In memory of George Gargov

How strange that the loss of close friends finds expression in their constant presence. We catch ourselves talking with people who have passed away, and we talk with them more often than when they were alive, even more often than they could have ever talked with us. Furthermore, in this inner dialogue, the departed always voice the truth: details are discussed with them, they answer our difficult questions ... Come to think of it, would those be their answers or are they ours, the ones we would have wished to hear from them, believing they would have been the right ones? Do we not seek in them the approval which should bring us the confidence we need so much? And we imagine that we have found it. At least as long as we live ...

Arithmetizations of Syllogistic *à la* Leibniz[1]

Vladimir Sotirov

Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
8, Acad. G.Bonchev Str., 1113 Sofia, Bulgaria

vlsot@math.bas.bg; sotirov@fmi.uni-sofia.bg
http://www.math.bas.bg/~vlsot

ABSTRACT. Two models of the Aristotelian syllogistic in arithmetic of natural numbers are built as realizations of an old Leibniz idea. In the interpretation, called Scholastic, terms are replaced by integers greater than 1*, and* s$\mathcal{A}$p *("Every* s *is a* p*") is translated as "*s *is a divisor of* p*",* s$\mathcal{I}$p *("Some* s *is a* p*") as "*g.c.d. (s, p) > 1*" (the same letters are used for the replacing numbers as well as for the terms). In the interpretation, called Leibnizian, terms are replaced by proper divisors of a special "Universe number"* u > 1 *(i.e.,* s < u*,* p < u*), and* s$\mathcal{A}$p *is translated as "*s *is divisible by* p*",* s$\mathcal{I}$p *as "*l.c.m. (s, p) < u*". Both interpretations are proved to be adequate to the Aristotelian syllogistic. They are extended to syllogistic including term negation and term conjunction as well (and, therefore, all Boolean operations with terms).*
KEY WORDS: syllogism, lattice, Boolean algebra, Aristotle, Leibniz.

1 The Prehistory

Leibniz's program for mathematization of human knowledge is mentioned in many of his manuscripts. Possibly, its best expression can be found in the following words: "Actually, when controversies arise, the necessity of disputation between two philosophers would not be bigger than that between two computists. It would be enough for them to take the quills in their hands, to sit down at their abaci, and to say (as if inviting each other in a friendly manner): Let's calculate!"[2] Lukasiewicz

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[2] "Quo facto, quando orientur controversiæ, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sidereque ad abacos, et sibi mutuo (accito si placet amico) dicere: Calculemus!" (*De Scientia universalis seu Calculo philosophico* [Phi, v. VII, p. 200]; [Rus, p. 497]). Other texts with the same content are cited in [Cou 01, p. 98, footnote 3]. About the system of citing see the **Bibliographical Note** at the end of this article.

had every reason to suppose that Leibniz's winged *Calculemus!* had been connected with the Aristotelian syllogistic [Luk 57, § 34]. Indeed, after Louis Couturat's pioneer efforts in commenting and publishing Leibniz's logical opuscula ([Cou 01]; [Log]), we have the opportunity to retrace the search for *Calculus Logicus*. If we lay aside the reminiscences of how the apparatus of *scientia generalis de instauratione et augmentis scientiarum* should have produced new truths and checked the old ones, more or less detailed definitions can be extracted from the manuscripts dated April 1679 ([Log, pp. 42–92; 245–247]; [Ger, pp. 170–240]; [Rus, pp. 506–546]).

The basic idea of the arithmetization of syllogistic was to establish a correspondence between terms and suitable integers (the *characteristic numbers* of notions), so that the logical truth of a proposition would turn into an arithmetical truth of a calculation. This idea had two realizations, one of them unsuccessful, the other one successful. Here I will justify the viability of the earlier (and less complicated) realization (with appropriate modifications, of course). The other realization, which uses pairs of co-prime numbers, will be mentioned further on.

The sole rule of the first correspondence proposed was formulated in the following words [Eng, p. 235]: "When the concept of a given term is composed directly out of the concepts of two or more other terms, then the characteristic number of the given term is to be produced by multiplying the characteristic numbers of the terms composing it."[3] The next example appears practically in all logical manuscripts: if the integer of the term 'animal' is 2, and the integer of 'rational' is 3, then the integer of 'man' being 'rational animal' will be $2 \cdot 3 = 6$. In such a way, the true proposition 'Every man is rational' comes to the fact that 6 is divisible by 3. (I reproduce the English reading of categorical propositions accepted in [Pri 62, p. 104].) If term letters are identified with the denotations of their characteristic numbers, and $s\mathcal{A}p$ denotes 'Every s is a p', the rule just cited automatically gives the

Leibniz criterion for Universal Affirmative (UA) propositions: $s\mathcal{A}p$ is true when s is divisible by p; in Leibniz's notation: $s = xp$. [4]

As it can be seen, the integer of a composite term is always greater than the integer of any of its components, i.e., the characteristic number of species is greater than the characteristic number of genus. This fact expresses the view that terms are sheaves of properties—a view laid down in Aristotle's *Organon* (see, e.g., *Anal. Pr.* A 24b25). However, if the term is treated as a class of objects—a view in better concordance with our set-theoretical intuition—then it would be more natural for the characteristic number of the species to be less than the characteristic number of the genus. In *Nouveaux Essais sur l'Entendement humain* (Book IV, Ch. XVII, § 8) Leibniz analyses in detail the reasons for both views: "La maniere d'enoncer vulgaire regarde plustost les individus, mais celle d'Aristote a plus d'egard aux idées ou universaux. Car disant 'tout homme est animal', je veux dire que tous les hommes sont compris dans tous les animaux; mais j'entends en même temps que l'idée de l'animal est comprise dans l'idée de l'homme. L'animal comprend plus d'individus que l'homme, mais l'homme comprend plus d'idées ou plus de formalités; l'un a plus d'exemples, l'autre plus de degrés de realité; l'un a plus d'extension, l'autre plus d'intension" [Phi, v. V, p. 469]. Clearly, Leibniz did note the possibility of two approaches to the term inclusion. The first is his own ("generis notio sit pars, speciei notio sit totum"); the second, accepted "in scholis",

[3] "Quando Termini dati conceptus componitur in casu rectu ex conceptibus duorum pluriumve aliorum terminorum, tunc numerus termini dati Characteristicus producatur ex terminorum termini dati conceptum componentium numeris characteristicis invicem multiplicatis" (*Elementa calculi* [Log, pp. 49–50]; [Ger, p. 180]; [Rus, p. 514]). See parallel texts in *Elementa Caracteristicæ universalis* ([Log, p. 42]; [Ger, p. 170]; [Rus, p. 506]) and in *Calculi universalis Elementa* ([Log, 60]; [Ger, pp. 194–195]; [Rus, p. 525]).

