

Vladimir Sotirov (Sofia)
Leibniz's "Calculemus!" at Work¹

1. Introductory Remarks

There are at least four aspects in which Leibniz's logical heritage can be considered. The first aspect is purely philosophical and the central topic here is the well known scene with the two disputing philosophers which finally take their quills, sit down at their abaci and inviting each other in a friendly manner begin calculating "the truth". According to the programme of Leibniz, human reasoning is to be transformed into a kind of arithmetic using *characteristic numbers* instead of notions. In such a way, the logical truth of a proposition should turn into an arithmetical truth of a calculation. E. g., if the number of *animal* were 2 and that of *rational* were 3 (Leibniz's beloved example) then the number of *man* being by definition a *rational animal* would be obtained by the multiplication $3 \cdot 2$. Then the answer to the question 'Is every man a rational being?' could be reduced to the fact that 6 is divisible by 3.

Some of the texts concerning this program can be seen in the editions of Gerhardt and Couturat. Their titles are very representative: *Discours touchant la Méthode de la Certitude et de l'Art d'Inventer pour finir les disputes, et pour faire en peu de temps des grands progrès* (GP VII, 174–183); *Projet et essais pour arriver à quelque certitude pour finir une bonne partie des disputes, et pour avancer l'art d'inventer* (C, 175–182), etc.

Two comments can be made here. First, all Leibniz's logic operates with atoms of the form 'A est B' and therefore is limited to syllogistic logic or, in at best, to the *monadic* predicate calculus. Both of them are decidable. On the contrary, nowadays we know that the predicate calculus with *dyadic relations* is not decidable. Second, no translation of arbitrary reasoning into arithmetic could be *ipso facto* a panacea for their correctness because of the undecidability of arithmetic itself. The business of the *computists* is no more algorithmic than that of the philosophers. Hence the plan to assign a few selected men the decomposition of all human knowledge to syllogistic relations (in a *quinquennium* according to Leibniz's dream!) and this way to liquidate any disputation had been radically unrealizable. Nevertheless, the fragment of arithmetic that Leibniz had had in mind to be used is decidable because it is the fragment containing *only multiplication* (without addition).

The second aspect of treating Leibniz's logic is algebraic. Its central theme is the abstract representation of the logical laws which (according to the initial language) leads to nowadays established algebraic structures such as partially ordered sets, semi-lattices, lattices, Boolean algebras, etc. Some of the relevant papers of Leibniz are: *De formis syllogismorum mathematice definiendis* (C, 410–416); *Generales Inquisitiones de Analysis Notionum et Veritatum* (C, 366–399); *Specimina calculi rationalis* (C, 259–264). A full picture of all Leibniz's approaches showing their exact place in a uniform algebraic scheme is given in my paper *Various syllogistics from an algebraic point of view*².

¹ I am deeply grateful to my German colleagues, especially to Prof. H. Schepers, Prof. H. Breger, Prof. M. Schneider, and Dr. Herma Kliege-Biller, for their consultations, for the opportunity to see some Leibniz's manuscripts, and for assigning me the extremely valuable Academy Edition of Leibniz. Due to the hospitality of Prof. W. Lenzen in Osnabrück I obtained access to the literature lacking in Bulgaria. All my papers devoted on Leibniz's logic are accessible from my web-page <http://www.math.bas.bg/~vlsot>.

² 2nd Panhellenic Logic Symposium (Delphi, 13–17 June 1999), Proceedings, 197–200.

The third aspect can be regarded as geometrical. The topic is the popular representation of notions using so-called Euler circles (sometimes incorrectly named Venn diagrams). The original Leibniz drawings can be seen in *De formae logicae comporatione per linearum ductus* (C, 292–321). Let us note here that the confusion in the arithmetical models proposed by Leibniz and analyzed in details in my paper *Arithmetizations of syllogistic à la Leibniz*³ becomes obvious when circles are used. The probable explanation of the contradiction between the arithmetical idea, which is in fact wrong, and the correct geometrical one is the 20 year span between them.

Finally, the fourth aspect includes the arithmetical models of the Aristotelian syllogistic built by Leibniz in April 1679. His most important ‘arithmetical’ papers are *Elementa characteristicae universalis* (A VI, II, 181–194); *Elementa calculi* (A VI, II, 195–205); *Calculi universalis elementa* (A VI, II, 205–216), etc. Strictly speaking, these papers are the only relevant ones to our task: to implement a natural and working translation of syllogistic into arithmetic. That is why, let us remind Leibniz’s attempts, his fault and his success.

