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MODAL THEORIES WITH INTUITIONISTIC LOGIC

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A few basic types of intuitionistic modalities are considered. The minimal intuitionistic modal logic (*IML*) is an expansion of the intuitionistic propositional calculus by a new operator M together with the rule $\frac{A \equiv B}{MA \equiv MB}$; the modality M may be interpreted as a "necessity" $\Box$ (with additional axiom $M[A \wedge B] \equiv MA \wedge MB$) or as a "possibility" $\Diamond$ (axiom $M[A \vee B] \equiv MA \vee MB$) with or without the rules of normalization ($\frac{A}{MA}$ and $\frac{\neg A}{\neg MA}$ respectively). Adequate algebraic, topological, Montague and Kripke semantics are introduced for these basic logics and for some extensions of theirs. A great number of *IML*-s are proved to have a finite model property (f. m. p.). Examples of *IML*-s without the f. m. p. are shown, too. A more important example is that of a logic which is an extension of Dummett's *LC* (it is well known that any non-modal extension of *LC* has the f. m. p.), its corresponding classical logic has the f. m. p. (moreover, $\Box A \equiv A$ is provable classically and so its classical analogue is fictiously modal), and finally $\Box A \equiv A$ is true on any finite model for that logic (so its proper modal theorems can be separated by infinite models only).

1. INTRODUCTION

If one looks through the existing literature on intuitionistic modal logics he will undoubtedly notice a peculiar regularity: there is an outburst of interest in this topic each decade.

As it seems to us the first relevant paper is the 1948 work of Fitch [19], where he proves by a purely syntactical method some theorems in the intuitionistic modal predicate calculus, extended by Barkan formulas for $\Box$ and $\Diamond$. In the paper [21] Bull considers the intuitionstic analogue of *S*5, introduced by Prior as early as 1957 [20], gives a characterization of its algebraic models and proves the finite model property. Later he achieved the reduction of this system to the monadic intuitionistic predicate calculus [23]. In 1968 Minc gave a deciding algorithm for the same system in [24], where he considered also the intuitionistic predicate calculus, extended with Prior's axioms. Besides this, in 1965 Bull [22] introduced an intuitionistic analogue of *S*5, which corresponds to our *IS*5 ($\Box + \Diamond$), and proved by the normal form method a theorem similar to our Corollary 5. He introduced also a system denoted by him *IM* (which we treat here under the name *Mon* 3 (□) in theorem 11), gave an algebraic characterization and using it proved the decidability of *IM*. In the same paper he investigated several extensions of the same system (amongst which we can find our *IS*4 (□)) and proved their decidability.

In 1975 there appeared two papers of Gabbay [25, 26]. In the second paper he gave a semantics for the case of *IK* (□). In a series of papers [27–31] Fisher-Servi considered the reduction of intuitionistic modal logics to bi-modal logics of *S*4-type: the *S*4-modality covers the intuitionistic connectives, and the additional modality represents the intuitionistic modality. In the first paper however she reduced only the intuitionistic *S*5 to the bi-modal *S*4-*S*5 system. In [29] analogously to the monadic algebras of Bull, she defined the monadic Heyting algebras, which serve at the same time to model the monadic predicate calculus and the modal logic from [21]. In the last paper [30] she tried to obtain a Kripke-style semantics from the reduction procedure. Given a bi-modal *S*4-logic and a classical modal logic *L*, she defined the intuitionistic analogue *IL* as the set of formulas, valid under the reduction into the bi-modal logic (when the second modality of the latter satisfies the axioms of *L*). This procedure generates a relational semantics. Her basic result is that a formula of the intuitionistic modal logic is valid in the obtained Kripke semantics iff it is provable in the above described sense, i. e. if its traslation is valid in the semantics for the bimodal logic. It is not clear however, whether this analogue is axiomatizable at all and can one get the most natural intuitionistic modal logics as, for example, *IS*4 from [22].

In his recent paper Ono considered several classically equivalent but intuitionistically different analogues of *S*4 and *S*5. He presented a Kripke semantics and proved the finite model property [32, 33].

As far as [34] and [35] are concerned it should be mentioned that while in the first paper we have a purely syntactical way of introducing intuitionistic modal logic (*IML*) in the second the author provides a semantics for a minimal non-normal *IML*. This paper however contains a serious error: the $\leq$ relation is not in any way connected with R and R* (for the modalities □ and ◇). So we cannot prove that from $x \Vdash A$ and $x \leq y$ it will always follow $y \Vdash A$. Hence, the correctness theorem formulated there is not true and therefore the semantics is not adequate.

This brief historical survey shows that there were at the time of the writing of the material, presented below, several important open questions in the field of *IML*:

1) Semantics for the weakest *IML*-s (non-normal and so on).
2) Syntax and semantics for logics with ◇ as well as □.
3) Connections with the classical bi-modal logic and with the bi-topological spaces.
4) General methods for proving decidability.
5) The finite model property of *IML*-s.

In the present paper we give solutions to some of this problems. Namely we present Montague, topological and Kripke semantics, prove several completeness results. In the last part we study the finite model property (f. m. p.) and show that it is in a way absolute. We give examples of intuitionistic modal extensions of *IS*4 for which the f. m. p. fails. We discuss also the question of decidability of *IML*-s.

2. MONTAGUE SEMANTICS

Here we investigate the intuitionistic propositional calculus, extended by a new unary connective M. We call it "modality", but as the only requirement for M is *the extensionality principle*: if A and B are equivalent, then MA and MB are also equivalent. It is clear that M can be interpreted not only as modality but also as, for example, deontic operator and so on. Different kinds of negation have this property too. Thus one may think a part of this paper as a study of the most general type of negation, added to intuitionistic logic. On the other hand, there are operators, which are clearly non-extensional, e. g. "I know that...": if I know that A and A is equivalent to B then it is not always the case that I know B because the logical equivalence could be unknown to me.

The system under consideration will be called intuitionistic modal propositional logic (***IMPL***).

The language of ***IMPL*** contains propositional variables: $p, q, r, \ldots$; the propositional constant 0, the connectives $\wedge$, $\vee$, $\Rightarrow$ and M; the sequent symbol $\leq$ and parentheses [,]. The formulas $A, B, C, \ldots$ are obtained from the atomic formulas (variables and 0) by applications of the connectives. The negation of a formula A denoted $\neg A$ is defined as $A \Rightarrow 0$. Further we set $1 = 0 \Rightarrow 0$.

If A and B are formulas, then $A \leq B$ is called a *sequent*. *Basic sequents* are the sequents of the following kind:

$$A \leq A,\ A \wedge B \leq A,\ A \wedge B \leq B,\ 0 \leq A,$$

$$A \leq A \vee B,\ B \leq A \vee B,\ [A \Rightarrow B] \wedge A \leq B.$$

$A \equiv B$ denotes the pair of sequents $\{A \leq B, B \leq A\}$.

The following are the *rules of inference*:

$$\frac{A \leq B,\ B \leq C}{A \leq C},\quad \frac{A \leq B,\ A \leq C}{A \leq B \wedge C},\quad \frac{A \leq C,\ B \leq C}{A \vee B \leq C},\quad \frac{A \wedge B \leq C}{A \leq B \Rightarrow C},\quad \frac{A \equiv B}{MA \leq MB}$$

(the last rule gives the extensionality principle $\frac{A \equiv B}{MA \equiv MB}$).

A *proof* is a finite sequence of sequents $S_1, \ldots, S_n$ each member of which S_i satisfies one of the conditions:

1. S_i is a basic sequent;
2. $(\exists j < i)\ \frac{S_j}{S_i}$ is an instance of a rule;
3. $(\exists j, k < i)\ \frac{S_j,\ S_k}{S_i}$ is an instance of a rule.

As usual we call a sequent provable if there exists a proof with last element equal to the sequent.

Definition. A formula A is *provable* ($\vdash A$) iff the sequent $1 \leq A$ is.

From now on when mentioning a sequent $A \leq B$ we shall consider it provable (unless the contrary is stated explicity).

Proposition 1. $A \leq B$ iff $\vdash A \Rightarrow B$.

We omit the easy proof.

The logical calculus just introduced will be denoted by ***M***. Usually we shall identify the logics under consideration with the set of formulas provable in them (and sometimes with the set of provable sequents) and employ the notation $L \vdash A$ for $A \in L$, if L denotes a logic.

We go on now to study the semantical characterization of M.

Definition. A *Montague frame* is an ordered triple $\langle K, \leq, R\rangle$ where:

1. $K \neq \emptyset$ is the set of states or possible worlds $x, y, z, \ldots$
2. $\leq$ is a binary reflexive and transitive relation in K;
3. $R \subseteq K \times 2^K$;
4. for $P \subseteq K$ we have: xRP and $x \leq y$ imply yRP.

Definition. A *Montague model* is an ordered quadruple $\langle K, \leq, R, \Vdash\rangle$ where:

1. $\langle K, \leq, R\rangle$ is a Montague frame;
2. $\Vdash$ (the forcing) is a relation between states and atomic formulas $(x \Vdash A)$;
3. $(\forall x \in K)$ (not $x \Vdash 0$);
4. $x \leq y$ and $x \Vdash A$ imply $y \Vdash A$, for all atomic A.

We can extend the forcing to a relation between elements of K and arbitrary formulas, by induction on the complexity of the formula: $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$; $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$; $x \Vdash A \Rightarrow B$ iff for all $y \geq x$, $y \Vdash A$ implies $y \Vdash B$; $x \Vdash \mathrm{M}A$ iff $xR\{y \mid y \Vdash A\}$. Let us note that it follows from the definition that $x \Vdash \neg A$ iff for all $y \geq x$, not $y \Vdash A$.

From now on we denote $\{x/x \Vdash A\}$ by $H(A)$.

Proposition 2. For any formula A, if $x \Vdash A$ and $x \leq y$, then $y \Vdash A$).

The proof is by induction on the complexity of A. The case of M follows from the monotonicity condition on R: if $x \Vdash \mathrm{M}B$ then $xRH(B)$, and from $y \geq x$ it follows that $yRH(B)$, i. e. $y \Vdash \mathrm{M}B$.

Proposition 3. In any Montague model for arbitrary x: $x \Vdash 1$.

Definition. A formula A is *true* in a model $\langle K, \leq, R, \Vdash\rangle$ iff for all $x \in K$, $x \Vdash A$. A sequent $A \leq B$ is true in a model iff for all $x \in K$, $x \Vdash A$ implies $x \Vdash B$.

Definition. A formula (sequent) is *valid* in a Montague frame $\langle K, \leq, R\rangle$ iff it is true in all models of the kind $\langle K, \leq, R, \Vdash\rangle$. A formula (sequent) is *Montague valid* iff it is valid in every Montague frame.

Proposition 4. A formula $A \Rightarrow B$ is true in a model (valid in the frame, valid) iff the sequent $A \leq B$ is true in the model (valid in the frame, valid).

The proof is easy. From it we get that a formula A is true in a model iff $1 \leq A$ is.

If we denote $\{y \mid x \leq y\}$ by $L(x)$, we can introduce set-theoretic operation $\Rightarrow$ as follows:

$$P \Rightarrow Q \underset{\mathrm{Df}}{=} \{x \mid L(x) \cap P \subseteq Q\}.$$

Now we can rewrite some of the facts about Montague models. Let x be a point in a model, A and B — formulas:

$$x \in H(A) \text{ implies } L(x) \subseteq H(A);$$

$$H(0) = \emptyset;$$

$$H(A \wedge B) = H(A) \cap H(B);$$
$$H(A \vee B) = H(A) \cup H(B);$$
$$H(A \Rightarrow B) = H(A) \Rightarrow H(B);$$
$$H(\mathsf{M}A) = \{x \mid xRH(A)\};$$
$$A \leq B \text{ is true iff } H(A) \subseteq H(B);$$
$$A \equiv B \text{ is true iff } H(A) = H(B).$$

Lemma 1. If a sequent (a formula) is provable, then it is valid.

The proof goes by checking the validity of all basic sequents and establishing the fact that the rules preserve validity.

For the proof of the converse we need the theory of pseudo-Boolean algebras (cf. [3]). Here we adopt the following view: a *pseudo-Boolean algebra* (pBa) is an algebra with three binary operations ($\wedge$, $\vee$, $\Rightarrow$), a constant 0 and a reflexive and transitive relation on it ($\leq$), so the difference from the well known definition lies in the fact that $\leq$ is not in general a partial order. But apparently postulating the relation $\equiv$ (where $a \equiv b$ is $a \leq b$ and $b \leq a$) to be a congruence, the quotient algebra of $\equiv$ leads to the usual notion of pBa. We prefer this more general approach in order to by-pass the construction of the so-called Lindenbaum algebra and work directly with formulas and not with equivalence classes.

Proposition 5. The set of all formulas in the language of the calculus M forms a pBa.

Definition. A subset x of a pBa is called a *filter* if:

1) $x \neq \emptyset$;
2) $a \in x$ and $a \leq b$ imply $b \in x$;
3) $a \in x$ and $b \in x$ imply $a \wedge b \in x$.

It is easy to check that these conditions can equivalently be replaced by:

1′) $a \equiv 1$ implies $a \in x$;
2′) $a \Rightarrow b \in x$, $a \in x$ imply $b \in x$.

Here 1 is $0 \Rightarrow 0$ or if we set $\neg a = a \Rightarrow 0$, 1 is $\neg 0$.

Definition. A filter is *prime* if it is *proper* (i. e. does not coincide with the whole pBa) and for all a, b: $a \vee b \in x$ implies $a \in x$ or $b \in x$.

It is shown in [3] that for any subset x of a pBa the set $\{c \mid (\exists a_1, \ldots, a_n \in x) a_1 \wedge \cdots \wedge a_n \leq c\}$ is a filter. We say that this is the filter, *generated* by x. For a given subset x and an element a in a pBa we say that a filter is generated by x and a if it is the filter, generated by $x \cup \{a\}$. We denote this filter by (x, a).

Lemma 2. If x is a filter and $a \Rightarrow b \notin x$, then $b \notin (x, a)$.

Lemma 3. If a filter y_0 does not contain an element b then it can be extended to a proper filter y such that $b \notin y$, too.

These are standard facts and can be found for example in [4].

Lemma 4. If x is a filter, $a \Rightarrow b \notin x$, then there is a prime y: $x \subseteq y$, $a \in y$, $b \notin y$.

The following notion is dual to the notion of filter. It is studied in detail in [5].

Definition. A subset x of a pBa is called a *co-filter* if:

1) it is a proper subset;
2) $a \in x$ and $a \leq b$ imply $b \in x$;
3) $a \vee b \in x$ imply $a \in x$ or $b \in x$.

It is easy to establish the following

Proposition 6. A subset of a pBa is a prime filter iff it is a filter and a co-filter at the same time.

Lemma 5 (interpolation). If x is a filter, m — a co-filter, $x \subseteq m$, then there is a prime filter y such that $x \subseteq y \subseteq m$.

Proof. Consider the set of all filters z, such that $x \subseteq z \subseteq m$. It is immediately clear that under inclusion this set is a partially ordered set satisfying Zorn conditions. Pick a maximal element y. This y is an extension of x and $y \subseteq m$. We are left to show that y is prime. If $a \vee b \in y$ but neither $a \in y$, nor $b \in y$, then (y, a) and (y, b) are extensions of y and (as can be easily shown) proper. Further $x \subseteq (y, a)$ and $x \subseteq (y, b)$. At least one of them is contained in m, because otherwise we can find $u \in (y, a)$ and $v \in (y, b)$ such that $u \notin m$, $v \notin m$ and consequantly $u \vee v \notin m$ (m is a co-filter). From here we get a contradiction by a familiar argument: $u \vee v \in y$ and at the same time $y \subseteq m$.