[4] "Si propositio Universalis Affirmativa est vera, necesse est ut numeris subjecti dividi possit exactè seu sine residuo per numerum prædicati"; in the margin: "$\frac{s}{p}$ succedit" ([Log, p. 42]; [Ger, p. 170]; [Rus, p. 506]). See also [Log, pp. 66; 69]; [Ger, pp. 204; 206]; [Rus, pp. 533; 535].

as he says in *Elementa Calculi* [Log, p. 53], is the opposite one. Respectively, a second (inverse) criterion for UA-propositions is possible [Eng, p. 238]: "Using fitting characters, we could demonstrate all the rules of logic by another kind of calculus than the one developed here, merely by an inversion of our own calculus."[5] But let us refer to Couturat [Cou 01, Ch. I, §§ 12–19] concerning Leibniz's reasons to prefer term *comprehension* to term *extension* (in the terminology of Couturat). And let us keep in mind that Leibniz assumes *two* equipollent variants of interpreting UA-propositions but accepts one of them as a working hypothesis.

We cannot, however, leave aside one detail from Leibniz's rule for calculating composite terms. If the rule is applied literally, the proposition 'Every man is a rational man' would not be true because $2 \cdot 3$ is not divisible by $2 \cdot 3 \cdot 3$. Couturat, too, has noted that the integer of any component may occur in the product to the power 1 only [Cou 01, p. 327]. This requirement is in accordance with Principle 129 in *Generales Inquisitiones de Analysi Notionum et Veritatum*: "Anything can be proved by numbers, it is enough for aa and a to be equivalent"[6]. Actually, traditional syllogistic has no means to make 'rational rational animal' coincide with 'rational animal'. Moreover, the very rule of attaching natural numbers to composite terms is in the competence of term conjunction. Nevertheless, the last Principle will be used in its arithmetical form given by our

First Addendum to Leibniz: *The integer of a term may not contain multiple factors.*

The interpretation of Particular Affirmative (PA) propositions in Leibniz's manuscripts is less clear and more problematic. In addition, one can find no sufficiently complete example. In the first logical manuscript (*Elementa Characteristicæ universalis*), the rule means that a PA-proposition is true when either the number of the predicate is divisible by the number of the subject or the number of the subject is divisible by the number of the predicate[7]. But the acceptance of such a rule is equivalent to a supposition that 'Some s is a p' always implies either 'Every s is a p' or 'Every p is an s'—an alternative which is not true, of course. In *Elementa Calculi* Leibniz himself observes this fact [Eng, p. 240]: "I now see that there are particular affirmative propositions also when neither term is genus or species"[8], and the manuscript finishes in pessimistic tones [*Ibid.*]: "We have been right about special cases; so we may now begin with the whole"[9]. Indeed, in the next manuscript, *Calculi universalis Elementa*, another rule is outlined ([Log, p. 58]; [Ger, pp. 192–193]; [Rus, p. 524]). It is concretized in *Calculi universalis investigationes*[10]. Thus, if $s\mathcal{I}p$ denotes 'Some s is a p', we can formulate the

Leibniz criterion for Particular Affirmative propositions: $s\mathcal{I}p$ is true when s, being multiplied by another integer, is divisible by p; in Leibniz's notation: $sx = yp$.

However, if we again understand the last rule literally, it will be trivially true because any integer s becomes divisible by any other integer p after multiplying it

[5] "Characteribus accomodatis possent omnes regulæ Logicæ a nobis demonstrari alio nonnihil calculo quam hoc loco fiet; tantum quadam calculi nostri inversione" ([Log, p. 53]; [Ger, p. 185]; [Rus, p. 518]). In *Regulæ de bonitate consequentiarum* the same remark is attached to the later procedure using *pairs* of co-prime integers ([Log, pp. 81–82]; [Ger, pp. 223–224]; [Rus, pp. 543]).

[6] "Omnia per numeros demonstrari possunt, hoc uno observato, ut aa et a æquivaleant" ([Log, p. 386]; [Ger, p. 284]; [Rus, p. 603]).

[7] "Si propositio particularis affirmativa est vera, sufficit ut vel numerus prædicati exacte dividi possit per numerum subjecti, vel numerus subjecti per numerum prædicati"; in the margin: "vel $\frac{s}{p}$ vel $\frac{p}{s}$ succedit" ([Log, p. 43]; [Ger, p. 171]; [Rus, p. 506]).

[8] "Video enim propositionem particularem affirmativam locum habere etiam cum neutrum est genus vel species" ([Log, pp. 56–57]; [Ger, p. 191]; [Rus, p. 522]).

[9] "Ergo specialiora justo diximus, adeoque de integro ordiemur" [*Ibid.*].

[10] "In omni propositione particulari affirmativa dividi potest numerus characteristicus subjecti, per alium numerum multiplicatus, per numerum characteristicum prædicati" ([Log, p. 69]; [Ger, p. 207]; [Rus, p. 535]).

by a suitable integer, e.g., by p itself. That is why it is natural to complete the criterion for PA by a

Second Addendum to Leibniz: *The multiplier in the criterion for PA-propositions must be less than the integer of the predicate.*

Thus we can obtain a more convenient form of both criteria by proving the following obvious facts:

Facts 1: The Leibniz criterion for UA-propositions is equivalent to: $s\mathcal{A}p$ is true when any divisor of p is also a divisor of s; on the basis of the First Addendum, it can be reduced to: any *prime* divisor of p is a divisor of s.

Fact 2: On the basis of the Second Addendum, the Leibniz criterion for PA-propositions is equivalent to: $s\mathcal{I}p$ is true when s and p have a common divisor greater than 1, or, g.c.d. $(s,p) > 1$; no matter whether the divisor is prime or not.

