Various Syllogistics from the Algebraic Point of View

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In this paper, we will consider different syllogistics and will show their places in a uniform scheme — the algebraic one.

Let us begin with the *traditional* (*Aristotelian*) *syllogistic*. We will follow its canonization given by Łukasiewicz in his celebrated book [Luk]: the language of the classical propositional calculus is extended by *term variables* (for short, *terms*) $t_1, t_2, \ldots$ together with two binary *term relations*: $\mathcal{A}$ and $\mathcal{I}$. Syllogistic *atoms* are all formulae of the kind $s\mathcal{A}p$ ("Every s is a p") or $s\mathcal{I}p$ ("Some s is a p") with s and p being terms. A *syllogism* is any propositional formula with all propositional letters replaced by syllogistic atoms. In particular, $s\mathcal{O}p$ ("Some s is not a p") and $s\mathcal{E}p$ ("No s is a p") are negations of $s\mathcal{A}p$ and $s\mathcal{I}p$, respectively.

The standard semantics of the Aristotelian syllogistic is that in the theory of sets: if S and P are arbitrary *non-empty* sets, $s\mathcal{A}p$ is translated into $S \subseteq P$ and $s\mathcal{I}p$ into $S \cap P \neq \emptyset$. As we know from the result of Słupecki [Słu], the syllogisms not admitting empty terms are axiomatized by the following four axioms of Lukasiewicz [Luk]: $s\mathcal{A}s$, $s\mathcal{I}s$, $(m\mathcal{A}p) \wedge (s\mathcal{A}m) \Rightarrow (s\mathcal{A}p)$ (*Barbara*), and $(m\mathcal{A}p) \wedge (m\mathcal{I}s) \Rightarrow (s\mathcal{I}p)$ (*Datisi*).

Let us consider $\mathcal{A}$ and $\mathcal{I}$ as binary relations $\leq$ and θ in an arbitrary class of objects, and take $\wedge$, $\Rightarrow$ in their informal sense. Then $s\mathcal{A}s$ and *Barbara* show that $\leq$ is *reflexive* and *transitive*, i.e., it is a *quasi-ordering relation*. To obtain an algebra, $\leq$ has to be *anti-symmetric*, i.e., an additional axiom $(s\mathcal{A}p) \wedge (p\mathcal{A}s) \Rightarrow (s = p)$ is needed. Strictly speaking, $=$ here belongs to the extended language of syllogistic but it is not difficult to carry out such extension formally. This innovation has been made for first time by Leibniz (see, e.g., *Elementa Calculi* [Lei, p. 52]). In such a way, the set of terms turns into a *partially ordered structure* [Bir]. The second relation θ is *reflexive* and *monotonic* with respect to $\leq$. So the traditional syllogistic may be shortly characterized by a pair $(\leq, \theta)$.

Actually, in the most of his logical manuscripts, Leibniz introduces into the syllogistic a *composition* of terms besides $\mathcal{A}$ and $\mathcal{I}$. We prefer this neutral name (adopted from Leibniz himself) in order to avoid any specifications of its "real" nature: is it a term conjunction or a term disjunction. Then some transpositions in the expressive means of the syllogistic language become possible. Let us denote by $\circ$ the algebraic analogue of the composition. Irrespective of what it is in the semantics, an intersection or a union, as a minimum it should be *idempotent*, *commutative*, and *associative*. To be formulated, these three properties require only term *equality* to appear explicitly. A structure with an operation like just described G. Birkhoff names *semi-lattice* [Bir]. An ordering relation $\leq$ we need in syllogistic may be defined in such a structure by $x \leq y \leftrightarrow x = x \circ y$. Indeed, its *reflexivity* follows from the idempotency of $\circ$, its *transitivity* from the associativity, and its *anti-symmetry* from the commutativity. On the other hand, with respect to $\leq$ so introduced, the composition has the properties of *inf*: $x \circ y \leq x$; $x \circ y \leq y$; $(x \leq y)\&(x \leq z) \rightarrow (x \leq y \circ z)$. These obvious results about the system $(=, \circ)$ are summarized in the following

Fact 1: Any operation $\circ$ which is *idempotent*, *commutative* and *associative* produces an ordering relation $\leq$ which is *reflexive*, *transitive* and *anti-symmetric*; moreover, $x \circ y = \mathit{inf}\,(x, y)$ with respect to $\leq$. A second ordering may be defined by $x \leq y \leftrightarrow y = x \circ y$; the composition will be a *sup* with respect to it.