This proves the lemma.

Now we are able to show the following

Lemma 6. If a sequent (formula) is Montague-valid then it is provable in the calculus ***M***.

Proof. We prove the contraposition: if a formula (or a sequent) is not provable then it is not true in a suitable Montague-model. In fact we can do a bit more — to construct an universal model in the sense that any non-provable formula is refuted in it. Such a model is the cannonical one. Moreover we shall build two models which differ in respect to the definition of R.

Let K be the set of all prime filters in the pBa of formulas. The relations $\leq$ is inclusion — clearly it is transitive and reflexive (in fact — a partial order). Let $h(A)$ denote the set of all prime filters containing A. We can define in two different ways R between prime filters and subsets of K:

$$xR^1P \text{ iff } (\exists A)[P = h(A) \text{ and } \mathsf{M}A \in x],$$

$$xR^2P \text{ iff } (\forall A)\ [\text{if } P = h(A) \text{ then } \mathsf{M}A \in x].$$

R^1 is familiar from other papers on Montague semantics [6, 7, 11, 13]. Unfortunatelly R^1 is not sufficient to cover all the logics (even in the classical case) in which M is interpreted rather as "possibility" then as "necessity". For the former we shall use R^2.

Now we show that $\langle K, \leq, R^1\rangle$ and $\langle K, \leq R^2\rangle$ are Montague frames. Assume that xR^1P, i. e. for some A, $P = h(A)$ and $\mathsf{M}A \in x$, and at the same time $x \leq y$ (that is $x \subseteq y$). Clearly we have $\mathsf{M}A \in y$, too. So, yR^1P. The monotonicity condition for R^2 is established in the same way. Note that xR^1P implies xR^2P.

We can define Montague models on these two frames, setting $x \Vdash A$ iff $A \in x$, for atomic formulas.

As all prime filters are proper it is clear that $(\forall x)$ (not $x \Vdash 0$). Again if $x \leq y$ and $x \Vdash A$, then $y \Vdash A$ (for atomic formulas). Extending $\Vdash$ to the set of all formulas following the inductive definition, we are able to show that for arbitrary formula F: $x \Vdash F$ iff $F \in x$. The proof of this fact is standard and exploits the properties of the prime filters. We mention explicity only the case of modality M. Consider first the model with R^1. If $x \Vdash \mathsf{M}A$, then $xR^1H(A)$ and there is a formula B such that $H(A)=h(B)$ and $\mathsf{M}B \in x$. But using the induction hypothesis we have $H(A)=h(A)$ and so $A \equiv B$ is true in the model. By the extensionality principle we have $\mathsf{M}A \equiv \mathsf{M}B$ true in the model, and $\mathsf{M}A \in x$. In the opposite direction: if $\mathsf{M}A \in x$ then, since $H(A)=h(A)$, we immediately get $xR^1H(A)$, therefore $x \Vdash \mathsf{M}A$. Now for the model with R^2: if $x \Vdash \mathsf{M}A$, then $xR^2H(A)$ and $\mathsf{M}A \in x$. In case $\mathsf{M}A \in x$, we have $h(A)=h(B)$ and $A \equiv B$ true in the model, so $\mathsf{M}A \equiv \mathsf{M}B$ is true. This implies that $\mathsf{M}B \in x$ and $xR^2h(A)$ but the induction hypothesis gives $h(A)=H(A)$ and therefore $x \Vdash \mathsf{M}A$.

To end the proof assume that $A \leq B$ is not provable. Then the filter x_0 generated by A is proper (B does not belong to it) and can be extended to a prime filter x, such that $B \notin x$. Then $x \Vdash A$ but $x \nVdash B$ by the previous consideration. Thus the sequent $A \leq B$ has at least two countermodels.

In this way we have proved

Theorem 1 (Montague-completeness). A sequent (formula) is provable in ***M*** iff it is valid.

M is a very general modal logic and the operator M has no special properties besides the extensionality. On the other side the relation R is arbitrary (but monotone with relation to $\leq$). It is clear that we should be able to treat more specific systems with the connective M containing axioms which describe it either as "necessity" or as a "possibility". We can extend our completeness proof to several such systems just by imposing additional conditions on the relation R. These systems are obtained from our basic system ***M*** by additional axioms and additional rules. Sometimes these will be called proper modal axioms and rules.

Definition. A logic ***L*** is *complete* for a class $\mathcal{M}$ of Montague frames if any formula (sequent) is provable in L iff it is valid in every frame from $\mathcal{M}$.

Now we introduce a two-sorted language of 1st order with three binary predicates: $\leq$, $\in$, R. The variables of the first sort range over elements of the frames, and variables of the second sort — over special subsets of the frames. A condition C on a frame is simply a formula (or a set of formulas) in this language.

Definition. We say that a sequent S (a rule P) is *adequate* to a condition C iff the logic with additional axiom S (additional rule P) is complete for the class of Montague frames determinated by the condition C.

In practice we have to check that all additional axioms are valid in all frames of the class, determined by C, that all additional rules preserve this property and, on the other hand, that any unprovable formula has a countermodel in the same class of frames, in order to prove the adequacy. In most of the cases we just show that cannonical frame of the logic belongs to the class of frames satisfying C.

Definition. A logic is *Montague-complete* (or simply complete) if it is complete for some class of frames. (It should be noted that sometimes the class of frames cannot be described in the language we have introduced.)

Theorem 2. The following axioms and rules are adequate to the conditions on the right:

$MA \wedge MB \leq M[A \wedge B]$	if xRP and xRQ then $xR(P \cap Q)$
$MA \leq A$	if xRP then $x \in P$
M1 (or $\frac{A}{MA}$)	xRK
$M[A \vee B] \leq MA \vee MB$	if $xR(P \cup Q)$ then xRP or xRQ
$A \leq MA$	if $x \in P$ then xRP
$\neg M0$ (or $\frac{\neg A}{\neg MA}$)	not $xR\varnothing$

The proof goes along the lines given above. It is left to the reader to check the validity of the sequents in all frames of the corresponding classes.

It can be seen that the first group of axioms are typical for "necessity" while axioms of the second group exhibit some of the features of "possibility".

For the proof of the remaining part of our theorem we use the cannonical frame with R^1 for the first group and with R^2 for the second. Let us consider an instructive example: take the first condition, and let xR^1P and xR^1Q in the cannonical frame, i. e. for some A and B we have $P=h(A)$, $Q=h(B)$, $MA \in x$, $MB \in x$. But x is a filter and so $MA \wedge MB \in x$ and using the axiom $M[A \wedge B] \in x$; besides this $P \cap Q = h(A) \cap h(B) = h(A \wedge B)$, hence $xR^1(P \cap Q)$.

We assume that the proof of Theorem 2 is clear now.

A typical rule that distinguishes modalities from other connectives (as negations for example) is *the monotonicity rule*: $\frac{A \leq B}{MA \leq MB}$. We call a logic *monotone* if it is closed under this rule.

Proposition 7. Monotonicity is adequate for the condition "if $P \subseteq Q$ and xRP, then xRQ".

For the proof of this fact we have to introduce new relations R in the cannonical frame:

$$xR^{10}P \text{ iff } (\exists A)(h(A) \subseteq P \text{ and } MA \in x),$$

$$xR^{20}P \text{ iff } (\forall A)(\text{if } h(A) \supseteq P \text{ then } MA \in x).$$

Now it is not difficult to prove the proposition.

The troubling feature of this method of proof which was originated by Gabbay [6] is that we in fact change the relations in the cannonical frame. Now if the logic is not monotone we have to use R^1 or R^2 but if we add the monotonicity rule, then we are forced to switch to R^{10} or R^{20}. That is why there follows an exposition of a more direct method of proof, that was invented (in the classical case) by Segerberg [7]. His proof however, contains a mistake, corrected in the paper of Chellas and Mc Kinney [8].

We give now the proof of Proposition 7 by a variant of this method. The validity of sequents provable by monotonicity rule in all frames with the corresponding condition was already established, so we treat only the second half of the proof. By Theorem 1 we have that all unprovable in M sequents are refuted in two models $\langle K, \leq, R^1, \Vdash\!\!- \rangle$ and $\langle K, \leq, R^2, \Vdash\!\!- \rangle$. On that base we can define two new models: $\langle K, \leq, R^1_*, \Vdash^* \rangle$ and $\langle K, \leq, R^2_*, \Vdash^* \rangle$ by postulating

$$xR^1_*P \text{ iff } (\exists P^-)(P^- \subseteq P \text{ and } xR^1P^-),$$
$$xR^2_*P \text{ iff } (\forall P^+)(P \subseteq P^+ \text{ implies } xR^2P^+)$$

and $x \Vdash^* A$ iff $x \Vdash\!\!- A$ for atomic formulas A. Clearly R^1_* and R^2_* are monotone. We are going to show that for an arbitrary A, $x \Vdash^* A$ iff $x \Vdash\!\!- A$, if the logic is a monotone extension of M. All other cases being trivial we check only formulas of the kind MA. Take R^1_*: if $x \Vdash\!\!-$ MA then automatically $x \Vdash^*$ MA; if $x \Vdash^*$ MA, then $xR^1_*H(A)$, i. e. for some P^-: $P^- \subseteq H(A)$ and xR^1P^-, but $P^- = H(B)$ for some formula B (from the definition of (R^1)) and M$B \in x$. Now the monotonicity implies that M$A \in x$ and so $x \Vdash\!\!-$ MA. If we are in the model with R^2_* it is easy to show that $x \Vdash^*$ MA implies $x \Vdash\!\!-$ MA; for the opposite implication: if not ($x \Vdash^*$ MA) then for some P^+, $H(A) \subseteq P^+$ and not xR^2P^+, then there exists B such that $P^+ = H(B)$ and M$B \notin x$. From this and monotonicity we get M$A \notin x$, i. e. not ($x \Vdash\!\!-$ M(A)).

By the same method we can prove that for any axiom or rule from Theorem 2, monotonicity rule + axiom are adequate for the condition from Proposition 7 plus the condition from Theorem 2 which corresponds to the axioms or rule. It should be noted that in the papers mentioned above there is no analogues of our R^2, R^{20} or R^2_*.

Now we change the relation R in a Montague frame $\langle K, \leq, R \rangle$ to an operation J, which maps 2^K into 2^K. Define: J$P = \{t \mid tRP\}$. We get the so called *modified Montague frame* $\langle K, \leq, \text{J} \rangle$ and J and R are connected by xRP iff $x \in$ JP. Note that we have the following property: if $x \in$ JP and $x \leq y$, then $y \in$ JP. Also H(MA) = J$H(A)$. It is easily seen that the two models $\langle K, \leq, R, \Vdash\!\!- \rangle$ and $\langle K, \leq, \text{J}, \Vdash\!\!- \rangle$ are equivalent. Using the new symbols we can simplify and extend Theorem 2 and Proposition 7.

Proposition 8. The following axioms and rules are adequate to the given conditions:

$\dfrac{A \leq B}{\text{M}A \leq \text{M}B}$,	if $P \subseteq Q$ then J$P \subseteq$ JQ
or M$[A \wedge B] \leq$ MA,	or J$(P \cap Q) \subseteq$ JP,
or M$A \leq$ M$[A \vee B]$	or J$P \subseteq$ J$(P \cup Q)$
M1 (or $\dfrac{A}{\text{M}A}$)	J$K = K$
$\neg$M0	J$\varnothing = \varnothing$
M$A \leq A$	J$P \subseteq P$
M$A \leq$ MMA	J$P \subseteq$ JJP
$A \leq$ MMA	$P \subseteq$ JJP
$\neg[A \wedge \text{M}A]$	$P \cap$ J$P = \varnothing$
M0	J$\varnothing = K$
M$A \wedge$ M$B \leq$ M$[A \wedge B]$	J$P \cap$ J$Q \subseteq$ J$(P \cap Q)$

$\neg M0$ (or $\frac{\neg A}{\neg MA}$)	$J\varnothing = \varnothing$
$A \leq MA$	$P \subseteq JP$
$MMA \leq MA$	$JJP \subseteq JP$
$M[A \vee B] \leq MA \vee MB$	$J(P \cup Q) \subseteq JP \cup JQ$

The proof is left to the reader. We just mention the definition of J^1 (which corresponds to R^1) in the cannonical frame for the axioms of the second group:

$$J^1P = \begin{cases} \{x \mid MA \in x\} \text{ if } P = h(A) \text{ for some } A, \\ \varnothing \text{ otherwise,} \end{cases}$$

and J^2 for the axioms of the third group:

$$J^2P = \begin{cases} \{x \mid MA \in x\} \text{ if } P = h(A) \text{ for some } A, \\ K \text{ otherwise.} \end{cases}$$

Note that due to the extensionality principle the formula A appears vacuously in the definitions of J.

Sometimes we are interested not only in adequacy (completeness with respect to some condition) but also in characterizability.

Definition. We say that a formula A (a sequent $A \leq B$) *characterizes* the condition C, if A ($A \leq B$) is valid in a frame iff this frame satisfies the condition C. A rule P characterizes C if it preserves validity in the frame just in case the frame satisfies C.

Now define a translat on S from the modal language into a set-theoretic language with constants K, $\varnothing$, $\cap$, $\cup$, $\Rightarrow$, J, $\subseteq$ and variables P, Q, $R, \ldots$, ranging over *cones* of K. A cone is a subset of K growing upwards w. r. t. $\leq$.

$S(p) = P$, $S(q) = Q, \ldots$ for propositional variables $p, q, \ldots$

$$S(A \wedge B) = S(A) \cap S(B),$$

$$S(A \vee B) = S(A) \cup S(B),$$

$$S(A \Rightarrow B) = S(A) \Rightarrow S(B),$$

$$S(MA) = JS(A),$$

$$S(0) = \varnothing, \; S(1) = K.$$

Theorem 3. A formula A characterizes the condition $S(A) = K$ in a Montague frame $\langle K, \leq, J \rangle$, a sequent $A \leq B$ characterizes the condition $S(A) \subseteq S(B)$.

The proof is clear. It is sufficient to establish by induction that if for a given $\Vdash$, $H(p) = S(p)$, then for an arbitrary formula A we have the same: $H(A) = S(A)$.

The connection between completeness and characterizability is quite interesting. If a logic $\boldsymbol{L}$ which is obtained by adding an axiom A to $\boldsymbol{M}$, is complete for any class of frames, then it is complete for the class of frames determined by the condition $S(A)=K$ over cones and this class is the greatest class with such a property. Analogously we can prove

Proposition 9. If a logic with additional axiom $A\leq B$ is complete then $S(A)\subseteq S(B)$ over cones determines the maximal class for which the logic is compiete.

In this way the question about the form of the adequate condition (if the logic is complete) is solved for the case of Montague frames and a useful description of one of the classes for which the logic is complete is given.

3. TOPOLOGICAL SEMANTICS

In this Section we investigate the possibility of topological interpretation for a fairly wide class of ***IML***-s and not only for the most "obviuos" one, such as for example intuitionistic ***S***4, taken moreover with "necessity" only. Our aim is to define a suitble translation into bi-modal classical logics, working in a variety of cases (not only in the case of ***S***4-***S***4 classical bi-modal logic).