The rest of the categorical propositions includes Universal Negative (UN) and Particular Negative (PN) propositions but they are not interesting for us, being negations of PA and UA, respectively.

Using the interpretation which has been just described, Leibniz has proved some evident properties of the categorical propositions but nothing more. The big problem connected with his interpretation is that some syllogisms stop being true in it. For example, take *Darii* as it was given in *De formæ logicæ comprobatione per linearum ductus*: "Omne C est B. Quoddam D est C. Ergo Quoddam D est B" [Log, p. 302]. The corresponding calculation is very elegant, indeed: "$C = XB, VD = YC$. Ergo $VD = YXB$ quod procedit." However, V may differ from C being not different from B, in which case the calculation, trivialized by $V = B$, does not work. Indeed, the values $C = 3 \cdot 2, B = 3, D = 2 \cdot 5$ reject *Darii*, following Leibniz's prescriptions, because there is not a number V to multiply $2 \cdot 5$ and to obtain a product divisible by 3, if only V itself is divisible by 3.

The cause of the failure of the primary Leibniz plan was in the confusion of both intuitions: his criterion for UA explored the *comprehensive* representation of terms, i.e., the genus is imagined as part of the species, while his criterion for PA explored the *extensive* one (the species is part of the genus). The prehistory of the arithmetization finished with building a radically different system, in which terms were interpreted by pairs of co-prime numbers[11]—a system that J. Słupecki would prove to be an adequate model of syllogistic ([Słu 48]; see also [Luk 57, § 34]).

I will show that Leibniz's primary plan was realizable in two variants at that. Two different yet not mixed variants! So it will be shown that the more complicated system of pairs has not been necessary.

The first to use one of the two correct interpretations (i.e., Leibniz's criterion for PA together with the inverse of his criterion for UA) was Słupecki [Słu 48]. He applied this model to prove the consistency of the axioms of syllogistic. He did not, however, prove its adequacy. The first and, as far as I know, the only attempt to attack the problem of adequacy of this interpretation was undertaken by S. Vieru [Vie 70]. However, his proof does not work: he has shown the interpretation of syllogistic in terms of divisors to be correct with respect to the theory of classes but he does not show how class operations can be translated in arithmetic. So his statement that "the arithmetical interpretation proposed by Słupecki is an isomorphic model of syllogistic" is not proved. Other methods of arithmetization will not be referred to here since they are not connected with Leibniz's ideas.

[11] See, e.g., *Regulæex quibus de bonitate consequentiarum formisque et modis syllogismorum categoricorum judicari potest, per numeros* ([Log, pp. 77–84]; [Ger, pp. 217–227]; [Rus, pp. 538–546]).

2 Arithmetizations of the Traditional Syllogistic

I will treat the Aristotelian syllogistic in the style that became canonical after Lukasiewicz's celebrated book [Luk 57]. For this purpose the language of the classical propositional calculus is extended by *term variables* (for short, *terms*) $t_1, t_2, \ldots$ together with two binary *term relations*: $\mathcal{A}$ and $\mathcal{I}$. Syllogistic *atoms* are all formulae of the kind $s\mathcal{A}p$ or $s\mathcal{I}p$ with s and p being terms. A *syllogism* is any propositional formula with all propositional letters replaced by syllogistic atoms.

The standard and the most intuitive semantics of the Aristotelian syllogistic is that in the theory of sets: if S and P are arbitrary *non-empty* sets, $s\mathcal{A}p$ is translated into $S \subseteq P$, $s\mathcal{I}p$ into $S \cap P \neq \emptyset$ (briefly, $S\theta P$), and the formal propositional connectives are replaced with the informal ones (especially, $\wedge$, $\Rightarrow$, $\Leftrightarrow$ will be used in the first rôle, and &, $\rightarrow$, $\leftrightarrow$ in the second). Thus any syllogism is translated into a sentence about non-empty sets. If this sentence is true, i.e., if the expression so obtained is a set-theoretical tautology, the syllogism is said to be *true*. It is *true in a given (non-empty) set* U when any replacement of its terms with (non-empty) subsets of U gives a true sentence. This semantics corresponds to the extensive view on terms, and I call it *Scholastic* following Leibniz. It is characterized by $(\subseteq, \cap \neq \emptyset)$.

Another semantics in the theory of sets is also possible; it corresponds to the comprehensive view on terms and will be named *Leibnizian* being (partially) accepted by him. When a non-empty set U is given, term variables are evaluated by subsets of U *different from* U. If S and P are such sets, $s\mathcal{A}p$ is interpreted as $S \supseteq P$, $s\mathcal{I}p$ as $S \cup P \neq U$ (briefly, $S\Theta P$), and the formal propositional connectives are replaced by informal ones. A syllogism is said to be *true in* U when the sentence obtained after any replacement of all term variables with subsets of U (different from U) is true. The syllogism is *true* when it is true in any set U. This semantics is characterized by $(\supseteq, \cup \neq U)$.

Obviously, both semantics are dual. This is the reason of the following

Fact 3: A syllogism is true in the Scholastic semantics *iff* it is true in the Leibnizian one.

The proof is based on the observation that when the sets S, P satisfy the condition of one of the semantics, their complements $\overline{S}$, $\overline{P}$ satisfy the condition of the other, and $S \subseteq P \leftrightarrow \overline{S} \supseteq \overline{P}$, $S\theta P \leftrightarrow \overline{S}\Theta\overline{P}$.

