of contraposition characterizes axiomatically the full variety of negations "hidden" inside the implication, both the intuitionistic and the classical one. Rearranging the matter and changing the emphases, we find these results in the paper of Vakarelov. One can conclude that they are not surprising taking into account the constructions by Peirce, Johansson–Kolmogorov, Wajsberg, and Kanger. However, the next result of Vakarelov was really surprising.

3. 4-valued semantics of the "hidden" negation

Vakarelov found that the logic of the "hidden" negation coincided with the 4-valued modal logic of J. Łukasiewicz [Łukasiewicz 53] and therefore it possessed a truth-table semantics. But a new peculiarity appeared in the proof: it used the presence of the classical negation. To carry out the proof strictly in the system of the plain implication, the multiplication of truth-tables will be applied.

I do not know who invented this method. Łukasiewicz used it in his main paper of 1953 without references. We meet it in an earlier paper of 1950 by J. Kalicki [Kalicki 50]. H. Rasiowa in 1955 applied the same method with a reference to Kalicki and to a paper of 1936 by S. Jaśkowski as well. Kalicki himself called the matrix multiplication "well-known" and referred to a paper of 1935 by Wajsberg. And so on. . . Obviously it was folklore of the Polish logicians. Anyway, the multiplication of two tables successfully works in a case when a new propositional operation has to combine the properties of two given operations. I shall explore it to build a 4-valued table for "hidden" negation. Suppose M_1 and M_2 are the matrices of two binary operations, I_1 and I_2 are the matrices of two unary operations, and 1 is the designated truth-value. Denote by T_1 and T_2 the sets of tautologies produced by the operations with matrices M_1, I_1, and M_2, I_2respectively.

LEMMA 6 (KALICKI–ŁUKASIEWICZ). — *The Cartesian products $M_1 \times M_2$ and $I_1 \times I_2$ define operations whose set of tautologies is the intersection of T_1 and T_2, (1,1) being the designated truth-value "true".*

In the system of $\sim$, its origin (the string $\Rightarrow\ s$) suggests that the tautologies of $\langle\Rightarrow, \sim\rangle$ coincide with those formulas which are tautologies when $\sim$ is treated both as a (classical) negation $\neg$ (in the case $s = 0$) and the constant "truth" t (in the case $s = 1$). The same can be observed in the fact that the axiom of $\sim$ remains true when $\sim$ is replaced both with $\neg$ and t. It will be convenient for the further to denote by $L(A) = (l(A), r(A))$ the value of A in the set of pairs of 0 and 1. Denote also by $\rightarrow$ and & the non-formal implication and conjunction expressed either by words or by numbers. Multiplying the corresponding 2-valued matrices of implication (obtaining the Cartesian square) as well as the matrices of $\neg$ and t, the 4-valued operations $\Rightarrow$ and $\sim$ are defined by $L(B \Rightarrow C) = (l(B \to C), r(B \to C)) = (l(B) \to l(C), r(B) \to r(C)) = (1 - l_B + l_B l_C, 1 - r_B + r_B r_C)$, where for short $l(B) = l_B$, etc. $L(\sim B) = (1 - l_B, 1)$. Numbering the pairs from (1,1) to (0, 0) with the figures from 1 to 4, the operations shape into Table 1:

$\Rightarrow$	1 2 3 4	$\sim$	$\wr$	$\neg$
1	1 2 3 4	3	2	4
2	1 1 3 3	3	1	3
3	1 2 1 2	1	2	2
4	1 1 1 1	1	1	1

Table 1

$\wedge$	1 2 3 4	$\square$	$\blacksquare$
1	1 2 3 4	2	3
2	2 2 4 4	2	4
3	3 4 3 4	4	3
4	4 4 4 4	4	4

Table 2

REMARK. — $\neg$ is the classical negation introduced with $L(\neg B) = (1 - l_B, 1 - r_B)$; $\wr$ is coming to be considered.