2. A short exposition of Leibniz’s manuscripts of April 1679

The example with the rational animal gives a criterion for Universal Affirmative propositions: “Every s is a p ” (sAp) is true when s is divisible by p . (We use the same letters for terms and their characteristic numbers.) In Leibniz’s notation: $s = xp$.

As it can be seen, the number of a composite term is always greater than the number of any of its components, i.e., the characteristic number of species is greater than the characteristic number of genus. This fact expresses the view that terms are sheaves of properties – a view laid down in Aristotle’s *Organon*. However, if the term is treated as a class of objects – a view in better concordance with our set-theoretical intuition – then it would be more natural for the characteristic number of the species to be less than the characteristic number of the genus. In *Nouveaux Essais sur l’Entendement humain* Leibniz analyses in details the reasons for both views:

“La maniere d’enoncer vulgaire regarde plustost les individus, mais celle d’Aristote a plus d’egard aux idées ou universaux. Car disant ‘tout homme est animal’, je veux dire que tous les hommes sont compris dans tous les animaux; mais j’entends en même temps que l’idée de l’animal est comprise dans l’idée de l’homme. L’animal comprend plus d’individus que l’homme, mais l’homme comprend plus d’idées ou plus de formalités; l’un a plus d’exemples, l’autre plus de degrés de réalité; l’un a plus d’extension, l’autre plus d’intension” (GP V, 496).

Clearly, Leibniz did note the possibility of two approaches to term inclusion. The first is his own (“generis notio sit pars, speciei notio sit totum”); the second, accepted “in scholis”, as he says, is the opposite one. According to the other approach, a second (inverse) criterion for UA-propositions is possible: “Using fitting characters, we could demonstrate all the rules of logic by another kind of calculus than the one developed here, merely by an inversion of our own calculus” (A VI, II, 200). Hence, let us keep in mind that Leibniz assumes two equipollent variants of interpreting UA-propositions, but accepts one of them as a working hypothesis. In the terminology of Couturat, he prefers term *comprehension* to term *extension*. (Sometimes the two semantics of term relations are called *intensional* and *extensional* respectively.) For this reason we name the interpretation just described *Leibnizian* in opposition to the dual interpretation that Leibniz himself named *Scholastic*.

³ Journal of applied non-classical logics, 9(1999), 387–405.

The interpretation of Particular Affirmative (PA) propositions in Leibniz's manuscripts is less clear and more problematic. Moreover, no sufficiently complete example can be found. In the first logical manuscript the rule means that a PA-proposition is true when either the number of the predicate is divisible by the number of the subject or the number of the subject is divisible by the number of the predicate (GP VII, 171). But the acceptance of such a rule is equivalent to a supposition that “Some s is a p ” (sIp) always implies either “Every s is a p ” or “Every p is an s ” – an alternative which is not true, of course, as Leibniz himself observes later. Indeed, in the next manuscript (GP VII, 192–193) another rule is outlined: sIp is true when s , being multiplied by another integer, is divisible by p ; in Leibniz's notation: $sx = yp$.

However, if we understand this rule literally, it will be trivially true because *any* integer s becomes divisible by *any other* integer p after multiplying it by a suitable integer, e.g., by p itself. That is why it is natural to complete the criterion for PA by the condition that the multiplier must be *less* than the number of the predicate. Now both criteria proposed by Leibniz can be formulated in a uniform manner: sAp is true when any divisor of p is also a divisor of s ; sIp is true when s and p have a common divisor greater than 1, or, g.c.d. (s, p) > 1.

Unfortunately, the procedure of checking just described sometimes gives a *negative* result when applied to a *true* syllogism. Such is the case with *Darii*: it ceases to be true (see the paper cited in footnote 3 for details). The cause for the failure of the initial Leibniz plan was in the confusion of the two intuitions: his criterion for UA explored the *comprehensive* representation of terms, i.e., the genus was understood as part of the species, while his criterion for PA explored the *extensive* one (the species as part of the genus).

The prehistory of the arithmetization ended with the building of a radically different system, in which terms were interpreted by *pairs of co-prime numbers*, i. e., numbers without common divisors greater than 1. The main corresponding papers are *Calculus consequentiarum* (A VI, II, 221–227); *Modus examinandi consequentias per numeros* (A VI, II, 228–236); *Regulae ex quibus de bonitate consequentiarum formisque et modis syllogismorum categoricum judicari potest, per numeros* (A VI, II, 242–251). In this interpretation, if the pair of co-prime numbers (s_1, s_2) is assigned to s , and (p_1, p_2) to p , then ‘Every s is a p ’ is true when s_i is divisible by p_i (for $i = 1, 2$), and ‘Some s is a p ’ is true when s_1 and p_2 as well as s_2 and p_1 are co-prime. This model was proved adequate by J. Ślupecki (see a detailed description of this model in the classical book of Łukasiewicz⁴). However it is obviously more sophisticated than the one initially planned. Moreover, its greater shortcoming is that it can not envelop syllogistic of term negation or that of term conjunction.