The following theorem expresses the decidability of syllogistic:

Theorem 1 *A syllogism with* n *term letters is true iff it is true in any set with no more than* 2^n *elements.*

The theorem is similar to the statement of decidability of the monadic predicate calculus. The *only-if* part is obvious. For the *if* part, let us suppose that a syllogism is not true in the Scholastic semantics, and let $T_1, \ldots, T_n$ be some non-empty sets rejecting it. Denote by U their union. Let us identify the elements of U following $x \approx y \leftrightarrow (\forall i)(x \in T_i \leftrightarrow y \in T_i)$. Denote by $|x|$ the equivalence class of x, i.e., $w \in |x| \leftrightarrow w \approx x$. The factorization of U produces no more than 2^n elements. For an arbitrary set $A \subseteq U$, let $A^* = \{|x| / x \in A\}$. It is easy to show that for any A and B, $A \subseteq B \leftrightarrow A^* \subseteq B^*$, $A\theta B \leftrightarrow A^*\theta B^*$. Then we see that syllogistic atoms $t_i\mathcal{A}t_j$ and $t_i\mathcal{I}t_j$, after evaluating terms t_i and t_j by T_i and T_j, respectively, obtain the same truth-values as after evaluating them by T_i^* and T_j^*. That is why the sets $T_1^*, \ldots, T_n^*$ will reject the syllogism too, all of them being non-empty subsets of a set contatining no more than 2^n elements[12]. To obtain the proof for the Leibnizian semantics, the complements of $T_1^*, \ldots, T_n^*$ to U^* may be taken.

[12] The upper bound 2^n can essentially vary depending on syllogistic relations and operations occurring in the concrete system. In our case of syllogistic containing only $\mathcal{A}$ and $\mathcal{I}$, the upper bound is $\frac{1}{2}n(n+1)$; see [Pie 94].

Scholastic arithmetical interpretation. Let $a_1, a_2, \ldots$ denote arbitrary integers *greater than* 1. Given a syllogism, replace $t_i\mathcal{A}t_j$ with $a_i|a_j$ ("a_i is a divisor of a_j"), $t_i\mathcal{I}t_j$ with a new relation a_iGa_j ("a_i and a_j have a common divisor greater than 1", or: g.c.d.$(a_i, a_j) > 1$), and the formal propositional connectives with informal ones. In short, this interpretation is characterized by (|, g.c.d.> 1). Call the syllogism *arithmetically true* (in the *Scholastic sense*) if the sentence so obtained is an arithmetical truth.

Theorem 2 (Adequacy of the Scholastic arithmetical interpretation): *A syllogism is true iff it is arithmetically true in the Scholastic sense.*

Let $\{a_i\}$ denote the set of all divisors of a_i greater than 1. Then $a_i|a_j$ is equivalent to $\{a_i\} \subseteq \{a_j\}$, and a_iGa_j to $\{a_i\}\theta\{a_j\}$. So, if the syllogism is true in the Scholastic semantics of sets, it will be arithmetically true as well. Conversely, if the syllogism is not true in the Scholastic semantics, and contains n term letters, then let $T_1^*, \ldots, T_n^*$ be the rejecting sets described in the proof of Theorem 1. Suppose all elements of U^* are $w_1, \ldots, w_k$ $(k \leq 2^n)$. Take $p_1, \ldots, p_k$ to be different prime numbers. If the elements of T_i^* are $w_{i_1}, \ldots, w_{i_m}$ (remember that $T_i^* \neq \emptyset$), pose $p_{i_1} \cdots p_{i_m} = a_i$. Any of the integers so obtained is greater than 1, and $a_i|a_j \leftrightarrow T_i^* \subseteq T_j^*, a_iGa_j \leftrightarrow T_i^*\theta T_j^*$. Ergo, the integers $a_1, \ldots, a_k$ will make the syllogism arithmetically false.

Note 1. As we see, when the rejecting numbers are building, the First Addendum to Leibniz is automatically accepted because no element of a set can occur twice in it. In such a way, a variant of the Scholastic arithmetical interpretation using integers *without multiple factors* will be adequate to syllogistic, too. Then $\{a_i\}$ in the proof of Theorem 2 may be reduced to the set of all *prime* divisors of a. This variant will be used to build a Scholastic semantics for syllogistic with negative terms. On the other hand, an additional syllogistic law, namely, $(s\mathcal{A}p) \wedge (p\mathcal{A}s) \Rightarrow (s = p)$ will be true according to the semantics so modified. This law does not figure in the traditional syllogistic (and, formally speaking, it is not in its language because it uses a new, third syllogistic relation: $=$). It figures, however, in Leibniz's logic [Eng, p. 237]: "Two terms containing each other and yet equal, I call *coincident* [...]; that is, exactly what is contained in one is also contained in the other"[13].

Note 2. If the empty set is admitted to evaluate terms in the Scholastic semantics, a new syllogistic will be obtained, which differs from the traditional one. For example, some of Aristotle's syllogisms like *Bramalip* stop being true. The law $s\mathcal{I}s$ introduced by Leibniz will not be true, either. The arithmetical interpretation can be modified to serve the syllogistic with empty terms by admitting 1 as their number value together with a sole modification in the proof of theorem 2: 1 has to be added to the list of divisors of a_i being the number corresponding to the empty terms; in the second part of the proof, if $T_i^* = \emptyset$ then pose $a_i = 1$.

The central problem connected with the use of empty terms is in the possibility of concluding a false $p\mathcal{I}s$ from a true $s\mathcal{A}p$, e.g., 'Some man is smiling' from 'Any smiling [thing] is a man' (an example by Leibniz). Evidently, this conclusion will not be true if no one is smiling and, in this way, the classical *conversio per accidens* would be violated. Leibniz insists on the reality of any term, and denies the existence of empty terms. The solution he proposed in the manuscript *Difficultates quædam logicæ* ([Phi, v. VII, pp. 211–217]; [Rus, pp. 623–631]) is in a hypothetical interpretation of UA- and PA-propositions. Then $s\mathcal{A}p$, being interpreted as 'Any thing supposed to be smiling is a man', does not contradict $p\mathcal{I}s$ 'Some man is supposed to be smiling'.

[13] "Duos Terminos sese continentes et nihilominus æquales voco *Coincidentes* [...], id est tantundem continentur in una, quantum in altera" (*Elementa Calculi* [Log, p. 52]; [Ger, p. 184]; [Rus, p. 517]).

Note 3. As we can see, it is impossible to avoid the theorem of decidability and to use directly the isomorphism between division and set-inclusion, as it was attempted in [Vie 70]: the rejecting sets may be infinite, and then no numbers will be constructed.