THEOREM 7. — *A formula A containing right s only is a theorem of $\langle \Rightarrow \rangle$ iff $\psi_s(A)$ is a theorem of $\langle \Rightarrow, Ax \rangle$.*

THEOREM 8. — *The system $\langle \Rightarrow, Ax \rangle$ is characterized by the 4-valued matrix.*

The proof is given by Lemma 6 and the construction of the matrices for $\Rightarrow$ and $\sim$.

We can notice after Łukasiewicz that the order of the coordinates of $(1 - l_B, 1)$ was of no importance and could be reversed. In that case a new operation would be obtained with the same grounds to be called "negation" as $\sim$. We denote it by $\wr$ and its values are shown in the table. Łukasiewicz observed a curious fact: there are "twins" in his logic — operations, which have the same properties when regarded separately, but different when they appear together. In our setting $\sim$ and $\wr$ are "twins". The symmetry between them can be derived also from the fact that both of them satisfy the axiom of the negation. As a result they take part in the same tautologies. However they are not equivalent.

The "twins" help us construct a new intuitive semantics, other than that of Łukasiewicz. Suppose the world is divided in two parts according to some criterion. To give some names, let us call them "here" and "there". Our sentences about the two "worlds" are classical, i. e., they are true or false but with an indication *where* they are true or false — "here", or "there". If, for example, it rains here, but not there, the situation is determined by the couple (1, 0), shortly denoted by 2. If it is cold there but not here, the situation is $(0, 1) = 3$. In such a case, is the sentence "If it rains, it's cold" true? It depends: the implication is false here but not there. Therefore its truth-value is $(0, 1) = 3$. In such a way the whole matrix of the implication can be filled up. It is natural to introduce a "global" negation $\neg$ inverting the truth-values by coordinates: the negation of "It rains here but not there" (1, 0) will be "On the contrary, it doesn't rain here but there" (0, 1). Imagine further that in the two halves of the world are two extremely dogmatic sects each of them denying every statement about "that" world but confirming as a truth everything concerning their own world.

So two "local" negations are obtained, $\sim$ and $\wr$. As we see, the 4-valued logic can serve in this manner the "bi-polar" thinking.

4. The "hidden" necessity

Now we turn to the system $\langle\Rightarrow,\wedge\rangle$, axiomatized as usual. This time the occurrence of a propositional letter s inside a conjunction will be read as a necessity $\Box$. For example, $(A\wedge C)\Rightarrow A$ gives $\Box A\Rightarrow A$ after replacing C with s . It is not difficult to realize that well-known tautologies produce some of the most popular laws of necessity, e. g., $\Box A\wedge\Box B\Rightarrow\Box(A\wedge B)$ together with the converse implication, $\Box A\Rightarrow\Box\Box A$, $\Box(A\Rightarrow B)\Rightarrow(\Box A\Rightarrow\Box B)$ and the stronger law $(A\Rightarrow B)\Rightarrow(\Box A\Rightarrow\Box B)$, and so on. Especially the standard representation of $\Box A$ as $A\wedge\Box T$ (T is a fixed theorem) will be used. The most important law missing in our list is $\Box T$ with T — a theorem. Respectively, the rule of necessitation inferring $\Box A$ from A is unsound, too. Because the conjunction is symmetric, we have only one possibility to define the necessity by s. At the same time we have to take care of some bad instances like $s\Rightarrow(s\wedge s)$, $(A\wedge s)\Rightarrow s$ or even $A\wedge(s\wedge s)\Rightarrow A\wedge s$.

It will be shown that the axioms of the "hidden" necessity are $(A\Rightarrow B)\Rightarrow(\Box A\Rightarrow\Box B)$ and $\Box A\Rightarrow A$. Denote this system by $\langle\Rightarrow,\wedge,\Box\rangle$. It is known as the Ł-modal logic of Łukasiewicz although his original axiom includes a variable functor δ and therefore has wider expressive capability. Łukasiewicz refers to an unpublished paper by Wajsberg about the completeness theorem with regard to the 4-valued truth tables. However the first published proof is that of T. Smiley [Smiley 61]. The adequacy of the 4-valued matrices for the plain $\Box$-axioms together with the s-interpretation of $\Box$ can be extracted from the second paper by Vakarelov because his L and D are inter-definable with $\Box$ (using $\neg$). His proofs are purely syntactic. Ten years later the paper [Porte 79] by J. Porte appeared. It contained the adequacy of the s-interpretation for the 4-valued matrices (in fact Porte used a constant instead of the variable s). His proof applied multiplication of matrices and obviously was independent of Vakarelov's papers.