3. Arithmetical models of syllogistic

In the Scholastic arithmetical interpretation terms are evaluated by integers greater than 1. sAp is replaced with ‘ s is a divisor of b ’, and sIp with ‘g.c.d. (s, p) > 1’. If empty terms are admitted, they are evaluated by 1. For the Leibnizian interpretation, an arbitrary integer $u > 1$ (a “universe”) is introduced and terms are evaluated by its arbitrary proper divisors (i. e., the divisors which are less than u); sAp is replaced by ‘ s is divisible by p ’, and sIp by ‘l.c.m. (s, p) < u ’. If empty terms are admitted, they are evaluated by u .

⁴ J. Łukasiewicz: *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford, 2nd ed. 1957.

When negation appears, the ‘universe number’ u without multiple factors must appear in both arithmetical semantics, terms are evaluated by divisors of u different from 1 and u (in case the empty term and the universal term are excluded), and ‘non- s ’ is interpreted as u/s . If term composition appears, which leads to full Boolean syllogistic, sp is modeled by g.c.d. (s, p) in the Scholastic arithmetical interpretation, and by l.c.m. (s, p) in the Leibnizian one.

Both arithmetical semantics are adequate to syllogistic in its full volume including arbitrary Boolean combinations of elementary syllogistic atoms (sAp and sIp) as well as arbitrary Boolean combinations of terms. It was also shown that the arithmetical interpretations can be extended to involve pure monadic predicate calculus (with or without equality).⁵

Although the Scholastic and the Leibnizian translations are dual and therefore, in theory, are equipollent, in practice they have quite different behaviour and different philosophy. This is the cause to present here the work of the Scholastic interpretation only and to leave Leibnizian interpretation for future investigation.

4. The Scholastic arithmetical interpretation in action

We shall test the Scholastic arithmetical model of the logic of notions as described above. Of course, this application has a purely hypothetical character because Leibniz, as we mentioned, did not leave us any comprehensive rules about the translation of the logical operations into arithmetical ones. Moreover, his sketches of such translation are incorrect. The material for our test is rather illustrative and artificial for the sake of simplicity and we do not claim adequacy to established taxonomy. Leibniz himself composed detailed classifications with definitions filling scores of pages (*Table de definitions*, C, 437–510). That is why it would be interesting to apply the arithmetical models to certain real biological classifications but this is another topic.

According to the Scholastic interpretation, the notions shall be treated as classes of objects. This means that the functional definition of a compound term (what it *is*) is not important in contrast with its composition (what it *contains*). In other words, we will regard the notions as lists of *objects* and not lists of *properties*.

Consider the extremely simplified diagram on Fig. 1: *mankind* is a compound term including the British, the French, the Germans, the Bulgarians, and “others”. *Mankind* is a part of the class of *mammals* which in our example consists of *apes*, *dogs*, and *bats*. Finally, the *mammals* together with the *birds*, *reptiles*, and *insects* comprise the class of *animals*. Let us stop here and attach prime numbers to prime notions: the nationalities just listed are assigned the numbers 2, 3, 5, 7, and 11, respectively. Then the characteristic number of the mankind will be $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. For the next prime notions (*apes*, *dogs*, and *bats*) we shall use the next prime numbers: 13, 17, and 19. Then *mammals* will be characterized by the number $2 \cdot 3 \cdot \dots \cdot 19$. Finally we attach 23 to *birds*, 29 to *reptiles*, and 31 to *insects*. Therefore the class of *animals* obtains the number $2 \cdot 3 \cdot \dots \cdot 31$.

Now it is easy to check the *universal affirmative sentences*. It is enough to use simple division. For example, “Every man is a mammal” is true because the number $2 \cdot \dots \cdot 11$ is a

⁵ V. Sotirov: *Monadic predicate calculus with equality arithmetized à la Leibniz*, in: *C. r. Acad. bulgare Sci.*, 54 (2001), n. 1, 9–10.

divisor of $2 \cdot \dots \cdot 11 \cdot 13 \cdot \dots \cdot 19$. Furthermore, we can observe that the model freely works “upward”, from species to genus. If one wants to expand the diagram adding for example the full configuration of *plants*, the prime numbers bigger than 31 can be used, and the diagram of the animals need not be changed. As a result, the vertex of the “vegetable part” will obtain the characteristic number $37 \cdot \dots \cdot p_k$ corresponding to the notion of plants and a new vertex over the animals and plants will appear representing the class of the *living organisms* with characteristic number $2 \cdot \dots \cdot p_k$.