The well known Gödel translation W of intuitionistic propositional claculus into S4 has the following characteristic property:

$$A \text{ is provable intuitionistically iff } \boldsymbol{S}4 \vdash W(A).$$

From this property and the usual Kripke semantics for ***S***4 one can obtain Kripke semantics for intuitionistic propositional calculus [11]. However, ***S***4 has Montague semantics, too. It is interesting to see what kind of semantics does the Gödel translation generate in this case. The answer is: The induced semantics is the topological interpretation, familiar from the days of Tarski and McKincey. Below we generalize this interpretation to ***IML***-s and especially to ***IS***4 (□) and ***IS***4 (◇). In fact we give a general translation into the bimodal ***M***-***S***4 logic and get a semantics for the ***IML***-s, which should be called "Montague-Montague" interpretation.

Definition. A *generalized topological space* is a set $K\neq\emptyset$, together with two operations on subsets of K: I and J, satisfying:

$$\mathrm{I}K=K,\ \mathrm{I}P\subseteq P,\ \mathrm{I}P\subseteq \mathrm{I}\mathrm{I}P,\ \mathrm{I}(P\cap Q)=\mathrm{I}P\cap \mathrm{I}Q,\ \mathrm{I}\mathrm{J}P=\mathrm{J}P.$$

Clearly I is an interior operation and generates a topology in which all sets JP are open. Given a generalized topological space $\langle K, \mathrm{I}, \mathrm{J}\rangle$ and an evaluation of the propositional variables $p, q, r, \ldots;\ v(p)$ — an open subset in $\langle K, \mathrm{I}\rangle$, we can extend it to evaluation $v(A)$ of an arbitrary formula A:

$$v(A\wedge B)=v(A)\cap v(B),\quad v(A\vee B)=v(A)\cup v(B),$$

$$v(A\Rightarrow B)=\mathrm{I}\,(\overline{v(A)}\cup v(B)),\quad v(\neg A)=\mathrm{I}\,(\overline{v(A)}),$$

$$v(\mathrm{M}A)=\mathrm{J}v(A)$$

(here $\overline{}$ is the complementation operation). A formula A is called *valid* in space $\langle K, \mathrm{I}, \mathrm{J}\rangle$ if $v(A)=K$ and *topologically valid* (top-valid) if it is valid in every generalized topological space.

In accord with the natural usage a space with evaluation in it should be called (generalized) *topological model.* It is easily seen that always $v(A)$ is open.

Now given a Montague model $\langle K, \leq, J, \Vdash \rangle$ we define the corresponding top-model by $t \in IP$ iff $L(t) \subseteq P$; $v(p) = H(p)$ and so on . . . These two models are equivalent in the following exact sense:

Lemma 7. If $\langle K, I, J, v \rangle$ corresponds to $\langle K, \leq, J, \Vdash \rangle$ then for any formula A we have $v(A) = H(A)$, i. e. $x \Vdash A$ iff $x \in v(A)$.

Proof. By induction on the complexity of A.

By Lemma 7 if a formula (or a sequent) is top-valid it is Montague valid and vice versa. Hence all provable in $\boldsymbol{M}$ formulas are top-valid.

Theorem 4 (topological completeness). The logic $\boldsymbol{M}$ is complete for the class of generalized topological spaces.

How can we go from spaces to frames. Given a space $\langle K, I, J \rangle$ we define $\langle K, \leq, J \rangle$ by $x \leq y$ iff $(\forall Q)$[if $x \in IQ$, then $y \in Q$]. $IQ \subseteq Q$ implies that $\leq$ is reflexive, $IQ \subseteq IIQ$ — the transitivity of $\leq$. Moreover since JP is open, if $x \in JP$ and $x \leq y$, then $y \in JP$. Unfortunatelly we cannot prove that the two notions — spaces and frames are equivalent, in particular because every quasi-order generates a topology, but certainly not every topology is generated by a quasi-order (in such topologies for examples we have $I \cap_i P_i = \cap_i IP_i$ which is not generally true). We are able however to prove the weaker (but nevertheless usefull)

Lemma 8. If a given additional axiom is without implication (and consequently without negation) and is valid in every Montague frame $\langle K, \leq, J \rangle$ from a class, defined by a condition C on J, then it is valid in every space in the class of spaces $\langle K, I, J \rangle$, determined by the same condition C on J.

Proof. If A is refuted in a certain topological model $\langle K, I, J, v \rangle$ with J satisfying C, we consider the generated Montague model $\langle K, \leq, J, \Vdash \rangle$, where $x \Vdash p$ iff $x \in v(p)$, for variables $p, \ldots$ Clearly the frame $\langle K, \leq, J \rangle$ is from the class given by C. By induction on the complexty of A we show that $H(A) = v(A)$ and so A is not true in this specific model. The same proof holds for sequents $A \leq B$, where A, B do not contain $\Rightarrow$, $\neg$.

This lemma shows that the connection between topological spaces and Montague frames is rather tricky. Nevertheless, it gives the opportunity to automatically transfer results about Montague-completeness to the case of spaces. In the following theorem we describe topologically a wide class of *IML*-s (for the first time):

Theorem 5. Any of the extensions of $\boldsymbol{M}$ mentioned in Proposition 8 is complete for the class of generalized topological spaces, satisfying the corresponding adequate condition.

The proof is an easy consequence of Lemma 7 and of the observation that all axioms can be written in the form, required by Lemma 8.

Now, we turn to a more detailed investigation of the intuitionistic analogues of $\boldsymbol{S}4$. From now on we use $\Box$ instead of $\boldsymbol{M}$. The classical Lewis' system $\boldsymbol{S}4$ is defined by certain axioms (see [11]), for $\boldsymbol{IS}4$ ($\Box$) we take the same axioms. In particular $\Box A \leq A$, $\Box A \leq \Box\Box A$, $\Box[A \wedge B] = \Box A \wedge \Box B$, $\Box 1$ are axioms of $\boldsymbol{IS}4$ ($\Box$).

By Theorem 5 we have the following adequate conditions for these axioms:

$$JP \subseteq P,\ JP \subseteq JJP,\ J(P \cap Q) = JP \cap JQ,\ JK = K.$$

It is clear that the operator J becomes a second interior operator and we get a second topology on K, which is weaker then the topology generated by I. If we adopt the name of *interior bi-topological spaces* for such generalized spaces, where JP is open in the other topology, we can formulate

Corollary 1. ***IS*4** ($\square$) is complete for the class of interior bi-topological spaces.

If we go from J back to relation R then we get the following:

$$xRK;\ \text{if } xRP, \text{ then } x \in P;$$

$$\text{if } xRP, \text{ then } xR\{y \mid yRP\};$$

$$\text{if } xRP \text{ and } xRQ, \text{ then } xR(P \cap Q);$$

$$\text{if } xRP \text{ and } P \subseteq Q, \text{ then } xRQ;$$

$$\text{if } xRP, \text{ then } x \in \mathrm{I}\{y \mid yRP\}.$$

Reading xRP as "P is a neighborhood of x" we readily recognize in the first five conditions the properties of neighborhoods of points [9]; the last condition expresses the fact that if P is a neighborhood of x, then x belongs to the interior of the set of all points, that have P as neighborhood. If we define the new topology as the collection of all sets that are neighborhoods of all its members, then we have the same topology as generated by J, but "described" in more "antique" way.

It is known that ***S*4** classically can be formulated with "possibility" $\Diamond$. Now ***IS*4** ($\Diamond$) is the system with additional axioms:

$$\neg \Diamond 0,\ \Diamond[A \vee B] \equiv \Diamond A \vee \Diamond B,\ A \leq \Diamond A,\ \Diamond \Diamond A \leq \Diamond A.$$

They can be equivalently replaced by:

$$\neg \Diamond 0, \frac{A \leq B}{\Diamond A \leq \Diamond B},\ \Diamond[A \vee B] \leq \Diamond A \vee \Diamond B,\ A \leq \Diamond A,\ \Diamond \Diamond A \leq \Diamond A.$$

Using Theorem 5 we obtain the following adequate conditions for ***IS*4** ($\Diamond$) (we use D for better readability):

$$D\varnothing = \varnothing,\ D(P \cup Q) = DP \cup DQ,\ P \subseteq DP,\ DDP \subseteq DP.$$

As we see D also generates new topology, but now D is interpreted more naturally as closure operator. The fact that $DP = \mathrm{I}DP$ shows that any closed set in the second topology is open in the main topology. Let us call such bi-topological spaces *closure bi-topological spaces*.

Corollary 2. ***IS*4**($\Diamond$) is complete for the class of closure bi-topological spaces.

Next to discuss is a translation of ***IML*-s** into the classical bi-modal logic.

Definition. ***MS*4**-*bi-modal logic* we call the classical ***S*4** extended by a new modality M together with the rule $\frac{A \equiv B}{MA \equiv MB}$ and the axiom $MA \leq \square MA$.

Lemma 9. $MS4$ is complete for the class of models $\langle K, R, J, \Vdash \rangle$ where $\langle K, R, \Vdash \rangle$ is a classical Kripke model for $S4$ (R — transitive and reflexive) and J satisfies "if $x \in JP$, then xRy implies $y \in JP$", and also $H(\mathsf{M}(A)) = JH(A)$.

Lemma 10. $MS4$ is complete for Montague-Montague models of the type $\langle K, I, J, \Vdash \rangle$ where $I(P \cap Q) = IP \cap IQ$, $IK = K$, $IP \subseteq P$, $IP \subseteq IIP$, $JP \subseteq IJP$ and $H(\Box A) = IH(A)$, $H(\mathsf{M}A) = JH(A)$.

The translation W from M into $MS4$ is defined as follows:

$$W(0) = 0,\ W(p) = \Box p,\ W(q) = \Box q, \ldots;$$

$$W(A \wedge B) = W(A) \wedge W(B),\ W(A \vee B) = W(A) \vee W(B),$$

$$W(A \Rightarrow B) = \Box(\neg W(A) \vee W(B)],\ W(\mathsf{M}A) = \mathsf{M}W(A),$$

$$W(A \leq B) = W(A) \leq W(B).$$

Lemma 11. A formula A (sequent $A \leq B$) is valid in Montague models for M iff $W(A)$ $(W(A) \leq W(B))$ is valid in models for $MS4$.

Lemma 12. A formula A (sequent $A \leq B$) is top-valid iff $W(A)$ $(W(A) \leq W(B))$ is valid in all Montague-Montague models.

The proofs of these two lemmas are based on transforming a countermodel of one type into a countermodel of the respective type, they do not present any special difficulty and are left to the reader.

As a consequence we have

Theorem 6. $M \vdash A$ iff $MS4 \vdash W(A)$ (and analogously for sequents)

4. KRIPKE SEMANTICS

This Section is devoted to those extensions of M, that possess a new kind of semantics — Kripke semantics. As we shall see the modality operator in M can be given two different meanings (adding different axioms) — that of "possibly" and that of "necessary". In the classical case they are interdefinable: $\Diamond$ is $\neg\Box\neg$ and, vice versa, $\Box$ is $\neg\Diamond\neg$. Here though, having an intuitionistic logic as base, we cannot expect $\Diamond$ and $\Box$ to be dual. Hence we are to expect interesting nontrivial relations between them. Unfortunately the existing works on intuitionistic modal logics either consider very strong systems where the modalities are too special, or do not mention $\Diamond$ at all.

Here we start with a minimal logic IK_0 $(\Box, \Diamond)$ with axioms (besides purely intuitionistic tautologies):

$$\Box A \wedge \Box B \leq \Box[A \wedge B],\quad \Diamond[A \vee B] \leq \Diamond A \vee \Diamond B$$

and rules (besides modus ponens):

$$\frac{A \leq B}{\Box A \leq \Box B},\ \frac{A \leq B}{\Diamond A \leq \Diamond B}.$$

It is clear that IK_0 $(\Box, \Diamond)$ is in fact M "taken twice". By $IK_0(\Box)$ we denote the logic with only modality $\Box$ and IK_0 $(\Diamond)$ concerns only $\Diamond$.

Definition. A 6-tuple $\langle K, M, N, \leq, \mathrm{R}, \mathrm{R}^*\rangle$ is called a *Kripke frame* if

1-3. K and $\leq$ satisfy the conditions from the definition of Montague frame;

4. $M \subseteq K$, $N \subseteq K$;
5. if $x \in M$ and $y \leq x$, then $y \in M$;
6. if $x \in N$ and $x \leq y$, then $y \in N$;
7. $\mathrm{R} \subseteq K^2$ (when $\langle x, y\rangle \in \mathrm{R}$ we shall write $x\mathrm{R}y$);
8. the same for R^*;
9. $x\mathrm{R}y$, $u \leq x$ and $y \leq v$ imply $u\mathrm{R}v$ ("extrapolation" of R);
10. R^* can be "interpolated": $x \leq u$, $v \leq y$, $x\mathrm{R}^*y$ imply $u\mathrm{R}^*v$.

On a Kripke frame one can define *Kripke models* by assigning cones of K to propositional variables and extending the assignment to all formulas by the familiar inductive definition, in particular the causes for $\Box$ and $\Diamond$ being:

$$x \Vdash \Box A \text{ iff } x \in N \text{ and } (\forall y) \text{ (if } x\mathrm{R}y, \text{ then } y \Vdash A),$$

$$x \Vdash \Diamond A \text{ iff } x \in M \text{ implies } (\exists y)(x\mathrm{R}^*y \text{ and } y \Vdash A).$$

Here $x \Vdash A$ (the "forcing" relation) means that x belongs to the subset of K assigned to A. A straight-forward argument shows that all such subsets are cones, i. e. $x \Vdash A$ and $x \leq y$ imply $y \Vdash A$. Now we can give literally the same definitions of truth in a Kripke model, validity in a Kripke frame and of Kripke validity, as in the case of Montague frames.

Before we proceed to the proof of the completeness theorem for IK_0 $(\Box, \Diamond)$ we mention the fact that it will be pieced together from two independent analogous proofs for IK_0 $(\Box)$ and for $IK_0(\Diamond)$. This will bring about certain advantageous properties, used later on.

Lemma 13. If a sequent (formula) is provable in IK_0 $(\Box)$ then it is Kripke valid.

The proof is evident by checking. We consider only the modal axiom: let $x \Vdash \Box A \wedge \Box B$, then $x \Vdash \Box A$ and $x \Vdash \Box B$, so for an y, such that $x\mathrm{R}y$, we have simultaniously $y \Vdash A$ and $y \Vdash B$ and (by the definition of $\Vdash$) $y \Vdash A \wedge B$. Hence we get $x \Vdash \Box[A \wedge B]$, as $x \in N$.

For the converse statement we again build up a frame from the prime filters in the formula algebra. Let K be the set of all such filters, $\leq$ be the set theoretical inclusion. Define $N = \{x \mid \Box 1 \in x\}$. It is clear that $x \in N$ and $x \leq y$ imply $y \in N$. Denoting by $\Box x$ the set $\{A \mid \Box A \in x\}$ we have the following

Proposition 10. If x is a filter and $\Box 1 \in x$ then $\Box x$ is again a filter.

Proof. Since $\Box 1 \in x$ then $1 \in \Box x$, so $\Box x \neq \emptyset$. If $A \in \Box x$ and $A \leq B$ then (by the monotonicity rule) $\Box B \in x$ and $B \in \Box x$. Finally the modal axiom gives the closure of $\Box x$ with respect to $\wedge$.