The procedures of the previous sections will not be given in detail but only the main points will be marked. Of course, it would be elementary to reduce the modality to the "hidden" negation using the classical negation, but we prefer not to introduce additional operations.

Define a translation φ_s of a formula of $\langle\Rightarrow,\Box\rangle$ into $\langle\Rightarrow,\wedge\rangle$ using a propositional letter s, according to the construction of the formulas:

$\varphi_s(X)=X$ when X is a propositional letter;

$\varphi_s(\Box X)=\varphi_s(X)\wedge s$ for $X\neq T$ (T is a fixed theorem);

$\varphi_s(\Box T)=s$ for T — the fixed theorem;

$\varphi_s(X\wedge Y)=\varphi_s(X)\wedge\varphi_s(Y)$;

$\varphi_s(X \Rightarrow Y) = \varphi_s(X) \Rightarrow \varphi_s(Y)$.

In such a way any theorem of $\langle\Rightarrow, \Box\rangle$ is translated into a theorem of $\langle\Rightarrow, \wedge\rangle$ because this is true for the axioms of $\langle\Rightarrow, \Box\rangle$, as we saw in our examples. For the converse translation ψ_s we have two possibilities. If s occurs somewhere in A out of conjunctions or in $s \wedge s$, then $\psi_s(A)$ could be $A(s/\Box T)$, T — a fixed theorem. It is the simplest way but then theorems like $(s \Rightarrow (s \wedge s)) \Rightarrow ((A \wedge s) \Rightarrow A)$ would lose their more informative image of the form $G \Rightarrow (\Box A \Rightarrow A)$ with an appropriate G. That is why we prefer to obtain a $\Box$-image which is maximally close to the original. The definition of ψ_s follows (the manipulation over T is not specially described):

$\psi_s(X) = X$ for X — a propositional variable or $s \Rightarrow s$;

$\psi_s(s \wedge X) = \psi_s(X \wedge s) = \Box\psi_s(X)$ for $X \neq s$;

$\psi_s(s \wedge s) = \Box T$;

$\psi_s(X \wedge Y) = \psi_s(X) \wedge \psi_s(Y)$ for $X, Y \neq s$;

$\psi_s(s \Rightarrow Y) = \Box T \Rightarrow \psi_s(Y)$ for $Y \neq s$;

$\psi_s(X \Rightarrow s) = \psi_s(X) \Rightarrow \Box T$ for $X \neq s$;

$\psi_s(X \Rightarrow Y) = \psi_s(X) \Rightarrow \psi_s(Y)$ for $X, Y \neq s$.

It has to be proven that if A is a theorem of $\langle\Rightarrow, \wedge\rangle$, then $\psi_s(A)$ is a theorem of $\langle\Rightarrow, \wedge, \Box\rangle$. Consider firstly the axioms. The axioms of implication do not contain $\wedge$ and come under the last case of the definition. The result is obvious. If the axiom is $(A \wedge B) \Rightarrow A$, the possible translations are $(\Box T \wedge \Box T) \Rightarrow \Box T$, $(\Box T \wedge \psi_s(B)) \Rightarrow \Box T, \Box\psi_s(A) \Rightarrow \Box T$, and $(\Box\psi_s(A) \wedge \Box\psi_s(B)) \Rightarrow \Box\psi_s(A)$ for $A, B \neq s$ (for the third formula use the standard representation). All three formulas are theorems of $\langle\Rightarrow, \Box\rangle$. The case of $(A \wedge B) \Rightarrow B$ is the same. The axiom $A \Rightarrow (B \Rightarrow (A \wedge B)$ produces $\Box T \Rightarrow (\Box T \Rightarrow \Box T)$, $\Box T \Rightarrow (\psi_s(B) \Rightarrow \Box\psi_s(B))$, $\psi_s(A) \Rightarrow (\Box T \Rightarrow \Box\psi_s(A))$, and $\psi_s(A) \Rightarrow (\psi_s(B) \Rightarrow (\psi_s(A) \wedge \psi_s(B))$ where $A, B \neq s$; the standard representation is needed again. To check that modus ponens preserves deducibility is not difficult because A and B in $A \Rightarrow B$ cannot be s. Therefore a theorem $\psi_s(B)$ is obtained from $\psi_s(A) \Rightarrow \psi_s(B)$ and $\psi_s(A)$ which are theorems by assumption. This time $\varphi_s(\psi_s(A))$ is *equivalent* to A because the strict coincidence of both formulas would require too complicated rules for φ_s and ψ_s. Therefore the system $\langle\Rightarrow, \wedge, \Box\rangle$ may be considered as an axiomatization of the necessity "hidden" in the system $\langle\Rightarrow, \wedge\rangle$:

THEOREM 9. — *A formula A is a theorem of $\langle\Rightarrow, \wedge\rangle$ iff $\psi_s(A)$ is a theorem of $\langle\Rightarrow, \wedge, \Box\rangle$.*

5. 4-valued semantics of the "hidden" necessity

To obtain a 4-valued matrix adequate for the necessity, almost the same observation can be made as in the case of negation. Following Łukasiewicz, note that the axioms of $\Box$ remain true when $\Box$ is replaced with both the constant "falsehood" and the operator of identity. The same conclusion comes from the nature of $\Box$: because $\Box A$ is generated from $A \wedge s$, $\Box A \equiv 0$ is obtained in the case of $s = 0$, and $\Box A \equiv A$ in the case of $s = 1$.

We find a third reason for such a treatment in the fundamental paper by S. Kripke [Kripke 65]. He deduces $\neg\Box A \vee (B \Rightarrow \Box B)$ as a theorem (in the system with $\neg$) where A and B may be assumed not to have variables in common. Then the theorems of $\langle\Rightarrow, \wedge, \Box\rangle$ are exactly those formulas, which are theorems simultaneously of the two systems obtained by adding respectively $\neg\Box A$ and $B \Rightarrow \Box B$ as a new axiom. The first additional axiom reduces $\Box$ to the operator "falsehood" and the second one reduces $\Box A$ to A. In other words, the system $\langle\Rightarrow, \wedge, \Box\rangle$ is the intersection of the *Falsum* and the *Trivial* systems. The last fact was first noticed by A. Prior [Prior 57] (Ch. 1). He appreciated it as an additional motivation for the two-component necessity. "Necessarily p, Prior proclaims, on no account asserts *less* than that p is actually true, and never asserts *more* than that p is at once true and false (for this last is a kind of upper limit to all assertions — if you'd believe that you'd believe anything)". His conclusion is that Łukasiewicz's logic is the logic of modalities each of them covering the maximal part of the natural spectrum of truthfulness.

Anyway the multiplication of the two 2×2 matrices for $\wedge$ gives $L(B \wedge C) = (l_B \& l_C, r_B \& r_C) = (l_B l_C, r_B r_C)$. The matrix for $\Box$ is obtained by $L(\Box B) = (l_B, 0)$. The inverse disposition $(0, r_B)$ gives $\blacksquare$, the "twin" of $\Box$. The results of renaming the pairs of 0 and 1 are presented in Table 2.

THEOREM 10. — *The system* $\langle\Rightarrow, \wedge, \Box\rangle$ *is characterized by the 4-valued matrix.*

The proof follows from Lemma 6 and the construction of the matrices for $\Rightarrow, \wedge$, and $\Box$.