Leibnizian arithmetical interpretation. From the mathematical point of view, this procedure is dual to the previous one and therefore is not essentially different. But from a philosophical point of view, the difference is considerable. The Scholastic procedure does not require any additional numbers for checking a syllogism besides the ones corresponding to the terms. On the contrary, for evaluating $s\mathcal{I}p$ according to the Leibnizian semantics a new integer will have to be introduced, and this integer will correspond to the Universe of reasoning. The cause of that asymmetry is in the asymmetry of the integer sequence: it contains a least number (namely, 1) but not a greatest one. Respectively, in the intuitive semantics of syllogistic there is a smallest set (namely, the empty one), and although this set can not be used for evaluating, it exists, so to say, by its very nature. On the contrary, a "largest" set does not need to exist initially.

Let u (the *Universe number*) be an arbitrary integer greater than 1, and let $a_1, a_2, \ldots$ be arbitrary its proper divisors, i.e., $a_i < u$ for any i (however, $a_i = 1$ is permitted). Replace $t_i\mathcal{A}t_j$ by a relation a_i/a_j ("a_i is divisible by a_j"; this notation is to remind of the *division* a_i/a_j), and $t_i\mathcal{I}t_j$ by a relation a_iLa_j ("the least common multiple of a_i and a_j is less than u", or: "there is a *prime* divisor of u dividing neither a_i nor a_j"). In short, the Leibnizian arithmetical interpretation is characterized by ($/$, l.c.m.$< u$). Finally, formal propositional connectives are replaced with their informal analogues. The syllogism is said to be *arithmetically true* (in the *Leibnizian sense*) *with respect to* u if the sentence so obtained is an arithmetical truth. The syllogism is *arithmetically true* (in the *Leibnizian sense*) if it is arithmetically true in the same sense with respect to any $u > 1$. So the Universe number actually occurs in the truth-value of the syllogism.

Theorem 3 (Adequacy of the Leibnizian arithmetical interpretation): *A syllogism is true iff it is arithmetically true in the Leibnizian sense.*

For proving Theorem 3 a bridge will be thrown to Theorem 2. Let a syllogism be arithmetically true in the Scholastic sense, and let a and b be arbitrary integers satisfying the conditions of the Leibniz procedure with respect to a Universe number u. Then $\frac{u}{a}$ and $\frac{u}{b}$ are integers satisfying the conditions of the Scholastic interpretation. Further, $\frac{u}{a}/\frac{u}{b} \leftrightarrow a|b$, and $\frac{u}{a}G\frac{u}{b} \leftrightarrow aLb$. So both evaluations—with a, b (in the Leibnizian interpretation), and with $\frac{u}{a}, \frac{u}{b}$ (in the Scholastic interpretation)—give equal logical values to $s\mathcal{A}p$ as well as to $s\mathcal{I}p$. Ergo, since the syllogism is arithmetically true in the Scholastic sense, it will be arithmetically true in the Leibnizian sense, too.

Conversely, let the syllogism be arithmetically true in the Leibnizian sense, and let $a_1, \ldots, a_n$ be arbitrary integers satisfying the Scholastic conditions (i.e., $a_i > 1$ for any i). Take $u = \text{l.c.m.}(a_1, \ldots, a_n)$. Then all $\frac{u}{a_i}$ satisfy the conditions of the Leibnizian interpretation, and $a_i|a_j \leftrightarrow \frac{u}{a_i}/\frac{u}{a_j}$, $a_iLa_j \leftrightarrow \frac{u}{a_i}G\frac{u}{a_j}$. Because the sentence obtained after evaluating the syllogism by $\frac{u}{a_1}, \ldots, \frac{u}{a_n}$ was supposed to be true, the sentence obtained after evaluating it by $a_1, \ldots, a_n$ will be true, too. Ergo, the syllogism is arithmetically true in the Scholastic sense.

Note 4. A remark analogous to **Note 2** may be made concerning empty terms: they have to be evaluated by the Universal number u. This is not surprising bearing in mind that, according to the intuition of the Leibnizian semantics, terms are collections of properties and then any non-existing object possesses *all* possible properties.

Note 5. In practice, the Universe number needed in Leibniz's arithmetical interpretation may be introduced *post factum*, after characteristic numbers had been attributed to terms, simply taking u to be their l.c.m. (of course, when the quantity of terms is finite). Nevertheless, as we see, the necessity of using an additional number not occurring initially in the evaluation of terms is a hard complication of the arithmetization procedure based on the Leibniz rule for UA-propositions. Besides, the relation l.c.m.$< u$ has no transparent arithmetical content. These facts have possibly been additional causes for the failure of Leibniz's primary plan. Fortunately, the introduction of a Universe of reasoning in parallel with a Universe number will serve us in investigating syllogisms with negative terms.

3 Arithmetizations of Syllogistic with Negative Terms

In contrast to the traditional syllogistic, there are many systems containing term negation, and none of them is canonized. To compare different semantics and their correspondent axiomatics is not our aim; see, e.g., [Pri 62] and [She 56]. Here I will examine only the system that is nearest to Leibniz's view. A big facilitation is the identical interpretation of term negation according to the extensive intuition as well as to the comprehensive one.

Expand the language of syllogistic by adding an operation of *term negation* $-$; then, if t is a term, $-t$ ("non-t") is a term, too. The definition of atoms is modified by permitting s and p to be arbitrary terms in $s\mathcal{A}p$ and in $s\mathcal{I}p$ as well. In both set-theoretical semantics, a *universal set* U is introduced. Terms are evaluated by subsets of U different from $\emptyset$ and U (obviously, U cannot be an one-element set). If a term t is evaluated by a set T, the value of $-t$ is the *complement* of T to U. The rest of the definitions of a true syllogism remains the same. In both arithmetical interpretations, a *Universe number* $u > 1$ *without multiple factors* is introduced together with the following rules: 1) all evaluating integers are divisors of u different from 1 and u (therefore, u cannot be prime); 2) if term t is evaluated by an integer a, then the term $-t$ is evaluated by $\frac{u}{a}$. As we see, if t is produced from a term t_0 by negations (in arbitrary quantity), and the value of t_0 is a, then the value of t is either a or $\frac{u}{a}$.