The possibility to build up syllogistic in this way has been noted many times by Leibniz. In fact, he does list idempotency and commutativity of the composition, defines $\mathcal{A}$ as it was shown (e.g., in *Primaria Calculi Logici fundamenta* [Lei, pp. 235–236]), and only associativity has not been mentioned explicitly. However, it is described in a significant example ([Lei, p. 258]): having three notions presented by integers b, c, d and denoting $l = bc, m = bd, n = cd$, their composition is written in four equivalent ways: $a = ld = mc = nb = abc$.

It would be natural to have *two* dual operations simultaneously, $\circ$ and $\bullet$, one of them playing the rôle of a *term conjunction*, and the other being a *term disjunction*. This means that the first one has to be an *inf* and the second to be a *sup* with respect to a *unique* ordering relation $\leq$. The coincidence of corresponding relations is ensured by two additional axioms connecting $\circ$ and $\bullet$:

Fact 2: In any structure with two semi-lattice operations $\circ$ and $\bullet$ satisfying the *laws of absorption*: $s \circ (s \bullet p) = s$ and $s \bullet (s \circ p) = s$, both operations define a common ordering, one of them being an *inf* and the other being a *sup* with respect to it.

Of course, the resulting system $(=, \circ, \bullet)$ coincides with the algebraic structure named now *lattice*, a structure for first time introduced and studied by Peirce [Pei].

Further, let us suppose that an *extreme* element e exists in the structure $(=, \circ)$ with the sole property $x \circ e = e$ for any x. We prefer a neutral name again in order to avoid a concretization about what is it "in reality": the empty term or the universal one. The axiom for e gives $e \leq x$ (in the sense of the ordering defined first). So e is the *least* element of the structure. The second syllogistic relation θ can be defined by $x\theta y \leftrightarrow x \circ y \neq e$. In such a case, if only elements different from e are admitted, $x\theta x$ is exactly the requirement $x \neq e$, and the analogue of the fourth Lukasiewicz's axiom, *Datisi*, follows using $e \leq m$. When the second ordering is taken, the only difference is that e appears as a *greatest* element of the structure. Let's summarize these results in the next

Fact 3: In any semi-lattice $(=, \circ, e)$ with an *extreme* element both syllogistic relations $\leq$ and θ can be defined.

Leibniz has used literally the same definitions of $\leq$ and θ calling our $\neq e$ "est Ens" (*Difficultates quædam Logicæ* [Phi, p. 212]); the requirement for any term to be non-empty has appeared as "A est Ens (ex hypothesi)" [*ibid.*, p. 213]. Obviously, if *term negation* $(-)$ appears in the semi-lattice with minimal suitable properties, then the extreme element becomes definable by Leibniz's "non-Ens" $b \circ (-b)$ (e.g., (8) in [Lei, p. 259]). The last basis could be characterized by a triple $(=, \circ, e(\circ, -))$ because e is represented here as a function of $\circ$ and $-$.

When the structure contains term negation, so to say, in its full volume, the relation θ becomes definable by $\leq$: $x\theta y \leftrightarrow x \not\leq -y$ (as well as $\leq$ by θ: $x \leq y \leftrightarrow \overline{x\theta - y}$). If empty and universal terms are prohibited then the full syllogistic of negative terms can be produced from the system $(\leq, -)$ using

Fact 4: If the negation satisfies the following three laws: $--x = x$, $x \leq y \rightarrow -y \leq -x$, $x \leq y \rightarrow x \not\leq -y$ (or, the last replaced with $x \not\leq -x$), the traditional syllogistic enlarged by term negation can be obtained.

The last system is axiomatized in [Wed] and [She], respectively. Leibniz has listed the first three laws of negation, e.g., in *Generales Inquisitiones de Analysi Notionum et Veritatum* (Principles 96, 93, 91(=100)) [Lei, p. 379–380]; another exposition including the fourth formula has been given in *Fundamenta Calculi Logici* [*ibid.*, p. 422]. When arbitrary terms are allowed, the last axiom has to be replaced with $x \leq -x \rightarrow x \leq y$ [She].

The next combination includes term composition and term negation (besides equality) with the usual axioms sufficient to obtain a Boolean algebra (see [Bir]). Then both relations $x \leq y$ and $x\theta y$ can be defined by $x \circ y = x$ and $x \circ (-y) \neq x$, respectively. Therefore, the system $(=, \circ, -)$ represents

Fact 5: Equality, composition, and negation are sufficient for building up the full Boolean syllogistic.