6. Kripke semantics of the "hidden" operations

What is the Kripke semantics for $\langle\Rightarrow, \wedge, \Box\rangle$? Does any ready to use result exist? Since $\Box(A \wedge B) \equiv (\Box A \wedge \Box B)$ is a theorem and $\Box T$ is not, the appropriate Kripke semantics should be searched among those ones which Kripke called "non-normal" [Kripke 65]. They include frames of the kind $\langle W, N, R\rangle$ where $W \neq \emptyset$ is the set of "possible worlds", $N \subseteq W$ is the set of "normal" worlds, and R is an "accessibility relation" in W. A valuation V is a function from worlds and propositional letters to truth-values 0 and 1. $V(p)$ is the whole "map of truthfulness" of p in the Universe of possible worlds: $V(p) = (v(x_1, p), v(x_2, p), \ldots)$ where $v(x_i, p)$ is the truth-value of p in x_i. When $v(x_i, p) = 1$ we write $x_i \Vdash p$; otherwise we write $x_i \nVdash p$, omitting the letter v. The valuation is extended to arbitrary formulas by induction on their construction: $x \Vdash (A \Rightarrow B)$ iff $x \Vdash A$ implies $x \Vdash B$; $x \Vdash (A \wedge B)$ iff $x \Vdash A$ and $x \Vdash B$; $x \Vdash \Box A$ iff both $x \in N$ and $y \Vdash A$ for all y such that xRy. Adding new axioms of $\Box$, corresponding additional conditions on R and N arise. So the first axiom of necessity, $\Box A \Rightarrow A$ has as a corresponding condition the reflexivity of R inside N.

The second axiom, $(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$ has a corresponding condition: R is a subrelation of identity, i. e., if xRy then $x = y$. The completeness theorem holds: a formula P is a theorem of $\langle \Rightarrow, \wedge, \Box \rangle$ iff P is valid in every world and any valuation. The adequacy of this semantics is described by J. Font and P. Hájek [Font, Hájek 02] with a reference to E. Lemmon [Lemmon 66].

The "hidden" negation was defined on the basis of the classical implication and it would be natural to look for semantics of the classical logic extended by an additional operation fulfilling Ax. A big quantity of negations with different basic logics were studied in the Ph. D. dissertation of Vakarelov [Vakarelov 77]. Unfortunately it is difficult of access. K. Segerberg [Segerberg 68] built Kripke semantics for many extensions of the minimal calculus of Kolmogorov–Johansson. To serve our logic, the minimal calculus has to be restricted to implication (without conjunction, etc.) but extended with the constant f. The Kripke frames for positive implication are of the form $\langle W, N, R \rangle$ where R is a reflexive and transitive relation and N is downward-closed: if xRy and $y \in N$ then $x \in N$. Valuation is closed under R: for any propositional variable, if $x \Vdash p$ and xRy then $y \Vdash p$. For the constant f we have $x \nVdash f$ for all x. The valuation is extended to arbitrary formulas by induction on their construction: $x \Vdash (A \Rightarrow B)$ iff for all y, xRy and $y \Vdash A$ imply $y \Vdash B$; $x \Vdash \sim A$ iff $x \Vdash (A \Rightarrow f)$ iff for all y, $y \in N$ and xRy imply $y \nVdash A$. In such a way, an adequate Kripke semantics of $\langle \Rightarrow^{+}, f \rangle$ is obtained. To obtain $\langle \Rightarrow, f \rangle$ from $\langle \Rightarrow^{+}, f \rangle$, positive implication has to be expanded to classical implication by adding, e. g., the Peirce law $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$. Then the corresponding condition for R is: R is a subrelation of identity. This is the system that Segerberg denoted by JP. The completeness theorem for $\langle \Rightarrow, f \rangle$ holds with respect to such structures.

Two notes are to be made here. The first one concerns $\langle \Rightarrow, f \rangle$. This system is not $\langle \Rightarrow, Ax \rangle$ although it is equipollent to it, as we saw. That is why the proof of adequacy of the same semantics requires some additional considerations. We will not adduce them. The second note is that in our case the basic logic is implicative and no property of disjunction or negation can be used in the demonstrations. That is why the machinery of the maximal implicative filters in implicative algebras is appropriate. The theorems needed can be taken from the monograph of H. Rasiowa 14.

