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In [1] Makinson constructed an extension of the classical modal logic T, which is not finitely approximable; there exists a formula which is not deducible in this extension but is generally valid in any finite model (it is, of course, refutable in some infinite model). In general, theorems in this system never differ from certain nondeducible formulas, which use only finite models. However, the existence of other recognition algorithms is not excluded, i.e., the system may be soluble. On the other hand, it is known [2] that any nonmodal extension of Dummett's superintuitionistic logic LC is finitely approximable and also tabular.

The modal system LCT^+ set out below has the following special properties. Firstly, it is a nonfinitely approximable modal extension of the logic LC. Thus we show that the property of Dummett's system described in [2] is intrinsic only in its nonmodal extensions. Secondly, on any finite model for LCT^+, the formulas A and $\Box A$ are equivalent. Hence it follows that with respect to finite models, the logic LCT^+ does not differ from the pure Dummett calculus without modality, i.e., we can only distinguish its properly modal theorems using infinite models. Finally (thirdly), the classical modal logic corresponding to LCT^+ is that obtained by adjoining the axiom scheme $A \vee \neg A$, which in a trivial way is finitely approximable and is even only modal fictitiously; in it, for any formula A, the equivalence $\Box A \equiv A$ is deducible.

For simplicity of notation, we shall use Kripke's models, but the arguments may easily be carried over to arbitrary algebraic models. Kripke's semantics for an intuitionistic modal logic were described briefly in [3], and here we shall give only the basic aspects. Denote by IT the intuitionistic analog of the classical modal logic T, which is obtained as an extension of the intuitionistic propositional calculus by a unary operator $\Box$ ("necessity"), together with the following axiom schemes and rules of deduction:

$$\Box A \wedge \Box B \leqslant \Box [A \wedge B],$$
$$\Box A \leqslant A, \quad \frac{A \leqslant B}{\Box A \leqslant \Box B}, \quad \frac{A}{\Box B}.$$

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($A \leqslant B$ is the abbreviation for $\vdash A \Rightarrow B$.) We denote the extension of IT by the axiom scheme

$$Ax: \Box A \Rightarrow \Box\Box A \leqslant A \Rightarrow \Box A$$

by IT$^+$, and LCT$^+$ is the extension of IT$^+$ by Dummett's axiom scheme

$$[A \Rightarrow B] \vee [B \Rightarrow A].$$

Kripke's model of IT is the ordered quadruple $\langle K, \leqslant, R, \Vdash \rangle$, in which $\langle K, \leqslant, \Vdash \rangle$ is Kripke's model for the intuitionistic propositional calculus, and R is an arbitrary binary relation in K which is reflexive and satisfies the "extrapolation" rule: from xRy, $u \leqslant x$ and $y \leqslant v$, we have uRv, and for a forcing of modality we have

$x \Vdash \Box A$ if and only if $(\forall y)$ [if xRy, then $y \Vdash A$]. We note that an adequate condition for Dummett's axiom is

$$(\forall x)(\forall y)[x \leqslant y \text{ or } y \leqslant x].$$

LEMMA 1. The formula $p \Rightarrow \Box p$ is generally valid on any finite model for LCT$^+$.

Proof. We show that if on some model for LCT$^+$ the formula Ax is generally valid, but $p \Rightarrow \Box p$ is refutable, then this model is infinite. Suppose that $x_0 \nVdash p \Rightarrow \Box p$, i.e., for some x_1, $x_0 \leqslant x_1$, $x_1 \Vdash p$ and $x_1 \nVdash \Box p$ are true. However, in this model the formula

$$[\Box p \Rightarrow \Box\Box p] \Rightarrow [p \Rightarrow \Box p];$$

is generally valid, and hence $x_1 \nVdash \Box p \Rightarrow \Box\Box p$; thus for any x_2 we have $x_1 \leqslant x_2$, $x_2 \Vdash \Box p$ and $x_2 \nVdash \Box\Box p$. But it follows from $x_1 \nVdash \Box p$ and $x_2 \Vdash \Box p$ that $x_1 \neq x_2$, which together with $x_1 \leqslant x_2$ we will denote by $x_1 < x_2$. We continue further by induction. Suppose that we have obtained a sequence $x_1 < x_2 < \ldots < x_n$ and $x_n \Vdash \Box^{n-1} p$, $x_n \nVdash \Box^n p$, and hence $x_n \nVdash \Box^{n-1} p \Rightarrow \Box^n p$, From the latter and Ax we obtain

$x_n \nVdash \Box^n p \Rightarrow \Box^{n+1} p$ and hence there exists x_{n+1} for which we have $x_n \leqslant x_{n+1}$, $x_{n+1} \Vdash \Box^n p$ and $x_{n+1} \nVdash \Box^{n+1} p$, and hence $x_n < x_{n+1}$. The sequence $x_1 < x_2 < x_3 < \ldots$ shows that the model contains an infinite number of elements.

LEMMA 2. The formula $p \Rightarrow \Box p$ is not deducible in LCT$^+$.