Because, as I noted, there are no traditional syllogisms with term negation which are obligatorily true, we may only accept some logical schemes met in the Leibniz manuscripts. First of all, some properties of the negation, such as $--t = t$ (or, using only the classical relation $\mathcal{A}$, $(--t)\mathcal{A}t$ and $t\mathcal{A}(--t)$), together with some new syllogisms like $s\mathcal{A}p \Leftrightarrow (-p)\mathcal{A}(-s)$ are obviously true in both set-theoretical semantics. These principles are mentioned by Leibniz as basic ones, indeed: in *Generales Inquisitiones...* ([Log, pp. 356–399]; [Ger, pp. 241–330]; [Rus, pp. 572–616]) they are listed under numbers 96 (170; 189; 198) and 77 (94), respectively. The two properties, being accepted, do not cause any change in either arithmetical interpretation because their arithmetical translations are automatically true. The only change we will need is if we suppose with Leibniz that $(-t)\mathcal{I}t$ is never true: Principle 171-*octavo* is "*A* non *A* non est res" [*Ibid.*]. That is the cause for the First Addendum to Leibniz to become obligatory, and for the Universe number u, to contain no multiple divisors.

Theorem 4 (Adequacy of both Scholastic and Leibnizian arithmetical interpretations of syllogistic with term negation): *A syllogism (possibly with negative terms) is true iff it is arithmetically true in the Scholastic as well as in the Leibnizian sense.*

Before proving this statement, Theorem 1 has to be modified in order to include occurrences of negative terms in atoms. First of all, note that U^*, produced from U, cannot be one-element, because in such a case all elements of U should belong (or not belong) to one and the same class of equivalence, in which case any T_i should be either U or $\emptyset$. So U^* may be used as a Universe. Suppose terms s and p are produced from term variables t_i and t_j, possibly with negations attached to them in some quantities. Suppose further, the values of s and p, after t_i and t_j have been evaluated by rejecting sets T_i and T_j, have become S and P. Then, as it was shown in the proof of Theorem 1, $S \subseteq P \leftrightarrow S^* \subseteq P^*$ and $S\theta P \leftrightarrow S^*\theta P^*$. But taking in consideration that $\overline{T_i^*} = \left(\overline{T_i}\right)^*$ and $\overline{T_j^*} = \left(\overline{T_j}\right)^*$ we see that S^* and P^* are exactly the values of s and p after t_i and t_j have been evaluated by T_i^* and T_j^*, respectively. So the syllogism will be rejected using the sets $T_1^*, \ldots, T_n^*$, too.

Theorem 4 will be proved about the Scholastic arithmetical interpretation first. For proving the *only-if* part suppose a syllogism of n term variables is true in the Scholastic set-theoretical semantics. Let $u > 1$ be a Universe number (i.e., without multiple factors), and let $a_1, ..., a_n$ be arbitrary evaluating numbers (i.e., divisors of u different from 1 and u). If x is an arbitrary integer, denote by $\{x\}$ the set of all its *prime* divisors. The following equality will be useful: $\{\frac{u}{x}\} = \overline{\{x\}}$. Further, take any term t of the syllogism under consideration, and let it be produced from a term variable, say t_i, possibly with negations in some quantity. Suppose the arithmetical value of t_i is a_i, and the value of t obtained according to the Scholastic arithmetical procedure is a. Let A be the set obtained from $\{a_i\}$ after applying *complements* to it in the quantity of the negations; such a set I call *corresponding to the number a*. Then it is easy to see, using the equality cited above (and applying induction over the number of negations), that $\{a\} = A$.

Suppose a second term p is produced from t_j, its arithmetical value obtained from a_j (the value of t_j) is b, and B is the set corresponding to b. Then $\{b\} = B$, and therefore $\{a\} \subseteq \{b\} \leftrightarrow A \subseteq B$. On the other hand, remember $a|b \leftrightarrow \{a\} \subseteq \{b\}$ (Fact 1). In such a way, the main connection between number-evaluation of terms and their set-evaluation is obtained: $a|b \leftrightarrow A \subseteq B$. It shows that the truth-value of a syllogistic atom $s\mathcal{A}p$, after evaluating its term variables by numbers a_i and a_j, coincides with its truth-value after evaluating them by sets $\{a_i\}$ and $\{a_j\}$. Analogously for the truth-value of $s\mathcal{I}p$. So, because the syllogism was supposed to be true in sets, it will be true according to the Scholastic arithmetical interpretation as well.

To prove the *if* part of the theorem concerning the Scholastic arithmetical interpretation, suppose a syllogism is not true in the Scholastic semantics, and let $T_1^*, ..., T_n^*$ be sets rejecting it according to Theorem 1 as just modified. As it was done in the proof of Theorem 2 for the traditional syllogistic, take different prime numbers and from any set $T_i^* = \{w_{i_1}, w_{i_2}, ...\}$ form the number $a_i = p_{i_1}p_{i_2}\ldots$. None of $a_1, ..., a_n$ so obtained contains all prime numbers, because none of $T_1^*, ..., T_n^*$ is U^*. If s and p are two terms produced from term variables t_i and t_j, respectively, and their rejecting set-values are T_i^* and T_j^*, denote by S and P the set-values of s and p; by a_i and a_j the numbers $p_{i_1}p_{i_2}\ldots$ and $p_{j_1}p_{j_2}\ldots$, respectively; by a and b the arithmetical values of s and p calculated from a_i and a_j; by A and B the sets correspondig to the numbers a and b. Then for any prime number, say p_k: $p_k \in A \leftrightarrow w_k \in S$. Further, $\{a\} = A$ was proved, and $p_k|a \leftrightarrow p_k \in \{a\}$. So $p_k|a \leftrightarrow w_k \in S$. In the same manner, $p_k|b \leftrightarrow w_k \in P$. Combining the last two equivalences, $a|b \leftrightarrow S \subseteq P$ is obtained, and, analogously, $aGb \leftrightarrow S\theta P$. Hence the syllogism will be rejected by integers $a_1, ..., a_n$.

The last part of the proof, the passage from the Scholastic arithmetical interpretation to the Leibnizian one, is not a problem.