We prefer to give a direct construction of a new Kripke semantics, avoiding in such a way the involvement of the relation R. The standard procedures including "saturation" of filters will be omitted. The frames are of the form $\langle W, N \rangle$ where $W \neq \emptyset$ and $N \subseteq W$. $x \Vdash \Box A$ iff $x \in N$ and $x \Vdash A$. It is elementary to check the validity of the two axioms. Hence any theorem of $\langle \Rightarrow, \wedge, \Box \rangle$ is valid. In the opposite direction, it has to be proved that if A is not a theorem, it is refuted in an appropriate model. As usual, this model will be the canonical one. W consists of all maximal filters of formulas, $N = \{x \mid \Box T \in x\}$. For a propositional variable p, define $x \Vdash p$ iff $p \in x$ and extend the canonical valuation by induction to arbitrary formula. Then $x \Vdash B$ iff $B \in x$ for any formula B. The proof for $B = C \Rightarrow D$ is routine. The only new case is $B = \Box C$. Let $\Box C \in x$. Using the standard representation, $\Box C$ is equivalent to $C \wedge \Box T$. Therefore $(C \wedge \Box T) \in x$, whence $C \in x$ and $\Box T \in x$, and then, by the induction step, $x \Vdash C$ and $x \in N$. In the opposite direction: let $x \Vdash C$ and $x \in N$, i. e., $\Box T \in x$. Then $(C \wedge \Box T) \in x$, that is $\Box C \in x$. Finally, if A is not a theorem, there is a maximal filter x_0 such that $A \notin x_0$, therefore $x_0 \nVdash A$. In such a way we proved the main theorem of adequacy of Kripke semantics:

THEOREM 11. — *Kripke semantics with frames $\langle W, N\rangle$ such that $W \neq \emptyset$ and $N \subseteq W$, and valuation $x \Vdash \Box A$ iff $x \in N$ and $x \Vdash A$, is adequate for $\langle \Rightarrow, \wedge, \Box\rangle$.*

The corresponding theorem about negation is

THEOREM 12. — *Kripke semantics with frames $\langle W, N\rangle$ such that $W \neq \emptyset$ and $N \subseteq W$, and valuation $x \Vdash \sim A$ iff $x \in N$ implies $x \nVdash A$, is adequate for $\langle \Rightarrow, \sim\rangle$.*

PROOF. — Define $N = \{x \in W \mid \sim T \notin x\}$. The crucial moment is to prove the standard representation: $\sim C \in x$ iff $(C \Rightarrow \sim T) \in x$. If $\sim C \in x$, take $C \in x$. From the theorem $\sim C \Rightarrow (C \Rightarrow \sim T)$, $\sim T \in x$ follows. Then from $\sim T \Rightarrow (C \Rightarrow \sim T)$, $A \Rightarrow \sim T$ is obtained. Conversely, if $\sim C \notin x$, then the theorem $(C \Rightarrow \sim C) \Rightarrow \sim C$ gives $C \in x$ (it is the implicative analogue of the law of excluded middle used in Boolean filters). Further on, $C \Rightarrow T$ gives $\sim T \Rightarrow \sim C$. But $\sim C \notin x$, whence $\sim T \notin C$ and $(C \Rightarrow \sim T) \notin x$. ■

Along with its simplicity, this form of Kripke semantics has an important advantage: it leads directly to the 4-valued semantics.

THEOREM 13. — *The Kripke frame $\langle \{x_1, x_2\}, \{x_1\}\rangle$ (with $x_1 \neq x_2$) determines $\langle \Rightarrow, \sim\rangle$ ($\langle \Rightarrow, \wedge, \Box\rangle$).*

PROOF 14. — Of course, each 2-point Kripke model is a Kripke model. We shall prove that an arbitrary model mentioned in Theorem 11 can be reduced to a 2-point one. Let us consider a formula A, which is not a theorem and therefore is refutable at a point x_0 of a model $\langle W_0, N_0\rangle$ by a valuation of the variables v_0. Replace each subformula of the form $\Box B$ with $B \wedge \Box T$ (T is a theorem; in some cases $T \wedge \Box T$ will appear); denote the result by A^*. A^* is equivalent to A . In this semantics neither implication, nor conjunction, nor necessity introduce a new variable when their truth-value is being calculated. That is why x_0 is the only letter occurring in v_0. Therefore everywhere inside A^* we have the following two possibilities: either $\Box T$ is true at x_0 and then $x_0 \in N_0$, or $\Box T$ is false at x_0 and then $x_0 \notin N_0$. In the first case we can rename x_0 to x_1 and the second point in W_0 to x_2. In the second case we rename x_0 to x_2 and the second point in W_0 to x_1. As we see, now A^* will be refuted in a Kripke model $\langle \{x_1, x_2\}, \{x_1\}\rangle$ containing no more then 2 points. The same reasoning is applicable to $\langle \Rightarrow, \sim\rangle$. ■