Define R by $x\mathrm{R}y$ iff $\Box x \subseteq y$. The "extrapolation" property is evident. Thus we have obtained a Kripke frame for the language with $\Box$. Defining a model on it by assigning to variables the set of all filters that contain them, we are able to prove the main feature: $x \Vdash A$ iff $A \in x$, not only for atomic but for arbitrary formulas. The proof is by induction on the complexity of the formula A. The only new case is the case $\Box A$. Assume

$\Box A \in x$, then $A \in \Box x$ and $\Box 1 \in x$, so $x \in N$. Now if xRy, then $\Box x \subseteq y$, $A \in y$ and (by the induction hypothesis) $y \Vdash A$, thus $x \Vdash \Box A$. In the opposite direction: if $\Box A \notin x$ and $\Box 1 \in x$, then, as just proved, $\Box x = y_0$ is a filter such that $A \notin y_0$. Then we extend y_0 to a prime y also excluding A. For this y we have xRy and not $y \Vdash A$ (by the induction hypothesis). Therefore not $x \Vdash \Box A$.

In this way it is shown that $\langle K, N, \leq, R, \Vdash \rangle$ is a Kripke model for the logic without $\Diamond$, where all unprovable formulas are refuted. So we get

Proposition 11. A sequent (formula) is provable in $IK_0(\Box)$ iff it is Kripke valid.

Next we consider the system without $\Box$.

Lemma 14. If a sequent (formula) is provable in $IK_0(\Diamond)$ then it is valid.

As with correctness lemmas, the proof is just checking the validity of axioms and the property of rules to preserve validity. For the converse statement we again use the described above structure K. Define $M \subseteq K$ by $M = \{x \mid \Diamond 0 \notin x\}$.

Proposition 12. If x is a co-filter, and $\Diamond 0 \notin x$, then $\Diamond x$ is a co-filter, too.

We omit the proof which is analogous to that of Proposition 9.

The relation R^* is introduced by setting xR^*y iff $y \subseteq \Diamond x$. In this case the "interpolation" property is evident. Using the familiar forcing we can as above prove by induction on the complexity of a formula A that $x \Vdash A$ iff $A \in x$. Thus we obtain

Proposition 13. A sequent (formula) is provable in $IK_0(\Diamond)$ iff it is Kripke valid.

Below we consider the question of additional axioms and additional properties of Kripke structures. Recall the notion of adequacy from Section 2.

Theorem 7 (the case of non-normal extensions). The following axioms and conditions are adequate:

$\Box 1$	$N = K$
$\neg\Diamond 0$	$M = K$
$\neg\Box 1$	$N = \varnothing$
$\Diamond 0$	$M = \varnothing$
$\Box 1 \leq \Diamond 0$	$N \cap M = \varnothing$
$\Diamond 0 \leq \Box 1$	$M \cup N = K$
$\Diamond 0 \vee \Box 1$	$M \subseteq N$
$\Box 1 \leq \neg\Diamond 0$	$N \subseteq M$
$\Diamond 0 \vee \neg\Diamond 0$	if $x \in M$ and $x \leq y$ then $y \in M$
$\Box 1 \vee \neg\Box 1$	if $x \in N$ and $y \leq x$ then $y \in N$
$\Box 1 \equiv \Diamond 0$	$M = \overline{N}$
$\Diamond 1$	if $x \in M$ then $(\exists y)[xR^*y]$
$\Box 0$	$N = K$ and $R = \varnothing$

For the proofs of all these, we shall use one and the same model — the cannonical model of the corresponding system. We treat only one example: the axiom $\Box 0$. Assume that in any model on a frame satisfying the conditions for some x, not $x \Vdash \Box 0$. This means that either $x \notin N$ (contradiction with the condition $N = K$) or for some y, $x\mathrm{R}y$ and not $y \Vdash 0$ (contradiction with $\mathrm{R} = \varnothing$). In the cannonical frame: $\Box 0 \in x$ and so $x \in N$, i. e. $K = N$; if there is an y such that $x\mathrm{R}y$, then we get $0 \in y$ — a contradiction, so $\mathrm{R} = \varnothing$.

Here we have chosen examples that exhibit properties typical for non-normal systems (concerning M and N). If we consider connections between R and R* we usually get (in the case of non-normal logics) conditions and the respective adequate axioms which have a rather complicated "apendexes":

$\Diamond \Box A \leq A$	$M = K$, and if $y\mathrm{R}^*x$ then $x\mathrm{R}y$
$\Box 1 \leq [A \Rightarrow \Box \Diamond A]$	if $x \in N$ and $x\mathrm{R}y$ then $y\mathrm{R}^*x$
$A \leq \Box \Diamond A$	$N = K$, and if $x\mathrm{R}y$ then $y\mathrm{R}^*x$
$\Box 1 \leq A \vee \Box \neg \Box A$	if $x \in N$ and $x\mathrm{R}y$ then $y\mathrm{R}x$

We omit the proofs (completely analogous to the ones presented above).

The independence of the completeness proofs for each of the "half" logics $\boldsymbol{IK}_0(\Box)$ and $\boldsymbol{IK}_0(\Diamond)$ allows us to establish easily

Theorem 8 (modal separation theorem). If a formula (sequent) contains only $\Box$ $(\Diamond)$ it is provable in $\boldsymbol{IK}_0(\Box, \Diamond)$ iff it is provable in $\boldsymbol{IK}_0(\Box)$ $(\boldsymbol{IK}_0(\Diamond))$.

Proof. If A contains only $\Box$, belongs to $\boldsymbol{IK}_0(\Box, \Diamond)$ and can not be proved in $\boldsymbol{IK}_0(\Box)$ alone, then we can construct a countermodel $\langle K, N, \leq, \mathrm{R}, \Vdash \rangle$ by Proposition 11. The expansion $\langle K, K, N, \leq, \mathrm{R}, K^2, \Vdash \rangle$ is clearly a model for $\boldsymbol{IK}_0(\Box, \Diamond)$ (here $M = K$ and $\mathrm{R}^* = K^2$) such that any formula B without $\Diamond$ is simultaneously forced or not forced by $x \in K$ in both models. Thus, the second model is a countermodel for A, too. A contradiction.

Theorem 9 (conservative extension theorem). $\boldsymbol{IK}_0(\Box, \Diamond)$ is a conservative extension of (pure) intuitionistic propositional calculus, i. e. if a formula A without $\Box$ and $\Diamond$ is provable in $\boldsymbol{IK}_0(\Box, \Diamond)$ then it is an intuitionistic tautology.

Proof. In view of the completeness theorem for the propositional intuitionistic logic, any A which is not an intuitionistic tautology is refuted in a model $\langle K, \leq, \Vdash \rangle$. As in the proof of the previous theorem we expand this model to a model for $\boldsymbol{IK}_0(\Box, \Diamond)$: $\langle K, K, K, \leq, K^2, K^2, \Vdash \rangle$, where A is refuted, too. So, A is not a theorem of $\boldsymbol{IK}_0(\Box, \Diamond)$.

Now we turn to *normal extensions* of the minimal logic. These are logics closed under the rules $\frac{A}{\Box A}$ and $\frac{\neg A}{\neg \Diamond A}$ or equivalently, containing the axioms $\Box 1$ and $\neg \Diamond 0$. The minimal normal logic is denoted by $\boldsymbol{IK}(\Box, \Diamond)$. By the above results (Theorem 7) we have for the adequate semantics a condition $M = N = K$.

Proposition 14. $\boldsymbol{IK}(\Box, \Diamond)$ is complete for frames of the kind $\langle K, \leq, \mathrm{R}, \mathrm{R}^* \rangle$.

We note that in this case the forcing relation is simplified: $x \Vdash \Box A$ iff $(\forall y)$ [if $x\mathrm{R}y$ then $y \Vdash A$], $x \Vdash \Diamond A$ iff $(\exists y)$ [$x\mathrm{R}^*y$ and $y \Vdash A$].

It can be said that in general the semantics for normal modal intuitionistic logics with $\Box$ was known, although there have not been published studies of $\Diamond$ and, what is more important, the considered cases were extremely simple (some variants of intuitionistic $S4$ and $S5$). Below we give a large number of examples of axioms with their corresponding adequate conditions. Among them there are some that are anything but evident. Besides, we answer some very natural questions concerning R and R* as for example what modal formulas guarantee the symmetry of the relation R*.

Theorem 10 (normal extensions). The folowing list consists of axioms and adequate conditions on the semantics:

$\Diamond\Box A\leq A$	if xR^*y then yRx
$A\leq\Box\Diamond A$	if xRy then yR^*x
$\Box A\leq A$	xRx (or: if $x\leq y$ then xRy)
$A\leq\Box A$	if xRy then $x\leq y$
$A\leq\Diamond A$	xR^*x (or: if $x\leq y$ then yR^*x)
$\Diamond A\leq A$	if yR^*x then $x\leq y$
$\Box A\leq\Box\Box A$	if xRy and yRz then xRz
$\Diamond\Diamond A\leq\Diamond A$	if xR^*y and yR^*z then xR^*z
$A\vee\Box\neg\Box A$	if xRy then yRx
$\Box\neg\Box A\vee\Box\neg\Box\neg A$	if xRy and xRz then $(\exists p)[yRp$ and $zRp]$
$\Diamond A\vee\neg\Diamond A$ and $\Diamond\neg\Diamond A\leq\neg A$	if xR^*y then yR^*x
$\Diamond A\vee\neg\Diamond A$	if xR^*y and $z\leq x$ then zR^*y
$\Box A\vee\neg\Box A$	if xRy and $x\leq z$ then zRy
$\Box[A\vee\neg A]$	if xRy and $z\geq y$ then $z=y$
$\Box A\vee\Box\neg A$	if xRy and xRz then $y=z$
$\Box\neg\Box A\vee\Box A$	if xRy and xRz then yRz
$\Box A\leq\Diamond A$	$(\exists y)[xRy$ and $xR^*y]$
$\Box[A\vee B]\leq\Box A\vee\Box B$	if xRy and xRz then $(\exists p)[p\leq y$, $p\leq z$ and $xRp]$
$\Diamond A\wedge\Diamond B\leq\Diamond[A\wedge B]$	if xR^*y and xR^*z then $(\exists p)[y\leq p$, $z\leq p$ and $xR^*p]$
$\Diamond\Box A\leq\Box\Diamond A$	if xR^*y and xRz then $(\exists p)[yRp$ and $zR^*p]$
$\Box\neg A\leq\neg A$	if $x\leq y$ then $(\exists t)[xRt$ and $y\leq t]$
$\Box\Box A\leq\Box A$	if xRy then $(\exists t)[xRt$ and $tRy]$
$\Diamond A\vee\Box\neg A$	if xRy then xR^*y
$\Box[\Box A\Rightarrow B]\vee\Box[\Box B\Rightarrow A]$	if xRy and xRz then yRz or zRy
$[\Box A\Rightarrow B]\vee[\Box B\Rightarrow A]$	if $x\leq y$ and $x\leq z$ then yRz or zRy
$\Box[A\Rightarrow B]\vee\Box[B\Rightarrow A]$	if xRy and xRz then $y\leq z$ or $z\leq y$
$A\leq\Box\Box A$	if xRy and yRz then $x\leq z$
$\Diamond A\leq\neg\neg A$	if xR^*y then $(\exists s)[x\leq s$ and $y\leq s]$
$\neg\Box A\vee\neg\Box\neg A$	if $x\leq y$ and $x\leq z$ then $(\exists p)[yRp$ and $zRp]$
$\Box A\leq\neg\neg A$	if $x\leq y$ then $(\exists p)[y\leq p$ and $xRp]$
$\Diamond\neg A\leq\neg\Box A$ (or $\Box A\leq\neg\Diamond\neg A$, or $\Diamond A\leq\neg\Box\neg A$, or $\Box\neg A\leq\neg\Diamond A$, or $\neg[\Box A\wedge\Diamond\neg A]$)	if xR^*y then $(\exists s)[xRs$ and $y\leq s]$

We are not going to check all these (most of them will be left to the reader) and take up a couple of (instructive) examples: (1). $\Diamond\Box A\leq\Box\Diamond A$ — first we show that the conditions imply the validity of the sequent. If in a model on a frame satisfying the conditions $x\Vdash\Diamond\Box A$ then there is an y, xR^*y and for all z, yRz implies $z\Vdash A$. Now if xRt we have to show that $t\Vdash\Diamond A$ or equivalently exhibit an w, tR^*w with $w\Vdash A$. By the condition from xR^*y and xRt we get an v such that yRv and tR^*v. But yRv gives $v\Vdash A$ by the choice of y. So v is what we need (put $w=v$). For the converse, consider the cannonical frame and take x, y, z — prime filters with xR^*y and xRz, i. e. $\Box x\subseteq z$, $y\subseteq\Diamond x$, and hence, $\Box y\subseteq\Box\Diamond y$, $\Diamond\Box x\subseteq\Diamond z$. On the other hand $\Box\Diamond x\subseteq\Diamond\Box x$ which is easily proved using the axiom. Thus $\Box y \subseteq \Diamond z$. But $\Box y$ is a filter, $\Diamond z$ — a co-filter, and applying the Interpolation Lemma there is a prime filter $p: \Box y \subseteq p \subseteq \Diamond z$, i. e. $y\mathrm{R}p$ and $z\mathrm{R}^*p$.

(2). $\Box\Box A \leq \Box A$. Clearly the conditions imply the validity of the axiom. Conversly, in the cannonical frame if $x\mathrm{R}y$ then there is a prime filter t with $x\mathrm{R}t\mathrm{R}y$. To show this we use Zorn lemma: let Φ be the set of all z — proper filters, that satisfy $\Box x \subseteq z$, $\Box z \subseteq y$. Φ is not empty: $\Box x \in \Phi$. If Φ is ordered by set-theoretic inclusion then it has all properties required by Zorn lemma. So Φ has at least one maximal element t. This filter is prime: if we assume the contrary, i. e. $A \vee B \in t$ but $A \notin t$, $B \notin t$, we can form (t, A) and (t, B) both of which are proper filters and extend t. At least one of them, (t, A) or (t, B) has the property $\Box(t, A) \subseteq y$ or $\Box(t, B) \subseteq y$. If not then we can find C and D, $\Box C \in (t, A)$, $\Box D \in (t, B)$, $C \notin y$, $D \notin y$. So $C \vee D \notin y$. But then for C', D' from $t: C' \wedge A \leq \Box C$, $D' \wedge B \leq \Box D$, $A \vee B \in t$, $C' \wedge D' \in t$ and so $[A \vee B] \wedge [C' \wedge D'] \leq \Box C \vee \Box D$; hence $\Box C \vee \Box D \in t$ and moreover $\Box[C \vee D] \in t$, therefore $C \vee D \in y$ — a contradiction.

We have just shown the completeness of several logics with respect to classes of Kripke frames. A proposition of a somewhat different nature is also true in theses cases: the axioms characterize the conditions, i. e. they are valid in a frame iff the frame satisfies the condition.

The problem of comleteness and the problem of characterization are related but may have different solutions. Any axiom characterizes a second order condition. The situation is identical with that in the case of Montague frame: we can effectively write down the (second order, with quantifiers ranging over cones) condition characterized by an axiom.

Sometimes (and these are the interesting instances) this second order sentence is equivalent to a sentence in the first order fragment of the same language, i. e. a sentence without quantifiers over cones.