Coming back to the traditional letters and using Leibniz's notation for the composition, we can find the corresponding symmetric definitions in *Primaria Calculi Logici fundamenta*: $s\mathcal{A}p$ is $s = sp$, $s\mathcal{E}p$ is $s = s(-p)$, $s\mathcal{O}p$ is $s \neq sp$, $s\mathcal{I}p$ is $s \neq s(-p)$ [Lei, p. 236].

Observing the consecutive variations in the language of syllogistic one can note a transfer from *syllogistic relations* to *term operations*, and this transfer means, in fact, a consecutive elimination of the traditional syllogistic: while only specific term relations ($\leq$ and θ, i.e., $\mathcal{A}$ and $\mathcal{I}$) have appeared at the beginning, only term operations ($\circ$ and $-$, i.e., composition and negation) with term equality appear at the end. Moreover, even the *relation* $=$ can be replaced by a *property* "$= e$" with e being the extreme element:

Fact 6: Composition, negation and equality to the extreme element are sufficient for building up the full Boolean syllogistic.

Indeed, if the empty term 0 is taken in the rôle of extreme element and if Leibniz's notation is used for composition (now, term conjunction), then four syllogistic relations corresponding to the traditional categorical propositions can be defined in the following symmetric manner: $s\mathcal{A}p$ is $s(-p) = 0$, $s\mathcal{O}p$ is $s(-p) \neq 0$, $s\mathcal{E}p$ is $sp = 0$, and $s\mathcal{I}p$ is $sp \neq 0$. These representations have been listed, e.g., in *Generales Inquisitiones...* (Principle 151) where our $\neq 0$ is named "est res" and $= 0$ is "non est res" [Lei, p. 393], and in *Primaria Calculi Logici fundamenta* where "est Ens" and "est non-Ens" stay for $\neq 0$ and $= 0$, respectively. It is curious that the same four equations would be proposed by G. Boole 160 years later in [Boo].

As we saw, various fragments of syllogistic had been considered in different periods of logic. The minimal fragment was the one introduced by Aristotle; it contained term relations only. The maximal system included all Boolean term operations besides the traditional relations. Most of that fragments had been introduced by Leibniz in a close connection with his idea to translate logical reasoning into arithmetic, i.e., to replace the logical relations between notions with arithmetical relations between integers (assuming logic to coincide with syllogistic). Figuratively speaking, according to Leibniz's plan syllogistic so algebraizied had to be arithmetized. His first trials to use divisibility of integers for this purpose were unsuccessful. Only the model using pairs of co-prime numbers was successful as Słupecki proved [Słu] (see also [Luk]) but this model was more complicated and could not be extended to syllogistic including term operations. Two realizations of Leibniz's primary plan applicable to all syllogistic systems described above have been exposed in our paper [Sot].

References

[Bir] BIRKHOFF G., *Lattice Theory*, Amer. Math. Soc., Providence, 3d ed. 1967.

[Boo] BOOLE G., *The Mathematical Analysis of Logic, Being an Essay towards a Calculus of Deductive Reasoning*, Macmillan, Cambridge, 1847.

[Lei] COUTURAT L., *Opuscules et fragments inédits de Leibniz, éxtraits des manuscrits de la Bibliothèque royale de Hanovre*, F. Alcan, Paris, 1903 (*reprint*: Olms, Hildesheim, 1961).

[Łuk] Łukasiewicz J., *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Clarendon Press, Oxford, 2nd ed. 1957.

[Pei] Peirce C., "On the algebra of logic", *Amer. J. Math.*, **3** (1880), pp. 15–57.

[Phi] Leibniz, *Die philosophischen Schriften von Gottfried Wilhelm...* (ed. C. Gerhardt), Bd. VII, Berlin, 1890.

[She] Shepherdson J., "On the interpretation of Aristotelian syllogistic", *J. Symb. Logic*, **21** (1956), pp. 137–147.

[Słu] Słupecki J., "Z badań nad sylogistiką Aristotelesa", *Travaux de la Société des Science et des Lettres de Wrocław, sér. 6*, Wrocław, 1948 (= "On Aristotelian syllogistic", *Studia Philosophica*, **4**, Poznań, 1949/50, pp. 275–300).

[Sot] Sotirov V., "Arithmetizations of syllogistic à la Leibniz", *J. Appl. Non-Class. Logics* (*to appear*).

[Wed] Wedberg A., "The Aristotelian theory of classes", *Ajatus*, **15** (1948), pp. 299–314.

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