REMARK. — $\sim T$ and $\Box T$ are not interrelated because of the lack of classical negation. That is why N cannot be common both for $\langle \Rightarrow, \sim\rangle$ and $\langle \Rightarrow, \wedge, \Box\rangle$ (in the general case). In other words, the theorem cannot be formulated for $\langle \Rightarrow, \wedge, \sim, \Box\rangle$.

Any valuation of a formula A in the 2-point Kripke semantics can be presented as a valuation L in the 4-valued logic in such a manner that $V(A) = L(A)$. To prove this recall the notations introduced for 4-valuations and use induction on the construction of A. For a propositional variable p define $l(p)$ and $r(p)$: $V(p) = (v(x_1, p), v(x_2, p)) = (l(p), r(p)) = L(p)$. Suppose $V(B) = (v(x_1, B), v(x_2, B)) = (l(B), r(B)) = L(B)$ and analogously for C. Then we obtain for $\Rightarrow$:

$$V(B \Rightarrow C) = (v(x_1, B \Rightarrow C), v(x_2, B \Rightarrow C)) =$$
$$(v(x_1, B) \rightarrow v(x_1, C), v(x_2, B) \rightarrow v(x_2, C)) = (l(B) \rightarrow l(C), r(B) \rightarrow r(C)) =$$
$$(l(B \Rightarrow C), r(B \Rightarrow C)) = L(B \Rightarrow C).$$

The chain of equalities for $\wedge$ is analogous. For $\sim$:

$$V(\sim B) = V(B \Rightarrow \sim T) = V(B) \rightarrow V(\sim T) =$$
$$(v(x_1, B), v(x_2, B)) \rightarrow (v(x_1, \sim T), v(x_2, \sim T)) =$$
$$(v(x_1, B), v(x_2, B)) \rightarrow (0, 1) = (1 - v(x_1, B), 1) = (1 - l(B), 1) = L(\sim B).$$

Finally, for $\Box$:

$$V(\Box B) = V(B \wedge \Box T) = V(B) \& V(\Box T) =$$
$$(v(x_1, B), v(x_2, B)) \& (v(x_1, \Box T), v(x_2, \Box T)) =$$
$$(v(x_1, B), v(x_2, B)) \& (1, 0) = (v(x_1, B), 0) = (l(B), 0) = L(B).$$

In such a way we proved the following

THEOREM 15. — *Any 2-point Kripke model generates the 4-valued tables of Łukasiewicz.*

COROLLARY 16. — *A formula A is a theorem of $\langle \Rightarrow, \sim \rangle$ ($\langle \Rightarrow, \wedge, \Box \rangle$) iff it is a 4-valued tautology.*

This is a new proof of the completeness theorem of the 4-valued logic avoiding the presence of classical negation. It also expresses the connection between the four truth-values of Łukasiewicz's logic and the distribution of the truthfulness between two possible parts of the human knowledge: one of them recognizing all that is actually happening as necessary, the other completely indeterministic and denying any necessity.

7. Concluding remarks

From the point of view of Kripke semantics — a point naturally unknown to Łukasiewicz, the operation $s \Rightarrow$ (Vakarelov's L) is a better candidate for "necessity" fulfilling the basic axioms for the "normal" Kripke semantics: $\Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$ together with $\Box T$ for T — a theorem. However the necessity so obtained possesses the implausible property $A \Rightarrow \Box A$ which is typical for the "possibility".

I will neither criticize nor justify Łukasiewicz's 4-valued logic. Most *pros* and *cons* are summarized in the recently published paper by Font and Hájek [Font, Hájek 02]. The main contribution in my paper, if any, is hidden in its title: this logic lives in classical logic *per se*. Whether we like it or not, whether we appreciate its modalities or not, it exists hidden in classical logic.