4 Arithmetizations of Syllogistic with Term Composition

Although term composition, as well as term negation, does not appear in the traditional syllogistic, it plays a central rôle in Leibniz's syllogistic. In his manuscripts even the basic syllogistic relation $\mathcal{A}$ becomes definable by term equality and term composition: $s\mathcal{A}p$ is reduced to $s = sp$ where sp is the Leibniz notation of the composition. I prefer to use the neutral name *composition* (adopted from Leibniz) in order to avoid undesirable specifications because it will be treated as term *intersection* (or, term *conjunction*) in the Scholastic semantics but in the Leibnizian semantics it will be a *union*. Besides, I will not discuss the well-known difficulties connected with introducing term intersection. The main problem is that the traditional requirement for terms to be not-empty is incompatible with the existence of the intersection of any two terms: if term intersection is everywhere defined, $s\mathcal{I}p$ becomes universally valid, and therefore trivial. The last fact follows even if term negation does not occur in the system; this problem was analyzed, e.g., in [She 56] and [Lem 58]. That is why the atavistic rejection of empty or universal terms seems to be irrelevant to syllogistic extended by term composition.

I will consider term composition through both set-theoretical semantics, and will look for arithmetical mechanisms in Leibniz's style that would be adequate. The treatment may be made independent of the presence of term negation. However, if negation does occur together with a composition (it does not matter whether it will be treated as an intersection or as a union), all Boolean term operations will be defined. That is why it will be better to consider the full Boolean algebra straight away. In this connection, it is interesting to quote the following Leibnizian thought anticipating the later Boole investigations [Eng, p. 243]: "If we mix the composition and division of terms in various ways, there arise many results until now untouched by logicians, especially if we add negative and particular propositions besides"[14] (by "division" Leibnis means both conclusions 's is a p' and 's is a q' from 's is a pq').

The composition will be noted by $\circ$. The class of terms now is the smallest class including term variables, and closed under *negation* and *composition*. Given a Universe $U \neq \emptyset$, an evaluation of a term t in U is a set T obtained after replacing all term variables in t with arbitrary subsets of U as well as term operations with their corresponding set-theoretical operations. Namely, in the Scholastic semantics $\circ$ is interpreted as an *intersection*, and in the Leibnizian semantics it is a *union*. Having terms evaluated, the translation of a syllogism into a set-theoretical sentence remains the same as in Section 2.

Further, in both arithmetical interpretations, term variables will be evaluated by arbitrary divisors of a Universe number $u > 1$ *without multiple factors*. The evaluation of the negation remains as it was defined in the previous Section. If terms s_1 and s_2 are evaluated by integers a and b, the composition $s_1 \circ s_2$ will be modelled by g.c.d.(a, b) in the Scholastic arithmetical interpretation, and by l.c.m.(a, b) in the Leibnizian one. Note that in the second case, if a and b have no common divisor, $s_1 \circ s_2$ is represented by their product ab — a fact demonstrating the felicitous choice of Leibniz's notation.

Theorem 1 of the decidability is again valid with the modifications in the definition of evaluation just described. The only addition, needed to prove it for the Scholastic semantics, is $T_i^* \cap T_j^* = (T_i \cap T_j)^*$.

[14] "Miscendo etiam varie compositiones et divisiones terminorum orientur multæ consequentiæ hastenus Logicis intactæ, præsentim si negativas, præterea et particulares propositiones adhibeamus" (*Ad specimen Calculi universalis addenda* [Phi, v. VII, p. 222]; [Ger, p. 135]; [Rus, p. 565]).

Theorem 5 (Adequacy of both Scholastic and Leibnizian arithmetical interpretations of syllogistic with all Boolean term operations): *A syllogism (possibly containing arbitrary Boolean term operations) is true iff it is arithmetically true in the Scholastic as well as in the Leibnizian sense.*

For proving the theorem I will describe only the modifications in the proof of Theorem 4 concerning the Scholastic interpretation. In the *only-if* part, suppose a term t is obtained from term variables $t_1, \ldots, t_k$ by a certain Boolean construction including $-$ and $\circ$. If $a_1, ..., a_k$ are the arithmetical values of that variables, denote by a the value of t, obtained following rules of the Scholastic arithmetical procedure (i.e., using g.c.d. and reciprocals with respect to u). Now, the *set corresponding to the number* a is the set A obtained from $\{a_1\}, ..., \{a_k\}$ *repeating the Boolean construction of the term* t, i.e., replacing $-$ with a complement, and $\circ$ with $\cap$. Then it is easy to check $\{a\} = A$ by induction over the construction of the term using the elementary equalities $\{\frac{u}{x}\} = \overline{\{x\}}$ and $\{\text{g.c.d.}(x, y)\} = \{x\} \cap \{y\}$ for arbitrary numbers x and y.

5 The Arithmetization of Syllogistic from an Algebraic Point of View

Let us begin with the *traditional syllogistic*, i.e., with syllogistic of $\mathcal{A}$ and $\mathcal{I}$. As we know from Słupecki's result [Słu 48], the syllogisms true in the set-theoretical semantics not admitting empty terms (i.e., in the Scholastic semantics), are axiomatized by the following four axioms of Łukasiewicz [Luk 57, § 25]: $s\mathcal{A}s$, $s\mathcal{I}s$, $(m\mathcal{A}p) \wedge (s\mathcal{A}m) \Rightarrow (s\mathcal{A}p)$ (*Barbara*) and $(m\mathcal{A}p) \wedge (m\mathcal{I}s) \Rightarrow (s\mathcal{I}p)$ (*Datisi*), where s, m, and p are arbitrary terms. Of course, the same formulas axiomatize the syllogisms true in the Leibnizian set-theoretical semantics non admitting universal terms. Consider $\mathcal{A}$ and $\mathcal{I}$ as binary relations $\leq$ and ϑ in the class of terms, and take $\wedge$, $\Rightarrow$ in an informal sense. Then $s\mathcal{A}s$ and *Barbara* show that $\leq$ is *reflexive* and *transitive*. In other words, it is a *quasi-ordering relation.* To obtain an algebra, $\leq$ has to be *anti-symmetric*, i.e., an additional axiom $(s\mathcal{A}p) \wedge (p\mathcal{A}s) \Rightarrow (s = p)$ is needed; this is a syllogism we have seen to be true. (Strictly speaking, $=$ here belongs to the extended language of syllogistic—see Note 1—but it is not difficult to carry out such extension formally.) So, the set of terms turns into a *partially ordered structure* [Bir]. Taking in account that $(s\mathcal{A}p) \wedge (p\mathcal{A}s) \Rightarrow (s = p)$ as well as $s\mathcal{A}s$ and $s\mathcal{I}s$ were introduced in syllogistic by Leibniz, we see that all his innovations work!