Proof. We construct a model (infinite) on which Ax is generally valid together with the remaining axioms of LCT$^+$, but $p \Rightarrow \Box p$ is refuted. Let $K = \{0, 1, 2, \ldots\}$. Write $x \leqslant y$ if $x \leqslant y$ in the arithmetical sense and xRy when $x \leqslant y + 1$. It is easily seen that R is reflexive and "extrapolable," and hence it follows that the axiom $\Box A \leqslant A$ is generally valid, while it follows from the comparability of any elements of K that Dummett's axiom is generally valid. Suppose now that for some forcing $x \nVdash Ax$, i.e., $x \Vdash \Box A \Rightarrow \Box\Box A$, but $x \nVdash A \Rightarrow \Box A$, and thus for any y we have $x \leqslant y$, $y \Vdash A$, and $y \nVdash \Box A$. Since $x \leqslant y$, then $y \Vdash \Box A \Vdash \Box\Box A$. In particular, it follows from $y \leqslant y + 1$ that if $(y+1) \Vdash \Box A$, then $(y+1) \Vdash \Box\Box A$. But if $(y+1) \Vdash \Box\Box A$, then it follows from (y + 1)Ry that $y \Vdash \Box A$, which is a contradiction. If, however, $(y+1) \nVdash \Box A$, then there exists a z such that $z \nVdash A$ and (y + 1)Rz; hence $y + 1 \leqslant z + 1$, i.e., $y \leqslant z$ which contradicts $y \Vdash A$. Thus all the axioms of LCT$^+$ are generally valid on this model. Moreover, we define the following forcing: $0 \nVdash p$, but if x > 0, then $x \Vdash p$. Then $1 \Vdash p$, but $1 \nVdash \Box p$, since 1R0, while $0 \nVdash p$. Thus $1 \nVdash p \Rightarrow \Box p$ and the formula is refuted.

THEOREM 1. The logic LCT$^+$ is not finitely approximable.

Proof. This follows from Lemmas 1 and 2, since on any finite model for LCT$^+$, the non-deducible formula $p \Rightarrow \Box p$ is generally valid.

As a corollary, we note that the formula $\Box A \equiv A$ is generally valid on all finite models of the logic LCT$^+$, and thus on such models LCT$^+$ in fact coincides with Dummett's superintuitionistic logic. In other words, the properly modal theorems of LCT$^+$ are not finitely distinguishable from the theorems of LC.

THEOREM 2. In the classical modal logic T the formulas Ax and $A \Rightarrow \Box A$ are equivalent.

Proof. We first show that from Ax we have $A \Rightarrow \Box A$, transforming Ax in a (classically) equivalent manner:

$$\Box A \Rightarrow \Box\Box A \Rightarrow [A \Rightarrow \Box A],$$
$$\neg[\Box A \Rightarrow \Box\Box A] \vee \neg A \vee \Box A,$$
$$[\Box A \wedge \neg\Box\Box A] \vee \neg A \vee \Box A,$$
$$\neg A \vee [\Box A \wedge [\Box A \vee \neg\Box\Box A]],$$
$$[\neg A \vee \Box A] \wedge [\neg A \vee \Box A \vee \neg\Box\Box A],$$
$$[A \Rightarrow \Box A] \wedge [A \wedge \Box\Box A \Rightarrow \Box A].$$

Conversely, it is sufficient to show that from $A \leqslant \Box A$ we have Ax, but this is obvious. The equivalence proved above, together with the axiom $\Box A \leqslant A$, gives $\Box A \equiv A$, so that the classical system T^+ is fictitiously modal.

Consider other logics. If we are only interested in finite approximability, irrespective of properties of modality, we may give examples of weaker systems, say without the axiom $\Box A \leqslant A$, and stronger systems, say with the additional axiom $\Box[\Box A \Rightarrow B] \vee \Box[\Box B \Rightarrow A]$; an adequate condition for R (if xRy and xRz, then yRz or zRy) is verified immediately in the model in Lemma 2, and therefore the proof of Theorem 1 is almost unchanged. We need to mention the logic IT^+ specially: it is also not finitely approximable, but on all finite models it coincides with the intuitionistic propositional calculus, and, moreover, its classical analog is fictitiously modal and thus trivially soluble.

The systems LCT^+ and IT^+ do not belong to any well-known classification of modal logics. Therefore we give an example of a nonfinitely approximable logic between IT and IS4; it is obtained by adjoining the following axiom to IT:

$$\Box\Box A \Rightarrow \Box\Box\Box A \leqslant \Box A \Rightarrow \Box\Box A.$$

The formula which is generally valid on all finite models is in this case $\Box\Box p \Rightarrow \Box p$. The classical logic corresponding to it coincides with S4 and is again finitely approximable.

We give another example of a logic which is classically, and intuitionistically, but not finitely approximable, taking Makinson's axiom in a suitable form for the intuitionistic case: extensions of T and IT by the axiom

$$\Box[\Box\Box A \Rightarrow \Box\Box\Box A] \leqslant \Box A \Rightarrow \Box\Box A$$

are not finitely approximable.

Finally, we give an example of a logic which is "nearer" to Dummett's logic, in the sense that its classical analog is contained in T, but nevertheless it is not finitely approximable. Denote by $LC(\Diamond)$ the extension of the intuitionistic propositional calculus by the symbol $\Diamond$ of "possibility," together with the axioms and rules

$$[A \Rightarrow B] \vee [B \Rightarrow A],$$
$$\Diamond[A \vee B] \leqslant \Diamond A \vee \Diamond B, \quad \frac{A \leqslant B}{\Diamond A \leqslant \Diamond B}, \quad \frac{\neg A}{\neg\Diamond A}.$$

Denote the extension of $LC(\Diamond)$ by the axiom scheme

$$\Diamond A \Rightarrow \Diamond\Diamond A \leqslant A \Rightarrow \Diamond A$$

by $LC(\Diamond)^+$. Then the nondeducible formula $p \Rightarrow \Diamond p$ is generally valid on all finite models. Thus, $LC(\Diamond)^+$ is not finitely approximable and is obviously contained in $LCT(\Diamond)$. At present, no example is known of a nonfinitely approximable extension of Dummett's logic which is contained in $LCT(\Box)$.

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