As to the direction “downward” from genus to species, the Scholastic interpretation meets some difficulties demanding re-numbering of almost all notions in use. If one wants to subdivide, e. g., the class of “other” nationalities and to separate explicitly, say, the class of all *Japanese*, the whole diagram will have to be reconstructed. Then it will be necessary to change the characteristic numbers of some prime notions and those of all composite notions. One possibility is to introduce a new prime number in the row of the nationalities preserving 11 for the class of “others”. However, the numbers of all “upper” classes – *mankind*, *mammals*, etc., will have to be multiplied by the new prime number. The same difficulty arises if one decides to list the subclasses of the *apes* – *gibbons*, *chimpanzees*, *orangutans*, and *gorillas*.

On the base of our example we can demonstrate the arithmetical translation of the Boolean operations with notions. In order to simplify the situation, suppose that the class of *animals* consists only of *reptiles*, *mammals*, *birds*, and *insects*. On their turn, let *mammals* be subdivided into *dogs*, *mankind*, and *bats*. Their characteristic numbers are shown on Fig. 2. Suppose now that one wants to introduce a new class, say, the class of the *winged*. In our simple classification this class will include *bats*, *birds*, and *insects*. The new class will have the number $5 \cdot 11 \cdot 13$ and will appear above *bats*, *birds*, and *insects* and below *animals* (see the right part of Fig. 3). The negation of *winged* is *wingless* (*apterouse*) and its number will be obtained by dividing $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ (for *animals*) by $5 \cdot 11 \cdot 13$ (for *winged*). The result is $2 \cdot 3 \cdot 7$ (see the left part of Fig. 3). The last number shows that the class *wingless* must be placed between *reptiles* and *animals*, and a new notion arises at the crossing point of the lines *wingless*–*dogs* and *mammals*–*mankind*. Obviously it is the notion of *wingless mammals*. It includes *dogs* and *mankind* and has the number $2 \cdot 3$.

Let us now check some *partially affirmative sentences*. For example, let us answer the question “Are there winged mammals?” We have to check if the number of mammals ($2 \cdot 3 \cdot 5$) and the number of winged ($5 \cdot 11 \cdot 13$) have a common divisor. Such divisor is 5 and it obviously corresponds to the class of *bats*, which appears below *mammals* and *winged* on the diagram. According to our scheme, the answer to the question “Are there winged reptiles?” is “no”, because 7 and $5 \cdot 11 \cdot 13$ are co-prime. As we see, nothing can be placed below *reptiles* and *winged*. If the question is disjunctive, e. g., “Which is the class of men and dogs?”, or in other words “What does unite men and dogs?”, the answer is: the class of the *wingless mammals* unites them because l.c.m. (2, 3) is $2 \cdot 3$ and the corresponding class is the nearest one placed above the *dogs* and *mammals*.

Unfortunately, the full Boolean diagram would contain $2^6 = 64$ vertices and is difficult to fit on a page. However, this is not an obstacle to answering various questions using characteristic numbers. So, it is not necessary to find on the diagram the place of the class corresponding to the negation of the notion of *mankind*. We know that its number is complementary to the number 2, namely it is $3 \cdot 5 \cdot \dots \cdot 13$. If we identify the *mankind* with the class of *rational animals* following Leibniz, then it will be true that rational mammals exist (because $2 \cdot 3 \cdot 5$ is divisible by 2) but rational winged do not exist (because $5 \cdot 11 \cdot 13$ is not divisible by 2).