Acknowledgements

I am grateful to the anonymous referees for the useful and benevolent report.

8. References

[Church 56] CHURCH A., *Introduction to Mathematical Logic I*, Princeton University Press, , Princeton, 1956.

[Curry 63] CURRY, H., *Foundations of Mathematical Logic*, McGraw-Hill, New York, 1963.

[Font, Hájek 02] FONT, J. M., HÁJEK, P., "On Łukasiewicz's Four-Valued Modal Logic", *Studia Logica*, vol. 70, no. 2, 2002, p. 157–182.

[Johansson 37] JOHANSSON, I., "Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus", *Compositio Matematica*, vol. 4, 1937, p. 119–136.

[Kalicki 50] KALICKI J., "Note on truth-tables", *The Journal of Symbolic Logic*, vol. 15, 1950, p. 174–181.

[Kanger 55] KANGER S., "A note on partial postulate sets for propositional logic", *Theoria*, vol. 21, 1955, p. 99–104.

[Kolmogorov 25] KOLMOGOROV A., "O principe tertium non datur", *Matematicheskiy sbornik*, vol. 32, 1924–1925, p. 646–667 (*Russian*).

[Kripke 65] KRIPKE S., "Semantical analysis of modal logic II. Non-normal modal propositional calculi", in: Addison J. *et al.*, Ed., *The Theory of Models,* Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1965, p. 206–220.

[Lemmon 66] LEMMON E., "Algebraic semantics for modal logics" (I and II), *The Journal of Symbolic Logic*, vol. 31, 1966, p. 46–65; 191–218.

[Łukasiewicz 53] ŁUKASIEWICZ J., "A system of modal logic", *The Journal of Computing Systems*, vol. 1, 1953, p. 352–390 (reprinted in: *Selected Works,* Borkowski L., Ed., Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam; Polish Academic Publishers, Warszawa, 1970.

[Porte 79] PORTE J., "The Ω-system and the Ł-system of modal logic", *Notre Dame Journal of Formal Logic*, vol. 20, no. 4, 1979, p. 915–920.

[Prior 57] PRIOR, A., *Time and Modalit*y, Clarendon Press, Oxford, 1957.

[Prior 62] PRIOR A., *Formal Logi*c, Clarendon Press, Oxford, 1962.

[Rasiowa 74] RASIOWA H., *An Algebraic Approach to Non-Classical Logics*, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam; Polish Academic Publishers, Warszawa, 1974.

[Segerberg 68] SEGERBERG, K., "Propositional logics related to Heyting's and Johansson's", *Theoria*, vol. 34, no. 1, 1968 , p. 26–61.

[Smiley 61] SMILEY, T., "On Łukasiewicz's Ł-modal system", *Notre Dame Journal of Formal Logic*, vol. 2, 1961, p. 149–153.

[Sotirov 01] SOTIROV V., "Negation Hidden in the Classical Implication", 2001 European Summer Meeting of the Association for Symbolic Logic, Logic Colloquium '01, Vienna, Austria, August 6–11, 2001; *Bulletin of Symbolic Logic*, vol. 8, no. 1, 2002, p. 177–178.

[Vakarelov 65] VAKARELOV, D., "Ein Aussagenkalkül mit Funktoren für 'Glaubwurdigkeit' und ‘Zweifel’ I, *Annuaire de l’Université de Sofia, Fac. de Math.*, Vol. 60, 1965–1966), p. 83–103 (*Bulgarian with German summary*).

[Vakarelov 66] VAKARELOV D., “Ein Aussagenkalkül mit Funktoren für ‘Glaubwurdigkeit’ und ‘Zweifel’ II, *Annuaire de l’Université de Sofia, Fac. de Math.*, Vol. 61, 1966–1967, p. 47–70 (*Bulgarian with German summary*) .

[Vakarelov 77] VAKARELOV D., Theory of Negation in Certain Logical Systems. Algebraic and Semantic Approach, Ph.D. dissertation, Warsaw University, 1977.