Proposition 15. The listed sequents characterize the following first order conditions:

$\neg\neg A \leq \Diamond A$	$(\exists y)[x \leq y$ and $(\forall z)[$if $y \leq z$ then $x\mathrm{R}^*z]]$
$\Box\Diamond A \leq A$	$(\exists y)[x\mathrm{R}y$ and $(\forall z)[$if $y\mathrm{R}^*z$ then $z \leq x]]$
$A \leq \Diamond\Box A$	$(\exists y)[z\mathrm{R}^*y$ and $(\forall z)[$if $y\mathrm{R}z$ then $x \leq z]]$
$\Diamond\neg A \vee \Box A$	if $x\mathrm{R}y$ then $(\exists t)[x\mathrm{R}^*t$ and $(\forall s)[$if $t \leq s$ then $s \leq y]]$
$\neg\Box\neg A \leq \Diamond A$	$(\exists p)[x \leq p$ and $(\forall y)[$if $p\mathrm{R}y$ then $x\mathrm{R}^*y]]$
$\neg\Diamond A \leq \Box\neg A$	if $x\mathrm{R}y$ then $(\exists z)[x \leq z$ and $z\mathrm{R}^*y]$
$\neg\neg A \leq \Box A$	if $x\mathrm{R}y$ then $(\exists p)[x \leq p$ and $(\forall q)[$if $p \leq q$ then $q \leq y]]$
$\neg\Diamond\neg A \leq \Box A$	if $x\mathrm{R}y$ then $(\exists z, t)[x \leq z$, $z\mathrm{R}^*t$ and $(\forall s)[$if $t \leq s$ then $s \leq y]]$

We omit the tedious but otherwise quite straithforward proof.

Again we should mention that if a logic is complete with respect to a class of Kripke frames, then if it is given by additional axioms, we can claim that it is complete also with respect to the class of Kripke frames, determined by the corresponding conditions. This class is the maximal class with such a property. Unfortunately there are incomplete logics, and we can not have a full duality between logics and classes of Kripke frames.

There is a narural connection between Kripke and Montague semantics. The former turns out to be a special case of the latter.

Proposition 16. If a logic is Kripke complete, then it is Montague complete. Moreover, it is complete with respect to a class of Montague frames $\langle K, \leq, \mathrm{J}, \mathrm{D}\rangle$ where $\mathrm{J}(P\cap Q)=\mathrm{J}P\cap \mathrm{J}Q$ and $\mathrm{D}(P\cup Q)=\mathrm{D}P\cup \mathrm{D}Q$.

The proof is based on the observation that given a Kripke frame $\langle K, M, N, \leq, \mathrm{R}, \mathrm{R}^*\rangle$ we can construct a Montague frame (the generated frame) $\langle K, \leq, \mathrm{J}, \mathrm{D}\rangle$ setting:

$$x\in \mathrm{J}P \text{ iff } x\in N \text{ and } (\forall y)[\text{if } x\mathrm{R}y \text{ then } y\in P],$$

$$x\in \mathrm{D}P \text{ iff } x\in M \text{ implies } (\exists y)[x\mathrm{R}^*y \text{ and } y\in P].$$

The verification of the properties of J and D required by the definition of Montague frame and in the proposition are left to the reader.

Now any model on the Kripke frame can be transformed into a model on the generated Montague frame, preserving the forcing for atomic formulas. It can be proved by induction on the complexity of formulas that the two forcings coincide for all formulas. Now in particular, if we consider only normal logics, we have

Proposition 17. If a normal logic is Kripke complete then it is Montague complete and moreover one may restrict his attention to frames where $\mathrm{J}K=K$ and $\mathrm{D}\varnothing=\varnothing$.

Proposition 18. If a logic is Kripke complete then it is topologically complete.

Here we have first to show that Montague completeness implies topological completeness and then apply Proposition 16.

Definition. $\boldsymbol{IS}5(\Box+\Diamond)$ is the extension of $\boldsymbol{IS}4(\Box, \Diamond)$ by the axioms $A\vee\Box\neg\Box A$, $\Diamond A\vee\neg\Diamond A$, $\Diamond\neg\Diamond A\leq\neg A$, $\Diamond\Box A\leq A$, $A\leq\Box\Diamond A$.

Here the sign $+$ emphasizes the fact that R and R* are not independent but coincide. By Theorem 10 the added axioms make $\mathrm{R}=\mathrm{R}^*$ reflexive and transitive; and the symmetry of the relation is guaranteed by $A\vee\Box\neg\Box A$, $\Diamond A\vee\neg\Diamond A$ and $\Diamond\neg\Diamond A\leq\neg A$.

Corollary 3. $\boldsymbol{IS}5(\Box+\Diamond)$ is complete with respect to Kripke frames where $\mathrm{R}=\mathrm{R}^*$ is an equivalence relation.

Corollary 4. $\boldsymbol{IS}5(\Box+\Diamond)$ is complete for the class of Kripke frames having the kind $\langle K, \leq, K^2, K^2\rangle$.

This can be established by fixing a countermodel for a non-provable in $\boldsymbol{IS}5(\Box+\Diamond)$ formula A taking an x such that not $(x\Vdash A)$ and forming the so-called generated submodel (in fact it will consist only of the members of the equivalence class of x with respect to R). This submodel will have the properties described in Corollary 4 and will be a countermodel for A.

We can say even more:

Corollary 5. $\boldsymbol{IS}5(\Box+\Diamond)$ is complete for frames of the kind $\langle K, \leq\rangle$ where $x\Vdash\Box A$ iff $(\forall y)[y\Vdash A]$ and $x\Vdash\Diamond A$ iff $(\exists y)[y\Vdash A]$.

This interpretation of $\boldsymbol{S}5$-modalities is the same as in the case of the classical modal system $\boldsymbol{S}5$ (with the appearance of $\leq$ of course). From it one can easily get a purely syntactic result:

Corollary 6. In $\boldsymbol{IS}5(\Box+\Diamond)$ $\Box A$ and $\Diamond A$ are Boolean, i. e. $\Box A\vee\neg\Box A$ and $\Diamond A\vee\neg\Diamond A$ are theorems.

Nevetheless, even in the presence of $\Box A\leq A$ we have

Corollary 7. $\boldsymbol{IS}5(\Box+\Diamond)$ is a conservative extension of the intuitionistic propositional calculus.

Corolary 8. (a separation theorem for $IS5(\Box+\Diamond)$). The connectives $\Box$ and $\Diamond$ can be separated in $IS5(\Box+\Diamond)$.

Corollary 9. $IS5(\Box+\Diamond)\vdash\Diamond A\equiv\neg\Box\neg A$.

The proof of the latter corollary is semantical, by the use of the completeness results on $IS5(\Box+\Diamond)$.

It must be noted that $\Box$ cannot be defined by means of $\Diamond$ in the same way. We can construct a model where $\Box p\equiv\neg\Diamond\neg p$ is not true.

This shows that in intuitionistical modal logic, even in the stongest systems, the duality between $\Diamond$ and $\Box$ is not complete,

5. DECIDABILITY RESULTS

In this Section of our paper we are going to apply the apparatus developed in the previous parts in order to show the decidability of some of the introduced logics. More precisely, we shall prove for some systems that they are complete with respect to classes of models of a special kinds, namely, of finite models. This property of a logic is usually termed "the finite model property" (f.m.p.). We prefer to call it *finite completeness*. If a logic is finitely (Montague, topological or Kripke) complete and axiomatized by a recursive set of axioms, then it is decidable. The decidability is a consequence of the familiar method establishing the theoremhood: list all proofs in an effective way and search for the proof of a formula A, parallel to that start another process of checking the truth of A in finite models. This procedure is always terminating and gives a definite answer, if the logic has the f.m.p. (cf. [10] and [7]). Moreover there is an a priori upper bound l (depending on the complexity of A) for the number of elements in the refuting model.

Below we list about 30 systems, among them intuitionistic analogues of all interesting classical modal systems, for which we prove decidability. The basic tool to be used is the well known method of filtration of algebraic models (to be described shortly). Of course, the intuitionistic case is more complicated. An example: for the classical logic of Brouwer a very simple filtration procedure yields finite completeness, while for the intuitionistic analogue it is necessary to carry out a systematic and non-trivial investigation of its syntactic properties before we can apply the method.

Definition. An 7-tuple $\langle\Psi, o, \leq, \wedge, \vee, \Rightarrow, \mathsf{M}\rangle$ is called *modal algebra* if $\langle\Psi, o, \wedge, \vee, \Rightarrow\rangle$ is a pseudo Boolean algebra, and M — an unary operation in Ψ (i. e. $a=b$ implies $\mathsf{M}a=\mathsf{M}b$). A modal algebra $\langle\Psi, o, \leq, \wedge, \vee, \Rightarrow, \mathsf{M}\rangle$ equiped with a valuation f (i. e. f is a function from all propositional variables to elements of the modal algebra) is called an *algebraic model*.

In an algebraic model the valuation f is inductively extended to all formulas A. We call A true in the model if $f(A)=1$. A sequent $A\leq B$ is true if $f(A)\leq f(B)$. Further, A is valid in a modal algebra if A is true in every model based on that algebra. Finally A is *algebraically valid* if it is valid in every modal algebra. Analogous definitions can be given for sequents.

It is clear that any logic L is algebraically complete, i. e. it is equal to the set of all formulas valid in every algebra where all theorems of L are valid. Indeed in one direction this claim is established trivially and in the other we use the Lindenbaum algebra of L.

This shows that algebraic semantics is very general, but at the same time not very informative because it differs insignificantly from the logic itself. Sometimes though, the existence of just one refuting model is sufficient for the decidability and an algebraic model of this kind always exists, so we are going to make use of this fact.

Each modal algebra $\langle \Psi, o, \leq, \wedge, \vee, \Rightarrow, \mathrm{M}\rangle$ generates a Montague frame $\langle K, \leq, J\rangle$ where K is the set of prime filters in Ψ, $x \leq y$ iff $x \subseteq y$, and $x \in JP$ iff $(\exists a)[P = h(a)$ and $\mathrm{M}a \in x]$. An algebraic model on $\langle \Psi, o, \leq, \wedge, \vee, \Rightarrow, \mathrm{M}\rangle$ with valuation f defines on the generated frame a Montague model: $x \Vdash p$ iff $f(p) \in x$, for a propositional variable p.

Lemma 15. If an algebraic model $\langle \Psi, o, \leq, \wedge, \vee, \Rightarrow, \mathrm{M}, f\rangle$ generates a Montague model $\langle K, \leq, J, \Vdash\rangle$, then for an arbitrary formula A: $x \Vdash A$ iff $f(A) \in x$.

We omit the easy proof which gives by induction.

For the extension of $IK_0(\Box, \Diamond)$ we introduce the *bi-modal algebras* $\langle \Psi, o, \leq, \wedge, \vee, \Rightarrow, \Box, \Diamond\rangle$. Given such an algebra the generated Kripke frame is $\langle K, M, N, \leq, R, R^*\rangle$ where K is again the set of all prime filters in Ψ, $N = \{x \in K \mid \Box 1 \in x\}$, $M = \{x \in K \mid \Diamond 0 \notin x\}$, $x \leq y$ if $x \subseteq y$, xRy if $\Box x \subseteq y$, and xR^*y if $y \subseteq \Diamond x$. Again we have

Lemma 16. If a bi-modal algebraic model generates a Kripke model, then for every A: $x \Vdash$ iff $f(A) \in x$.

We have now four kind of models: algebraic, Kripke, Montague and topological. We shall call a model *finite* if its underlying set is finite.

Definition. A logic L is called *finitely complete* if there is a class $\mathcal{M}$ of finite frames (or algebras) such that all theorems of L are valid in all stuctures of $\mathcal{M}$ and any non-provable formula is refuted in a model based on a structure from $\mathcal{M}$.

Lemma 17. (i) If an extension of $IK_0(\Box, \Diamond)$ is finetely complete with respect to bi-model algebras, then it is finitely complete for Kripke frames.

(ii) If a logic is finitely complete for Kripke frames, then it is finitely complete for Montague frames.

(iii) If a logic is finitely complete for Montague frames, then it is finitely complete with respect to generalized topological spaces.

(iv) If a logic is finitely complete for topological frames, then it is finitely complete for modal algebras.

Proof. The proof of (i)—(iv) are quite straightforward. We use the generated structures and by the above lemmas establish the desired result. Only in case (iv) we have to define an algebra given a generalized topological space $\langle K, I, J\rangle$. The algebra is $\langle K^*, \varnothing, \subseteq, \cap, \cup, \Rightarrow, \mathrm{M}\rangle$ where K^* is the class of all I-open subsets of K; $\cap$, $\cup$, $\Rightarrow$ are the usual set-theoretical operations $(a \Rightarrow b = \mathrm{I}(\bar{a} \cup b))$ and $\mathrm{M}a = Ja$.

Thus we obtain the following absoluteness property:

Proposition 19. The extensions of $IK_0(\Box, \Diamond)$ are equivalently finitely complete with respect to:

(i) Kripke frames; (ii) Montague frames; (iii) generalized topological spaces; (iv) bi-modal algebras.

Analogously we can show

Proposition 20. The extensions of M are equivalently finitely complete with respect to:

i) Montague frames; ii) generalized topological spaces: iii) modal algebras.

We turn now to the description of the method of filtration applied in the case of algebraic semantics. We shall follow H a n s s o n and G ä r d e n f o r s [13], [14] where one can find a clear exposition of the construction of finite refutation algebras. In the classical case a finite subset Ψ of a Boolean algebra will generate a finite Boolean subalgebra, while in case of pseudo Boolean algebras the generated algebra may be infinite.

Nevertheless, we can include Ψ into a finite pseudo Boolean algebra such that the restrictions of all operations in it to Ψ coincide with the operations in the original algebra — this is the well-known McKinsey — Tarsky theorem (cf. [3]).

We shall apply their result according to the following scheme: if a logic $\boldsymbol{L}$ is an extension of $\boldsymbol{M}$, then it is characterized by the class of modal algebras in each one member of which all theorems of $\boldsymbol{L}$ are valid. In particular the Lindenbaum algebra of $\boldsymbol{L}$ is in this class, so, when a formula A is not provable it is refuted at least in one modal algebra of this kind. In order to construct a finite algebra in this class we take a finite set Ψ' of formulas, containing all subformulas of A and consider its image in the Lindenbaum algebra (i.e. the set of equivalence classes of members of Ψ'). The image is finite, too. By McKinsey — Tarski theorem it is a part of a finite pseudo Boolean algebra Ψ^0 and all operations within Ψ' are preserved as well as 1 and 0. In general, though, Ψ^0 will not be well defined modal algebra. We solve this problem by defining for each particular logic a special operation M which coincides with the original operation on the relevant elements and in such a way that the modal algebra (Ψ^0, M) is in the class, defined by the logic. This last feature makes of (Ψ^0, M) the refutation algebra we need.

For logics having Kripke semantics we shall employ the filtration method. Assume $\langle K, \leq, \mathrm{R}, \mathrm{R}^*, \Vdash\rangle$ is a model for a normal extension of $\boldsymbol{IK}(\Box, \Diamond)$ and Ψ is a set of formulas closed under subformulas (0 is subformula of any formula). Define $\approx$ on K as follows: $x \approx y$ iff $(\forall A \in \Psi)[x \Vdash A$ iff $y \Vdash A]$. Clearly $\approx$ is an equivalence relation. Denote by $|x|$ the class $\{y \mid y \approx x\}$ and by K^0 the set of classes $\{|x| \mid x \in K\}$. The elements of K^0 will usually be denoted by Greek letters: $\alpha, \beta, \ldots$ Set $\alpha \leq^0 \beta$ iff $(\forall x \in \alpha)(\exists y \in \beta)$ $(\forall A \in \Psi)$ [if $x \Vdash A$, then $y \Vdash A$]. Evidentely $\leq^0$ is reflexive and transitive and $x \leq y$ implies $x \leq^0 y$. Assume further, that in K^0 we have relations R^0 and R^{*0} with the properties:

$$\text{if } x\mathrm{R}y, \text{ then } |x|\,\mathrm{R}^0\,|y| \ (\text{if } x\mathrm{R}^*y \text{ then } |x|\,\mathrm{R}^{*0}\,|y|);$$

if $\alpha \mathrm{R}^0 \beta$ then $(\forall x \in \alpha)(\forall y \in \beta)(\forall A)$[if $\Box A \in \Psi$ and $x \Vdash \Box A$ then $y \Vdash A$]

(analogously for R^*).

