Monadic Predicate Calculus with Equality Arithmetized à la Leibniz*

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1. Introduction. In the style of his epoch, Leibniz has identified the whole logical reasoning with the Aristotelian syllogistic. In such a way, Leibniz's "Calculemus!" has meant to construct an adequate model of the basic syllogistic relations into the arithmetic of integers. All his trials to find some natural model of syllogistic can be retraced from the first publications of his logical papers made by L. Couturat ([1], [2]) as well as from the new academic edition of his philosophical manuscripts [3]. Unfortunately, that trials have been unsuccessful. Finally Leibniz has radically changed the arithmetical interpretation (in a series of manuscripts dated April 1679): on the place of simple relations between integers based on divisibility, relations between *pairs* of (relatively prime) numbers appeared. In 1946, J. Słupecki proved that the last model is adequate to the Aristotelian syllogistic [4]; see also the celebrated book of J. Łukasiewicz [5]. However, the model of pairs possesses two big disadvantages. The first one is that the initial naturalness has died and is not fatal from the mathematical point of view. However, the second disadvantage is principal: it is not clear how the model of pairs of integers can be extended so to involve the generalizations of the Aristotelian syllogistic containing, e. g., term negation or term conjunction.

We showed in [6] that Leibniz's primary plan to explore divisibility of integers had been vital and built up two kind of arithmetical models adequate to many systems of syllogistic depending on the term relations and operations used. Later on we extended the arithmetical interpretation so to cover the entire monadic predicate calculus: an abstract of this interpretation was announced in [7] and the proof of the adequacy was published in [8]. It will be shown now that Leibniz style arithmetization can be extended on the monadic predicate calculus containing equality as well.

Two methodological notes may be added here. First of all, any variant of syllogistic is decidable while the whole arithmetic is not. In such a way, speaking *in abstracto*, the calculation is not able to increase the "algorithmicity" of any disputation according to Leibniz's dreams. Nevertheless, that fragment of arithmetic in which all variants of syllogistic were imbedded is decidable because it contains *multiplication* only; see [9] and § 25 of [10]. The same fragment was used for the pure monadic predicate calculus (without equality) and it will be used now for the monadic predicate calculus with equality. The second note is that the full predicate calculus containing arbitrary binary relations is undecidable. Therefore, we cannot expect a natural Leibniz style translation of the logic of relations into arithmetic, i. e., a translation using divisibility of integers only. In other words, the arithmetization of the monadic predicate calculus with equality achieves the maximum in some sense.

2. The Main Result. The language of the calculus contains *individual variables* $x_1, x_2, \ldots$, one-place *predicate symbols* $P_1, P_2, \ldots$, the only two-place predicate

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=, quantifiers, any complete set of propositional connectives together with brackets. Formulas are defined as usually. For this language, *interpretations* of the predicates in non-empty domains, *evaluations* of the individual variables, *values* of formulas in given interpretation under given evaluation, and predicate *tautologies* are introduced in the standard manner (see § 29 of [11]).

For the *arithmetical* models, let $N > 1$ be an integer *without multiple factors*. Any unary predicate P_i is interpreted by arbitrary divisor of N (possibly 1 or N) denoted with $d(P_i)$, and any individual variable x_i is evaluated by $d(x_i)$, a *prime* divisor of N (as usual, 1 is not supposed to be prime). Following the construction of a formula F, the *arithmetical statement* $\mathcal{AR}[F]$ corresponding to given evaluation will be obtained. Namely, for atomic formulas, $\mathcal{AR}[P_i(x_j)]$ is the statement "$d(x_j)$ divides $d(P_i)$", and $\mathcal{AR}[x_i = x_j]$ is "$d(x_i) = d(x_j)$". For a subformula G, $\mathcal{AR}[(\forall x)G]$ is the statement "for any prime divisor d, $\mathcal{AR}^x_d[G]$" where $\mathcal{AR}^x_d$ differs from $\mathcal{AR}$ attaching d to x. Finally, all propositional connectives have to be replaced with their non-formal analogues. If $\mathcal{AR}[F]$ is a true arithmetical sentence for any N under arbitrary evaluation of the individual variables, F is named *arithmetically true*. This semantics is relevant:

Theorem. Any formula of the monadic predicate calculus with equality is a tautology *iff* it is arithmetically true.

Proof. First of all, a predicate formula is a tautology *iff* its universal closure is a tautology. Formulas without free variables are called *propositions*. Because of the decidability of the monadic predicate calculus with equality, the domains may be supposed to be finite. Namely, if a given proposition containing k predicates and r individual variables is not a tautology then it can be refuted in a domain with no more then $N = 2^k \cdot r$ elements (see, e. g., Theorem 25.1 in [10]). Any model built on such a domain can be transformed in an isomorphic arithmetical model: if all elements of the domain D are $a_1, \ldots, a_N$, take an integer $u = p_1 \cdot \ldots \cdot p_N$ where $p_1, \ldots, p_N$ are arbitrary but *different* prime numbers. If the interpretation of the predicate P_i is the *non-empty* subset of D $\{a_{i_1}, \ldots, a_{i_n}\}$, let $d(P_i)$ be the product $p_{i_1} \cdot \ldots \cdot p_{i_n}$; if the subset is empty, $d(P_i)$ is 1. The evaluation of individual variables is obvious: if the value of x_i in the domain is a_s, its arithmetical value is p_s. Conversely, if the formula is not arithmetically true for a given integer $N > 1$ without multiple factors, a set of its prime divisors may be taken on the place of each its divisor (the set obtained will be empty when the divisor is 1). Further on it will be sufficient to use the isomorphism between the Boolean algebra of *all subsets* of the set of the divisors of an integer $N > 1$ without multiple factors, and the Boolean algebra of *all divisors* of N with l.c.m., g.c.d. and reciprocals in the rôle of the Boolean operations. As it seems, this isomorphism has been for first time used by E. Bunitsky in [12].

There is a well-known translation of both basic syllogistic relations into the pure monadic predicate calculus: $s\mathcal{A}p$ ("every s is a p") is interpreted as $\forall x(S(x) \Rightarrow P(x))$ and $s\mathcal{I}p$ ("some s is a p") is interpreted as $\exists x(S(x)\&P(x))$. Applied to the predicate formulas just written, the arithmetical interpretation described above will produce the arithmetical semantics of the Aristotelian syllogistic named in [6] *Scholastic*. Its dual semantics was named there *Leibnizian*. The last semantics could be extended on the whole monadic predicate calculus (with or without equality) but then the dual arithmetical interpretation of any formula would simply coincide with the normal interpretation of *the negation of its dual*.

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