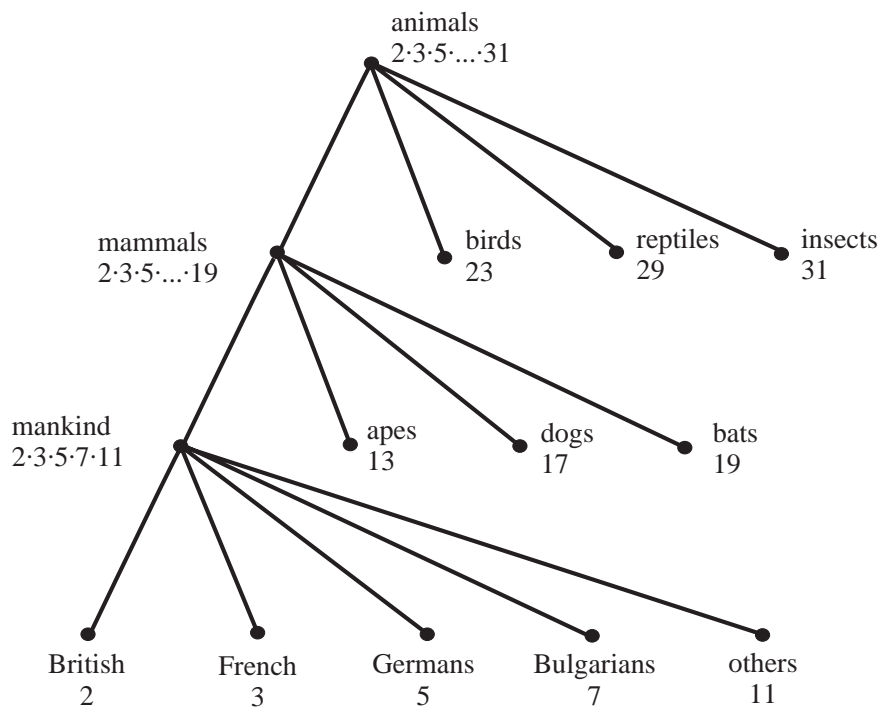


Figure 1.

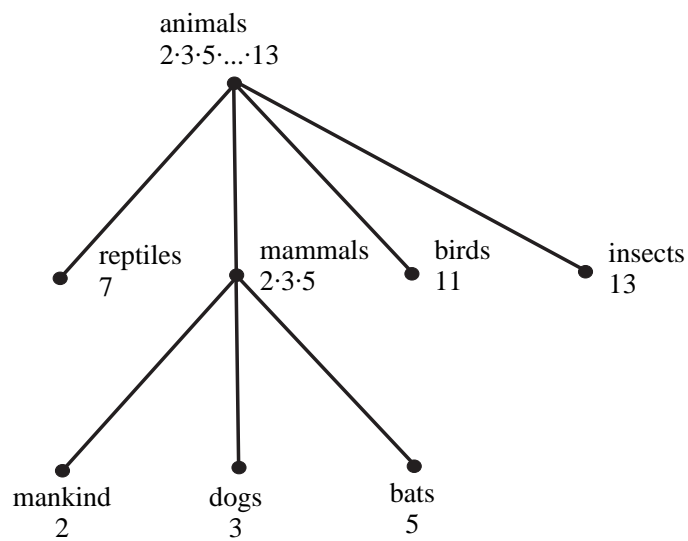


Figure 2.

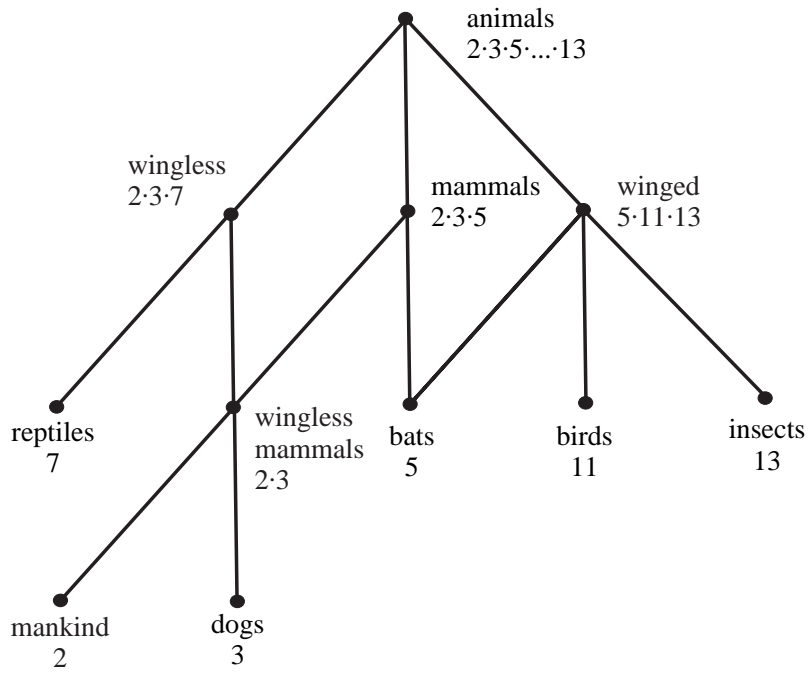


Figure 3.