Unfortunately R^0 and R^{*0} are not always the relations we need in order to have a Kripke frame in contrast to the classical case. Sometimes though we are able to prove that they satisfy the conditions imposed on Kripke frames.

If we define: $\alpha \Vdash^0 p$ iff $(\forall x \in \alpha)[x \Vdash p]$ for variables in Ψ and otherwise arbitrary, then extending $\Vdash^0$ to all formulas we get

Lemma 18. If $\langle K^0, \leq^0, \mathrm{R}^0, \mathrm{R}^{*0}, \Vdash^0\rangle$ is a Kripke model, then for $A \in \Psi$ and $x \in K: x \Vdash A$ iff $|x| \Vdash^0 A$.

In cases when $\langle K^0, \leq^0, R^0, R^{*0}, \Vdash^{0} \rangle$ is a Kripke model we say that it is the model obtained by *filtering* $\langle K, \leq, R, R^*, \Vdash \rangle$ *through* Ψ. The classical theory of filtration can be found in [6] and [10]. The proof of Lemma 18 is by induction on the complexity of A. The most interesting case of modal operators is handled by means of the conditions for R^0 and R^{*0}.

Assuming this lemma proved we are able to describe the method of filtration: let L be an extension of $IK(\Box, \Diamond)$ adequate for the condition C. If a formula A is not provable in L, then it is refuted in some Kripke model $\langle K, \leq, R, R^*, \Vdash A \rangle$ satisfying C, i. e. there is $x_0 \in K$ such that not $x_0 \Vdash A$. We shall choose a finite set Ψ of formulas including all subformulas of A, closed under taking subformulas. Then we try to define R^0 and R^{*0} in such a way that we get a Kripke frame satisfying C. In case of a success, we will have a finite countermodel (because if Ψ is finite, then K^0 will always be finite).

If the above described constructions are not applicable, another procedure is available. In Section 2 we considered some extensions of the logic M complete with respect to monotone Montague semantics. Here we are going to show the decidability of several such logics using the cannonical frame $\langle K, \leq, R^{20} \rangle$.

Let Ψ be a finite set of formulas with the already familiar property of being closed under subformulas. Define $K^0, \leq^0$ and $\Vdash^{0}$ as in Lemma 18 and for $P \subseteq K$:

$$P^0 = \{\alpha \mid (\exists x)[x \in \alpha \text{ and } x \in P\}.$$

For $Q \subseteq K$, $\widehat{Q} = \bigcup \{x \mid x \in \alpha \text{ and } \alpha \in Q\}$. Easily, $P \subseteq \widehat{P^0}$. Assume now that the relation R^0 satisfies: $R^0 \subseteq K^0 \times 2^{K^0}$; if $xR^{20}P$ then $|x| R^0 P^0$; if $\alpha R^0 Q$, then $(\forall x \in \alpha)(\forall A)$[if $MA \in \Psi$ and $\widehat{Q} \subseteq h(A)$, then $MA \in x$].

Lemma 19. If $\langle K^0, \leq^0, R^0, \Vdash^{0} \rangle$ is a Montague model, then for $A \in \Psi$: $x \Vdash A$ iff $|x| \Vdash^{0} A$.

The proof uses the properties of R^0 assumed above. Generally speaking this lemma will be exploited in the same way as the last one. If we succeed to define R^0 which is monotone with respect to both arguments, satisfies the rest of the conditions on Montague frames and has the properties from the lemma, then the finite completeness of the logic will follow.

This lemma exibits some of the complications arising from the intuitionistic basis of the systems under consideration. For some monotone logics we are able to construct filtrations only of the cannonical model and even here we can do it only for the choice of R^{20}.

Nevertheless, the technique we have introduced is enough to establish the following

Theorem 11 (on decidability). The logics listed on the next page are with the finite model property.

Remarks. In the first half of this table only logics which have just Montague semantics appear, and the logics in lower part are monotone. The second half of the table includes logics — with Kripke semantics, and the systems in the lower part are normal. $\frac{M}{\Box}$ and $\frac{M}{\Diamond}$ show that M in the language is changed to $\Box$ or $\Diamond$ respectively.

Proof of the theorem. We consider $IK(\Box, \Diamond)$ first. As is already known if A_0 is not provable in $IK(\Box, \Diamond)$ then it is refuted in a model

Name	Additional axioms and rules
$\boldsymbol{M}$	$\dfrac{A\equiv B}{\mathsf{M}A\equiv\mathsf{M}B}$
$\boldsymbol{M}1(\Box)$	$\boldsymbol{M}+[\mathsf{M}1]$
$\boldsymbol{M}1(\Diamond)$	$\boldsymbol{M}+[\neg\mathsf{M}0]$
$\boldsymbol{M}2(\Box)$	$\boldsymbol{M}+[\mathsf{M}A\leq\mathsf{MM}A]$
$\boldsymbol{M}2(\Diamond)$	$\boldsymbol{M}+[\mathsf{MM}A\leq\mathsf{M}A]$
$\boldsymbol{MT}(\Box)$	$\boldsymbol{M}+[\mathsf{M}A\leq A]$
$\boldsymbol{MT}(\Diamond)$	$\boldsymbol{M}+[A\leq\mathsf{M}A]$
$\boldsymbol{Mon}$	$\dfrac{A\leq B}{\mathsf{M}A\leq\mathsf{M}B}$
$\boldsymbol{Mon}1(\Box)$	$\boldsymbol{Mon}+[\mathsf{M}1]$
$\boldsymbol{Mon}1(\Diamond)$	$\boldsymbol{Mon}+[\neg\mathsf{M}0]$
$\boldsymbol{MonT}(\Box)$	$\boldsymbol{Mon}+[\mathsf{M}A\leq A]$
$\boldsymbol{MonT}(\Diamond)$	$\boldsymbol{Mon}+[A\leq\mathsf{M}A]$
$\boldsymbol{Mon}3(\Box)$	$\boldsymbol{MonT}(\Box)+[\mathsf{M}A\leq\mathsf{MM}A]$
$\boldsymbol{Mon}3(\Diamond)$	$\boldsymbol{MonT}(\Diamond)+[\mathsf{MM}A\leq\mathsf{M}A]$
$\boldsymbol{IK}_0(\Box)$	$\boldsymbol{Mon}\dfrac{\mathsf{M}}{\Box}+[\Box A\wedge\Box B\leq\Box[A\wedge B]]$
$\boldsymbol{IK}_0(\Diamond)$	$\boldsymbol{Mon}\dfrac{\mathsf{M}}{\Diamond}+[\Diamond[A\vee B]\leq\Diamond A\vee\Diamond B]$
$\boldsymbol{IS}2(\Box)$	$\boldsymbol{IK}_0(\Box)+[\Box A\leq A]$
$\boldsymbol{IS}2(\Diamond)$	$\boldsymbol{IK}_0(\Diamond)+[A\leq\Diamond A]$
$\boldsymbol{IS}3(\Box)$	$\boldsymbol{IS}2(\Box)+[\Box A\leq\Box\Box A]$
$\boldsymbol{IS}3(\Diamond)$	$\boldsymbol{IS}2(\Diamond)+[\Diamond\Diamond A\leq\Diamond A]$
$\boldsymbol{IK}(\Box,\Diamond)$	$\boldsymbol{IK}_0(\Box)+[\Box 1]+\boldsymbol{IK}_0(\Diamond)+[\neg\Diamond 0]$
$\boldsymbol{IT}(\Box,\Diamond)$	$\boldsymbol{IK}(\Box,\Diamond)+[\Box A\leq A]+[A\leq\Diamond A]$
$\boldsymbol{IS}4(\Box)$	$\boldsymbol{IS}3(\Box)+[\Box 1]$
$\boldsymbol{IS}4(\Diamond)$	$\boldsymbol{IS}3(\Diamond)+[\neg\Diamond 0]$
$\boldsymbol{IS}4(\Box,\Diamond)$	$\boldsymbol{IS}4(\Box)+\boldsymbol{IS}4(\Diamond)$
$\boldsymbol{IS}5(\Box)$	$\boldsymbol{IS}4(\Box)+[A\vee\Box\neg\Box A]$
$\boldsymbol{IS}5(\Diamond)$	$\boldsymbol{IS}4(\Diamond)+[\Diamond A\vee\neg\Diamond A]+[\Diamond\neg\Diamond A\leq\neg A]$
$\boldsymbol{IS}5(\Box,\Diamond)$	$\boldsymbol{IS}5(\Box)+\boldsymbol{IS}5(\Diamond)$
$\boldsymbol{IS}5(\Box+\Diamond)$	$\boldsymbol{IS}5(\Box,\Diamond)+[\Diamond\Box A\leq A]+[A\leq\Box\Diamond A]$
$\boldsymbol{IB}(\Box,\Diamond)$	$\boldsymbol{IT}(\Box,\Diamond)+[A\vee\Box\neg\Box A]+[\Diamond A\vee\neg\Diamond A]+[\Diamond\neg\Diamond A\leq\neg A]$
$\boldsymbol{IB}(\Box+\Diamond)$	$\boldsymbol{IB}(\Box,\Diamond)+[\Diamond\Box A\leq A]+[A\leq\Box\Diamond A]$

$\langle K, \leq, \mathrm{R}, \mathrm{R}^*, \Vdash\rangle$ with no additional requirements on R and R*. Let Ψ be the set of all subformulas of A_0. Then K^0 is finite. Define

$$\alpha\mathrm{R}^0\beta \text{ iff } (\forall x\in\alpha)(\forall y\in\beta)(\forall A)[\text{if } \Box A\in\Psi \text{ and } x\Vdash\Box A \text{ then } y\Vdash A],$$

$$\alpha\mathrm{R}^{*0}\beta \text{ iff } (\forall x\in\alpha)(\forall y\in\beta)(\forall A)\ [\text{if } \Diamond A\in\Psi \text{ and } y\Vdash A \text{ then } x\Vdash\Diamond A].$$

From $\alpha\mathrm{R}^0\beta$ and $\xi\leq^0\alpha$, $\beta\leq^0\zeta$ it follows that $\xi\mathrm{R}^0\zeta$. If $u\in\xi, v\in\zeta$, $\Box A\in\Psi$ and $u\Vdash\Box A$, then $x\Vdash\Box A$, as $|u|\leq^0\alpha$ and $\alpha=|x|$; if $\beta=|y|$, then $\alpha\mathrm{R}^0\beta$ implies $y\Vdash A$, therefore $v\Vdash A$, hence $\xi\mathrm{R}^0\zeta$. Analogously it can be shown that $\alpha\mathrm{R}^{*0}\beta$, $\alpha\leq^0\xi$, $\zeta\leq^0\beta$ imply $\xi\mathrm{R}^{*0}\zeta$, and so on. We have proved that $\langle K_0, \leq^0, \mathrm{R}^0, \mathrm{R}^{*0}\rangle$ is a Kripke frame. Then we can apply Lemma 18 and get a finite model where A_0 is not true.

By Theorem 10 the logic $\boldsymbol{IT}(\Box, \Diamond)$ is characterized by Kripke frames where R and R* are reflexive. The above filtration works, we only need to show that R^0 and R^{*0} are reflexive in this case. Indeed, if $\Box A \in \Psi, x \in \alpha$ and $x \Vdash \Box A$ it follows from the axioms that $x \Vdash A$ and we get $\alpha R^0 \alpha$; the same argument applied to R^{*0} gives its reflexivity.

Again by Theorem 10 the logic $\boldsymbol{IS}4(\Box, \Diamond)$ is the logic of Kripke frames with reflexive and transitive relations R and R*. We could use the Lemmon — Scott filtration (cf. [10]) adopting the following definitions:

$$\alpha R^0 \beta \text{ iff } (\forall x \in \alpha)(\forall y \in \beta)(\forall A)[\text{if } \Box A \in \Psi \text{ and } x \Vdash \Box A \text{ then } y \Vdash \Box A],$$

$$\alpha^* R^0 \beta \text{ iff } (\forall x \in \alpha)(\forall y \in \beta)(\forall A)[\text{if } \Diamond A \in \Psi \text{ and } y \Vdash \Diamond A \text{ then } x \Vdash \Diamond A].$$

It is not difficult to see that R^0 and R^{*0} defined in this way are reflexive and transitive.

However, another method, due to Segerberg [12] is also applicable here. Let us go back to the definitions of R^0 and R^{*0} given for $\boldsymbol{IK}(\Box, \Diamond)$ but change the set Ψ, now demanding that it contains all subformulas of A_0 together with their modalized forms, i. e. if $B \in \Psi$ then $\Box B$ and $\Diamond B$ should be in Ψ. K^0 is still finite because $\Box\Box B \equiv \Box B$ and $\Diamond\Diamond B \equiv \Diamond B$ in $\boldsymbol{IS}4$. To see that R^0 is transitive, take $\Box A \in \Psi$ and $x \Vdash \Box A$. Then $x \Vdash \Box\Box A$, but $\Box\Box A \in \Psi$. So if $\alpha R^0 \beta$, $\beta R^0 \gamma$, $\alpha = |x|$, $\beta = |y|$, $\gamma = |z|$, then $y \Vdash \Box A$ and $z \Vdash A$, hence $\alpha R^0 \gamma$; analogously R^{*0} is transitive.

In $\boldsymbol{IS}5$-group of logics our first examples are the systems which expose no connections between R and R*. Corollary 3 shows that $\boldsymbol{IS}5(\Box, \Diamond)$ is characterized by Kripke frames where R and R* are equivalence relations. Again we use the set Ψ of Segerberg and define:

$$\alpha R^0 \beta \text{ iff } (\forall x \in \alpha)(\forall y \in \beta)(\forall A)[\text{if } \Box A \in \Psi \text{ then } x \Vdash \Box A \text{ iff } y \Vdash \Box A],$$

$$\alpha R^{*0} \beta \text{ iff } (\forall x \in \alpha)(\forall y \in \beta)(\forall A)[\text{if } \Diamond A \in \Psi \text{ then } x \Vdash \Diamond A \text{ iff } y \Vdash \Diamond A].$$

R^0 and R^{*0} are reflexive, transitive and symmetric as can be easily shown.

For the decidability of $\boldsymbol{IS}5(\Box + \Diamond)$ it is sufficient to note that $\Diamond A$ there is equivalent to $\neg\Box\neg A$ and we can reduce the problem to the same one for $\boldsymbol{IS}5(\Box, \Diamond)$.

Next consider $\boldsymbol{Mon}$ — the minimal monotone system. If $A_0 \notin \boldsymbol{Mon}$ let Ψ be the set of all subformulas of A_0. Let R^0 be defined from the cannonical model for $\boldsymbol{Mon}$ by the following definition:

$$\alpha R^0 P \text{ iff } (\forall x \in \alpha)(\forall A)[\text{if } \mathsf{M}A \in \Psi \text{ and } \widehat{P} \subseteq h(A) \text{ then } \mathsf{M}A \in x].$$

Now we show that $\alpha R^0 P$ and $\alpha \leq^0 \beta$ imply $\beta R^0 P$. Let $\alpha = |x|$, $\beta = |y|$, $\mathsf{M}A \in \Psi$ and $\widehat{P} \subseteq h(A)$, then $\mathsf{M}A \in x$, but $\mathsf{M}A \in \Psi$, $|x| \leq^0 |y|$ and therefore $\mathsf{M}A \in y$, so $\beta R^0 P$. We show that $\alpha R^0 P$ and $P \subseteq Q$ imply $\alpha R_0 Q$. Let $\alpha = |x|$, $\mathsf{M}A \in \Psi$ and $\widehat{Q} \subseteq h(A)$: from $P \subseteq Q$ it follows that $\widehat{P} \subseteq \widehat{Q}$, so $\mathsf{M}A \in x$, therefore $\alpha R^0 Q$. Finally, if $x R^{20} P$, then $|x| R^0 P^0$: if $\mathsf{M}A \in \Psi$ and $\widehat{P^0} \subseteq h(A)$, then from $P \subseteq \widehat{P^0}$ it follows that $P \subseteq h(A)$, then $x R^{20} P$ gives $\mathsf{M}A \in x$. The second condition on R^0 is automatically satisfied. Thus if $A_0 \notin x_0$ in the cannonical model, then not $|x_0| \Vdash^{\circ} A_0$ in the obtained model, which is finite.

For $\boldsymbol{MonT}(\Diamond)$ note that an adequate condition is "if $x \in P$, then xRP" (see Proposition 7). The model defined above (for $\boldsymbol{Mon}$) is appropriate here, too. We only show that $\alpha \in P$ gives $\alpha R^0 P$. Let $MA \in \Psi$ and $\widehat{P} \subseteq h(A)$. If $\alpha = |x|$ and $\alpha \in P$, then $x \in \widehat{P}$ and hence $x \in h(A)$, i. e. $A \in x$, but then by the axiom $MA \in x$, therefore $\alpha R^0 P$.

Mon1$(\Diamond)$ is characterized by the condition "non $xR\emptyset$". In order to get an useful filtration add to Ψ the formula M0. K^0 remains finite. Let R^0 be the same relation. Assume $\alpha R^0 \emptyset$ for $\alpha = |x|$. From $\widehat{\emptyset} \subseteq \emptyset$, $h(0) = \emptyset$ and $M0 \in \Psi$ it follows that $M0 \in x$ — a contradiction.

The system ***Mon***3$(\Diamond)$ is characterized by Montague frames with "if $x \in P$ then xRP" and "if $xR\{y \mid yRP\}$ then xRP". Denoting by JP the set $\{y \mid yRP\}$ the latter condition is expressed by "if $xRJP$ then xRP". Take now as Ψ the Segerberg set of all subformulas of a non-provable formula A_0 together with all modalized variants. From $MMA = MA$ it follows that K^0 is still finite. The definition of R^0 is unaltered, so there is no need to check again the first condition. For the second assume that $\alpha R^0 J^0 P$, where $J^0 P = \{\beta \mid \beta R^0 P\}$, $MA \in \Psi$ and $\widehat{P} \subseteq h(A)$. Then $\widehat{J^0 P} \subseteq h(MA)$: if $y \in \widehat{J^0 P}$, then $y \in \beta$ and $\beta R^0 P$, and from $\widehat{P} \subseteq h(A)$ we have $MA \in y$, i. e. $y \in h(MA)$. So, in this case $\alpha R^0 J^0 P$ gives that $x \in \alpha$ implies $MMA \in x$, but from the axioms it follows that $MA \in x$. Thus we get $\alpha R^0 P$.

As for the logic ***IK***$_0(\Diamond)$ (which should bear the name ***I—C***2$(\Diamond)$ in a more traditional terminology), one has to take Ψ to be a set containing all subformulas of A_0 together with all disjunctions of its elements and all formulas of the kind $\Diamond[A \vee B]$ for $\Diamond A \in \Psi$ and $\Diamond B \in \Psi$. Clearly K^0 is finite in this case, too. Let us check the condition "if $\alpha R^0(P \cup Q)$ then $\alpha R^0 P$ or $\alpha R^0 Q$". Assume the contrary, that $\alpha R^0(P \cup Q)$ and not $\alpha R^0 P$ and not $\alpha R^0 Q$. This means that there are $\Diamond A \in \Psi$, $\Diamond B \in \Psi$ such that $\widehat{P} \subseteq h(A)$, $\widehat{Q} \subseteq h(B)$, but $\Diamond A \notin x$ and $\Diamond B \notin y$ for $x \in \alpha$ and $y \in \beta$. Then from $x \approx y$ we get $\Diamond B \notin x$ and so $\Diamond A \vee \Diamond B \notin x$. By the axiom $\Diamond[A \vee B] \notin x$ and by the construction of Ψ: $\Diamond A \vee \Diamond B \in \Psi$ and $\Diamond[A \vee B] \in \Psi$. Then by the easily established inclusions:

$$\widehat{P \cup Q} \subseteq \widehat{P} \cup \widehat{Q} \subseteq h(A) \cup h(B) \subseteq h(A \cup B),$$

and from $|x| R^0(P \cup Q)$ it would follow that $\Diamond[A \vee B] \in x$: a contradiction.

Joining the results for ***MonT***$(\Diamond)$ and ***IK***$_0(\Diamond)$ we can prove the decidability of ***IS***2$(\Diamond)$ — a non-normal analogue of ***T***. Analogously, from the result for ***Mon***3$(\Diamond)$ and ***IK***$_0(\Diamond)$ we can obtain the decidability of ***IS***3$(\Diamond)$— a non-normal variant of ***S***4.

Finally, we present applications of the method of filtration of algebras. These applications solve the problem of decidability for the rest of the logics in the table. For the logic ***M*** we can take as Ψ' the set of all subformulas of A_0 and the operation M^0 can be defined arbitrarily outside Ψ'. For ***MT***$(\Box)$ Ψ' is the same, $M^0 A = MA$ if $MA \in \Psi'$ and $MA = 0$ otherwise. Clearly $M^0 A \leq A$. Varying this construction the decidability of a great number of extensions of ***M*** can be proved (for example extensions with additional axioms $MA \leq MMA$, M1, etc., defining $M^0 A$ as 0, 1, A etc., when $MA \notin \Psi'$, and as MA, when $MA \in \Psi'$).

The cases of monotone extensions of ***M*** are more interesting, though. Following Lemmon, we say that $A \in \Psi^0$ *covers* B if $B \in \Psi'$, $B \leq A$, $MB \in \Psi'$, where Ψ' is the set of all subformulas of A_0 plus M0. If A covers only $A_1, \dots, A_n$, we define $\mathrm{M}^0 A$ as $\mathrm{M}A_1 \vee \dots \vee \mathrm{M}A_n$. A always covers 0, so the definition is correct. If the logic is monotone, then $\mathrm{M}^0 A \leq \mathrm{M}A$: if A covers A_i then $A_i \leq A$, so $\mathrm{M}A_i \leq \mathrm{M}A$, therefore $\mathrm{M}A_1 \vee \dots \vee \mathrm{M}A_n \leq \mathrm{M}A$. Moreover, if $\mathrm{M}A \in \Psi'$, then $\mathrm{M}A \leq \mathrm{M}^0 A : \mathrm{M}A \in \Psi'$ implies $A \in \Psi'$ and hence A covers A and so $\mathrm{M}^0 A = \mathrm{M}A \vee \mathrm{M}A_1, \vee \dots \vee \mathrm{M}A_n$. It follows that $\mathrm{M}A \in \Psi'$ implies $\mathrm{M}^0 A = \mathrm{M}A$.

As a first application of the just described operation M^0 on Ψ^0 we give another proof of the finite model property for ***Mon***. Let us check the validity of the monotonicity rule in Ψ_0. If A covers A_i, then if $A \leq B$, B covers A_i, too. Therefore $A \leq B$ implies $\mathrm{M}^0 A \leq \mathrm{M}^0 B$.

For ***MonT***$(\Box)$: $\mathrm{M}^0 A \leq A$ is valid because $\mathrm{M}^0 A \leq \mathrm{M}A$ and $\mathrm{M}A \leq A$. For ***Mon***$1(\Box)$: add M1 to Ψ', the 1 covers 1 and from $1 \leq \mathrm{M}1$ it follows that $1 \leq \mathrm{M}1 \vee \mathrm{M}A_1 \vee \dots$ For ***Mon***$3(\Box)$: add to Ψ' all modalized forms plus all disjunctions; Ψ' remains finite and $\mathrm{M}^0 A \leq \mathrm{M}^0\mathrm{M}^0 A$: if $\mathrm{M}^0 A = \mathrm{M}A_1 \vee \dots \vee \mathrm{M}A_n$, then $\mathrm{M}A_i \leq \mathrm{M}^0 A$, so $\mathrm{M}\mathrm{M}A_i \leq \mathrm{M}\mathrm{M}^0 A$, hence $\mathrm{M}A_i \leq \mathrm{M}\mathrm{M}^0 A$ and $\mathrm{M}^0 A \leq \mathrm{M}\mathrm{M}^0 A$; then since $\mathrm{M}^0 A \in \Psi'$, $\mathrm{M}\mathrm{M}^0 A = \mathrm{M}^0\mathrm{M}^0 A$; thus $\mathrm{M}^0 A \leq \mathrm{M}^0\mathrm{M}^0 A$.

Another application is the finite completeness of ***IK***$_0(\Box)$ where Ψ' is enriched with M0 and all formulas of the kind $\Box[A \wedge B]$ for $\mathrm{M}A, \mathrm{M}B \in \Psi'$.

Combining the proofs we just carried out we can establish the finite model property of ***IS***$2(\Box)$, ***IS***$3(\Box)$ and ***IS***$4(\Box)$.

We have found a quite interesting proof of the finite completeness and consequently of decidability of the intuitionistic analogue of the Brouwer's modal logic ***IB***$(\Box, \Diamond)$. We have shown above that this logic is characterized by frames with reflexive and symmetric R and R*. Now we use the filtrations known from the proof for ***IT***$(\Box, \Diamond)$, enlarging Ψ by stipulation: if $\Box A \in \Psi$ then $\Box\neg\Box A \in \Psi$. K^0 remains finite as follows from the axioms of ***IB***$(\Box, \Diamond)$. Analogously, if we want to include $\Diamond\neg\Diamond A$ in Ψ every time $\Diamond A$ is in Ψ, we can do it without making K^0 infinite. It is easy to prove in ***IB***$(\Box)$ the sequent $\neg\Box\neg\Box A \leq A$ and hence $\Box\neg\Box\neg\Box A \leq \Box A$. On the other hand using the axiom with $\Box$ we get $\neg\Box\neg\Box\neg\Box A \leq \neg\Box A$, but by the law of double negation for formulas of the kind $\Box B$ (which can be proved analogously to the case of ***IS***$5(\Box + \Diamond)$ — see Corollary 6), it follows that $\Box A \leq \Box\neg\Box\neg\Box A$. The equivalence $\Box\neg\Box\neg\Box A = \Box A$ shows that the operation of adding $\Box\neg$ is idempotent. At the same time, due to the equivalence $\Diamond\neg\Diamond\neg\Diamond A = \Diamond A$ we can assume that Ψ contains $\Diamond\neg\Diamond A$ together with each $\Diamond A \in \Psi$, without becoming infinite. We prove now the symmetry of R^0. If $\alpha \mathrm{R}^0 \beta$ but not $\beta \mathrm{R}^0 \alpha$ we have $x \in \alpha$, $y \in \beta$, $\Box A \in \Psi$, $y \Vdash \Box A$, but not $x \Vdash A$. By the axioms and the latter $x \Vdash \Box\neg\Box A$ and by the definition of Ψ: $\Box\neg\Box A \in \Psi$. Then $\alpha \mathrm{R}^0 \beta$ gives $y \Vdash \neg\Box A$ — a contradiction. In the same way we can show that R^{*0} is symmetric.

6. INTUITIONISTIC MODAL LOGICS WITHOUT THE FINITE MODEL PROPERTY

Now we turn to negative examples. First we are going to present a modified version of Gabbay's example [6, § 24]. His example was a simplification of Fine's logic [16]. In this way we answer affirmatively the question about existence of extensions of ***IS***$4(\Diamond)$ without the f.m.p. It should be noted that it is an open problem whether there is an analogous construction for ***IS***$4(\Box)$.

Let us introduce the following notations:

$$B = \Diamond A_1 \wedge \Diamond A_2 \wedge \Diamond A_3 \Rightarrow \Diamond[A_1 \wedge \Diamond[A_2 \vee A_3]] \vee \Diamond[A_2 \wedge \Diamond[A_1 \vee A_3]] \vee \Diamond[A_3 \wedge \Diamond[A_1 \vee A_2]],$$

$$C = \Diamond[\neg \Diamond A_1 \wedge \Diamond A_2 \wedge \Diamond A_3] \wedge \Diamond[\Diamond A_1 \wedge \neg \Diamond A_2 \wedge \Diamond A_3] \wedge \Diamond[\Diamond A_1 \wedge \Diamond A_2 \wedge \neg \Diamond A_3],$$

$$Ax = B \vee C, \quad \boldsymbol{IS}4^+ = \boldsymbol{IS}4(\Diamond) + Ax.$$

Denote by b the formula built up from the propositional variables p_1, p_2, p_3:

$$\Diamond p_1 \wedge \Diamond p_2 \wedge \Diamond p_3 \Rightarrow \Diamond[p_1 \wedge \Diamond[p_2 \vee p_3]] \vee \Diamond[p_2 \wedge \Diamond[p_1 \vee p_3]] \vee \Diamond[p_3 \wedge \Diamond[p_1 \vee p_2]].$$

Further we set $P_{i,n}$ ($i = 1, 2, 3$ and $n \geqq 0$) as follows:

$$P_{1,0} = p_1, \; P_{2,0} = p_2, \; P_{3,0} = p_3;$$

$$P_{1,n+1} = \neg \Diamond P_{1,n} \wedge \Diamond P_{2,n} \wedge \Diamond P_{3,n}$$

$$P_{2,n+1} = \Diamond P_{1,n} \wedge \neg \Diamond P_{2,n} \wedge \Diamond P_{3,n}$$

$$P_{3,n+1} = \Diamond P_{1,n} \wedge \Diamond P_{2,n} \wedge \neg \Diamond P_{3,n}.$$

Lemma 20. If in a Kripke model for $\boldsymbol{IS}4^+$ there is an x such that not $x \Vdash b$ then for any $n \geqq 1$, $x \Vdash \Diamond P_{i,n}$ ($i = 1, 2, 3$).

The easy proof by induction on n is left to the reader.

Lemma 21. $P_{1,n} \leq \neg \Diamond P_{1,n-k}$ for $k \geqq 1$, $n \geqq 1$.

Lemma 22. The formula b is true in any finite model for $\boldsymbol{IS}4^+$.

Proof. If b is refuted in a model for $\boldsymbol{IS}4^+$ then this model is infinite (use Lemmas 20 and 21).

Lemma 23. Not $\boldsymbol{IS}4^+ \vdash b$.

Proof. Gabbay [6, pp. 121—122] gives a *classical* model for $\boldsymbol{S}4^+$ where $\neg b$ is true. Hence b cannot be provable in $\boldsymbol{IS}4^+$ which is a subsystem of $\boldsymbol{S}4^+$. In this way we have (by Lemmas 22 and 23):

Proposition 21. The logic $\boldsymbol{IS}4^+$ does not have the finite model property.

We shall describe briefly another logic without the f. m. p., because this logic was published in [37]. It is an extension of the well known intermediate propositional logic of Dummett (obtained by adding the axiom $[A \Rightarrow B] \vee [B \Rightarrow A]$ to the intuitionistic propositional calculus) and at the same time an extension of $\boldsymbol{IT}(\Box)$. We construct the logic by adopting a modified version of Makinson's example [17] of a classical extension of the system $\boldsymbol{T}$.

Our logic has three remarkable properties:

1) As is well known (cf. [18]) all (non-modal) extensions of Dummett's logic are tabular and so have the f. m. p. Our example shows there are *modal* extensions of Dummett's logic not finitely complete.

2) The corresponding *classical* logic is trivially finitely complete: it is simply equivalent to the classical propositional logic since $\Box A = A$ is among

its theorems. So we produce a logic without f. m. p. whose classical counterpart has the f. m. p.

3) In contrast to the previous example, the formula valid in all finite models but not provable is not an *ad hoc* introduced formula, but a very natural one: $\Box A = A$. Consequently, on finite models our logic coinsides with non-modal Dummett's logic and its proper modal axioms can be distinguished only by infinite models.

The logic is defined as follows: let $\boldsymbol{DumT}^+$ be the extension of $\boldsymbol{IT}(\Box)$ by the axioms

$$Ax = \Box A \Rightarrow \Box\Box A \leq A \Rightarrow \Box A \text{ and } [A \Rightarrow B] \vee [B \Rightarrow A],$$

Lemma 24. The formula $p \Rightarrow \Box p$ is valid in any finite model of $\boldsymbol{DumT}^+$.

Lemma 25. $p \Rightarrow \Box p$ is not a theorem of $\boldsymbol{DumT}^+$.

From these two lemmas we obtain

Proposition 22. $\boldsymbol{DumT}^+$ does not possess the f. m. p.

Property 3) follows from Lemma 24 and the axiom $\Box A \leq A$ of $\boldsymbol{DumT}^+$; property 2) is given by

Proposition 23. Ax is equivalent to $A \leq \Box A$ in the classical logic.

7. CONCLUDING REMARKS

The author is indebted to D. Vakarelov for the advice to undertake the above investigations. The main results connected with Montague and Kripke semantics were obtained during the period December 1975 — May 1976 and were shortly announced in [36]. The decidability results and the examples of logics without the f. m. p. were presented in November 1977; the most interesting examples of logics without the f. m. p. are published in [37]. In April 1978 the author gave a lecture on the intuitionistic modal logic at the Banach Mathematical Centre in Warsaw, where he described the central results of this paper. This work is a part of the author's Ph. D. Thesis, defended at Moskow State University (July 1978).

Some additional results not mentioned in this paper are as follows. The topological characterization of $\boldsymbol{IS}4(\Box)$ (our Corollary 1) was obtained by Radev [38] analogously to the proof for $\boldsymbol{S}4$ in [3].

Mihajlova proved [39] that $\boldsymbol{IS}4(\Box)$ has exactly 31 different modalities, $\boldsymbol{IS}4.2(\Box)$ and $\boldsymbol{IS}4.3(\Box)$ have 19 each, and $\boldsymbol{IS}5(\Box)$ has 9 modalities, namely p, $\neg p$, $\neg\neg p$, $\Box p$, $\neg\Box p$, $\Box\neg p$, $\neg\Box\neg p$, $\Box\neg\neg p$, $\neg\Box\neg\neg p$ (before that she had proved in her M. Sc. Thesis, 1978, that $\boldsymbol{S}5(\Diamond)$ has only 7 modalities: p, $\neg p$, $\neg\neg p$, $\Diamond p$, $\neg\Diamond p$, $\Diamond\neg p$, $\neg\Diamond\neg p$).

Vakarelov showed [40] that the implication (and negation) free fragment of the intuitionistic modal logic may be used to obtain quite simple examples of incomplete logics. An unexpected result was presented by the same author in [41]: there exists a continuum of intuitionistic modal logics which don't admit the law of excluded middle, i. e. they are consistent but became inconsistent after adding $A \vee \neg A$ as an axiom, and so they are strongly intuitionistic. Later on Tselkov showed [42] there exists a continuum logics incompatible with any single formula in the Rieger-Nishimura lattice. An open question is whether there exists an absolutely strongly intuitionistic logic, incompatible with any classically provable but intuitionistically unprovable formula.

Popov gave in [43] Gentzen-type calculi for $IK(\Box)$, $IT(\Box)$, $IK4(\Box)$ (i. e. $IK(\Box)+\Box A\leq\Box\Box A$), and $IS4(\Box)$; the cases with $\Diamond$ are open.

Kirov studied in [44] the intuitionistic analogue of the modal logic of provability Q (or $K4W$ in Segerberg's notation). He proved its completeness with respect to the class of all finite structures with irreflexive and transitive R. The logic IGL is an extension of $IK(\Box)$ by the axiom $\Box[\Box A\Rightarrow A]\Rightarrow\Box A$. With help of this result he obtained some properties of IGL: closure under several rules, like $\frac{\Box A}{A}$ or $\frac{A\vee B}{A \text{ or } B}$. Ursini developed an alternative treatment of the same logic in [45] and [46]. In [47] the so-called "strong modality" $\Box! A = A\wedge\Box A$ in IGL is studied. The logic of this new operator turns out to be $IS4Grz$ — the extension of $IS4(\Box!)$ by the axiom of Sobociński $\Box![\Box![A\Rightarrow\Box! A]\Rightarrow A]\Rightarrow A$.

In some unpublished work Vakarelov showed that $\Box$ and $\Diamond$ can be introduced strictly analogously to the classical case — with one relation R, if we add $\Box\neg A\vee\neg\Box\neg A$ as an additional axiom; then $(x\Vdash\Diamond A$ iff $(\exists y)[xRy$ and $y\Vdash A])$ (R is the relation for $\Box$) may be used as a definition of $\Diamond A$, and $\Diamond A\equiv\neg\Box\neg A$ will be true. The same observation was made in [48].

Finally, the author showed in [49] that the intuitionistic double negation can be considered as a modality. Namely, $\neg\neg$ satisfies $[\neg\neg A\wedge\neg\neg B]\leq\neg\neg[A\wedge B]$ and $\frac{A\leq B}{\neg\neg A\leq\neg\neg B}$, since it is an IK-"necessity" with some additional properties, e. g. $A\leq\neg\neg A$ and $\neg\neg[\neg\neg A]\leq\neg\neg A$. That is why it was interesting to obtain a full axiomatization of this new intuitionistic modality. Let we denote $\neg\neg$ by $\Box$; $I^+(\Box)$ is the positive (without negation) fragment of $IK(\Box)$ extended by axioms $A\leq\Box A$, $\Box[\Box A\Rightarrow A]$ and $\Box[A\vee[A\Rightarrow B]]$. If A is a formula of the intuitionistic propositional logic (IPL) without groups of odd number of adjacent symbols $\neg$ we shall denote by $A\frac{\neg\neg}{\Box}$ the "pairing' of the negations in A (i. e. the replacement of every double negation by $\Box$). Then the answer of our question is given by the following

Theorem. If A contains only even negations then it is provable in IPL iff $A\frac{\neg\neg}{\Box}$ is provable in $I^+(\Box)$.

$I^+(\Box)$ possesses an adequate Kripke semantics. Some superintuitionistic logics may be treated in the same style. E. g. the axiomatics of the double negation of Dummett's KC (with additional axiom $\neg A\vee\neg\neg A$ is $I^+(\Box)$ plus $\Box[A\vee B]\leq[\Box A\vee\Box B]$.

REFERENCES

General

1. Church, A. Introduction to Mathematical Logic. V. 1. Princeton Univ. Press, 1956.
2. Kleene, S. Introduction to Metamathematics. New York — Toronto, D. van Norstrand Comp., 1952.

3. Rasiowa, H., R. Sikorski. The Mathematics of Metamathematics. Warszawa, PWN, 1963.
4. Fitting, M. Intuitionistic Logic, Model Theory and Forcing. Amsterdam, North-Holland, 1969.
5. Vakarelov, D. Theory of a Negation in Certain Logical Systems. Algebraic and Semantical Approach. Ph. D. Thesis, Warsaw Univ., 1976.
6. Gabbay, D. Investigations in Modal and Tense Logics with Applications to Problems in Philosophy and Linguistics. Dordrecht, North-Holland, 1976.
7. Segerberg, K. An Essay in Classical Modal Logic. Uppsala, 1971.
8. Chellas, B., A. McKinney. The completeness of monotonic modal logics. — Z. Math. Logik u. Grundlagen Math. **21**, 1975, 379—383.
9. Bourbaki, N. Topologie générale, ch. 1—2. 3 éd., Paris, Hermann, 1961.
10. Минц, Г. Система Льюиса и система T. — В: Фейс, Р. Модальная логика, М., Наука, 1974, 254—303.
11. Schütte, K. Vollständige Systeme modaler und intuitionistischer Logik. — Ergebnisse der Mathematik und ihrer Grenzgebiete, **42**, 1968.
12. Segerberg, K. Decidability of S4. I. — Theoria, **34**, 1968, 7—20.
13. Hansson, B., P. Gärdenfors. A guide to intentional semantics. — In: Modality, Morality and Other Problems of Sense and Nonsense. Lund, 1973, 151—167.
14. Hansson, B., P. Gärdenfors. Filtrations and the finite frame property in Boolean semantics. — In: Proc. Third Scandinavian Logic Symposium. Amsterdam, 1975, 32—39.
15. Lemmon, E. Algebraic semantics for modal logics I. — J. Symb. Logic, **31**, 1966, 46—65.
16. Fine, K. Logics containing S4 without the finite model property. — In: Lecture Notes Math., **255**, 1972, 98—102.
17. Makinson, D. A normal modal calculus between T and S4 without the finite model property. — J. Symb. Logic, **34**, 1969, 35—38.
18. Dunn, J., R. Meyer. Algebraic completeness results for Dummett's LC and its extensions. — Z. Math. Logik u. Grundlagen Math., 1971, 225—230.

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19. Fitch, F. Intuitionistic modal logic with quantifiers. — Portugaliae Mathematica, **7** 1948, 113—118.
20. Prior, A. Time and Modality. Oxford, 1957.
21. Bull, R. A modal extension of intuitionistic logic. — Notre Dame J. Formal Logic, **6**, 1965, 142—146.
22. Bull, R. Some modal calculi based on IC. — In: Formal Systems and Recursive Functions. Amsterdam, 1965, 3—7.
23. Bull, R. MIPC as the formalization of an intuitionist concept of modality. — J. Symb. Logic, **31**, 1966, 609—616.
24. Минц, Г. О некоторых исчислениях модальной логики. — Труды Матем. Инст. АН СССР, **98**, 1968, 88—111.
25. Gabbay, D. Modal theory for tense logics. — Ann. Math. Logic, **8**, 1975, 185—236.
26. Gabbay, D. Decidability results in non-classical logics. — Ann. Math. Logic, **8**, 1975, 237—295.
27. Fischer-Servi, G. On modal logics with an intuitionistic base. — Bull. Section Logic, **6**, 1977, 32—34. (Preliminary report on [28].)
28. Fischer-Servi G. On modal logics with an intuitionistic base. — Studia Logica, **36**, 1977, 141—149.
29. Fischer-Servi, G. Un' algebrizzazione del calcolo monadico intuizionista (to appear in Mathematiche, Catania).
30. Fischer-Servi, G. Semantics for a class of intuitionistic modal calculi. — Bull. Section Logic, **7**, 1978, 26—30. (Preliminary report on [31].)
31. Fischer-Servi, G. Semantics for a class of intuitionistic modal calculi. — In: Italian Studies in the Philosophy of Science, Boston Studies, 1977.
32. Ono, H. On some intuitionistic modal logics. — Bull. Section Logic, **6**, 1977, 182—185. (Preliminary report on [33].)
33. Ono, H. On some intuitionistic modal logics. — Publ. Res. Inst. Math. Sci. Kyoto Univ., **13**, 1977, 687—722.
34. Акимов, А. Интуиционистский аналог системы S4 Льюиса. — В: Модальные и интенсиональные логики. М., Наука, 1978, 3—5.

35. Булавинов, А. Пропозициональные модальные исчисления на интуиционистской основе. — В: Модальные и интенсиональные логики. М., Наука, 1978, 24—27.
36. Sotirov, V. Modal theories with intuitionistic logic. — In: Sixth Balkan Mathematical Congress. Varna, 3-9 June 1977, Summaries, p. 91.
37. Сотиров, В. Не финитно аппроксимируемые интуиционистские модальные логики. — Математические заметки, **27**, 1980, вып. 1, 89—94.
38. Radev, S. Intuitionistic modal propositional calculus (M. Sc. Thesis, Sofia Univ., Sofia, 1973).
39. Mihajlova, M. Reduction of modalities in several intuitionistic modal logics. — C. R. Acad. bulg. Sci., **33**, 1980, No 6, 743—745.
40. Vakarelov, D. Simple examples of incomplete logics. — C. R. Acad. bulg. Sci., **33**, 1980, No 5, 587—589.
41. Vakarelov, D. Intuitionistic modal logics incompatible with the law of the excluded middle. — Studia Logica, **40**, No 2, 1981, 103—111.
42. Tselkov, V. Intuitionistic modal logics contradicting the Rieger-Nishimura logics.— In: Mathematics and Education in Mathematics. Proc. 12th Spring Conf. Union Bulg. Math., Sunny Beach, April 6—9, 1983. Sofia, Bulg. Acad. Sci., 1983, 133—138.
43. Popov, W. Gentzen-type calculi for some intuitionistic modal logics. — In: Mathematics and Education in Mathematics. Proc. 11th Spring Conf. Union Bulg. Math., Sunny Beach, April 6—9, 1982. Sofia, Bulg. Acad. Sci., 1982, 274—279.
44. Kirov, K. An intuitionistic analogue of the modal logic of provability in Peano's arithmetic. — Proc. Conf. Math. Logic, Dedicated Memory A. A. Markov (1903-1979), Sofia, 1980, Sofia, Bulg. Acad. Sci., 1984, 65—72.
45. Ursini, A. Intuitionistic diagonizable algebras. — Algebra Universalis, **9**, 1979, 229—237.
46. Ursini, A. A modal calculus analogous to K4W based on intuitionistic propositional logic I. — Rapp. Matematico, No 4, Inst. di Mat. dell' Univ. di Siena, 1979.
47. Gargov, G., K. Kirov. The logic of "strong box" in IGL is IS4Grz. — In: Mathematics and Education in Mathematics. Proc. 11th Spring Conf. Union Bulg. Math., Sunny Beach, April 6—9, 1982. Sofia, Bulg. Acad. Sci., 1982, 154—159.
48. Шехтман, В. Семантика типа Крипке для пропозициональных модальных логик с интуиционистской базой. — В: Модальные и временные логики. II Советско-финский коллоквиум по логике, Москва, 3—7 декабря 1979. М., 108—109.
49. Sotirov, V. The intuitionistic double negation is a modality. — In: 7th Intern. Wittgenstein Symp., 22-29 August 1982, Kirchberg am Wechsel, Austria. (Abstracts p. 58).

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