The second relation, ϑ (produced from $\mathcal{I}$), is *reflexive*, *commutative* and *monotonic* with respect to $\leq$, i.e., $(m \leq p)\&(m\vartheta s) \rightarrow (p\vartheta s)$ (the last two properties follow from *Datisi*). Then, the problem of arithmetization of the Aristotelian syllogistic becomes a problem of embedding a partially ordered set with an additional reflexive, commutative and monotonic relation into arithmetic, i.e., of finding an appropriate integer system that would possess two, more or less natural, relations with the properties just described.

Leibniz's solution was: having all terms replaced with their *numeri characteristici*, to take s/p for $s\mathcal{A}p$, and sGp for $s\mathcal{I}p$. Indeed, / is reflexive, transitive and anti-symmetric; however, the relation $\text{g.c.d.}(s, p) > 1$, being reflexive and commutative, is *not* monotonic with respect to /: see above the rejection of *Darii* and use the same integers. To sum up: if, following Leibniz, sGp (i.e., $\text{g.c.d.}(s, p) > 1$) will be conserved for $s\mathcal{I}p$, then $s|p$ ("s is a divisor of p") will serve for $s\mathcal{A}p$; this arithmetical interpretation was called *Scholastic*. However, if Leibniz's s/p ("s is divisible by p") will be conserved for $s\mathcal{A}p$, then sLp (i.e., the relation $\text{l.c.m.}(s, p) < u$) would have to be used for $s\mathcal{I}p$; this interpretation was called *Leibnizian.* Note that the Scholastic interpretation may be used in the case of infinitely many terms while the Leibnizian one is limited to finite structures. This difference is unimportant if the only purpose is to check syllogisms (because, according to the Decidability Theorem, only a finite quantity of integers is sufficient) but for Leibniz's global plan it is of importance. Namely, there is no reason to refuse the infiniteness (at least potentially) of the class of all elementary notions, and if one wants to attribute (prime) numbers to all of them simultaneously, then the Leibniz procedure will be inappropriate.

Actually, from the very beginning, Leibniz introduces into the syllogistic a *composition* of terms $\circ$. Then some transpositions in the expressive means of the syllogistic language become possible. Let us suppose the term equality to be taken as a sole primary relation; note that equality belongs rather to the general logical relations then to the specific syllogistical ones. If the composition has the suitable properties (namely, being it *idempotent*, *commutative* and *associative*), the relation $\mathcal{A}$ becomes definable by $s\mathcal{A}p \leftrightarrow s = s \circ p$. Further, for introducing the second syllogistic relation, $\mathcal{I}$, the existence of an *extreme* (i.e., a least or greatest) element e is sufficient because $s\mathcal{I}p \leftrightarrow s \circ p \neq e$. In this way, the system $(\mathcal{A}, \mathcal{I})$ is built on the basis $(=, \circ, e)$. Obviously, the extreme element can be expressed, if the system besides $\circ$ contains term *negation* with some minimal properties: then $e = t \circ (-t)$ for some term t. Let us denote the last basis by $(=, \circ, e(\circ, -))$ because e is a function of $\circ$ and $-$.

On the other hand, the negation may appear on the place of $\circ$ in the combination $(\mathcal{A}, \circ)$. Then $\mathcal{I}$ becomes definable by $s\mathcal{I}p \leftrightarrow \neg s\mathcal{A}(-p)$, and syllogistic of $(\mathcal{A}, \mathcal{I}, -)$ is built on the basis $(\mathcal{A}, -)$. If a composition and a negation occur with required axioms including term equality, then the full Boolean syllogistic will be obtained on the basis $(=, \circ, -)$. Especially, four traditional syllogistic relations are defined in the following symmetric manner: $s\mathcal{A}p$ is $s = s \circ p$, $s\mathcal{E}p$ is $s = s \circ (-p)$, $s\mathcal{I}p$ is $s \neq s \circ (-p)$, $s\mathcal{O}p$ is $s \neq s \circ p$. Finally, the *relation* $=$ may be replaced by a *property* "$= 0$" (where 0 is the empty term). Then the following symmetric definitions give all syllogistic relations on the basis $(\circ, -, = 0)$: $s\mathcal{A}p$ is $s \circ (-p) = 0$, $s\mathcal{E}p$ is $s \circ p = 0$, $s\mathcal{I}p$ is $s \circ p \neq 0$, $s\mathcal{O}p$ is $s \circ (-p) \neq 0$. In this way, one may observe a transfer from *syllogistic relations* to *term operations*, and this transfer—algebraic in character—means, in fact, a consecutive elimination of the traditional syllogistic: while only specific term relations ($\mathcal{A}$ and $\mathcal{I}$) appear at the beginning, only term operations ($\circ$ and $-$) together with term equality appear at the end.

All variants sketched here can be found in Leibniz's manuscripts, and many correlations between them could be demonstrated but this is a subject for a separate study.

Theorem 5 establishes an isomorphism between the *finite* Boolean algebra of terms and a suitable Boolean algebra of integers. In the case of the Scholastic interpretation, the second algebra consists of all divisors of an integer u without multiple factors. It has 1 as the least element, u as the greatest one, and g.c.d., l.c.m. and the reciprocals in the rôle of Boolean operations. For first time this algebra—now used as a popular example in many text-books—has been used by S. Bernstein to prove the consistency of his axiomatics of elementary events (which he calls *propositions*) [Ber 17]. As for the nature of the term composition in the Leibnizian semantics: what is it—an intersection, or a union?—the answer is a little surprising. All Leibniz's examples like 'Homo est animal rationale', i.e., 'Man is an animal *and* a rational being'; his terminology in sentences like "diversa prædicata in unum conjungi possunt", and even his notation ab suggest that we treat the composition as a term *conjunction* (respectively, as an *intersection* in the semantics). However, if according to Leibniz's instructions $s\mathcal{A}p$ is categorically $S \supseteq P$, then the semantics is fixed to be the Leibnizian one, and the composition turns into a term *disjunction* (a *union*, respectively)! In this interpretation, the example above means that 'man has all properties of the animals *and* all properties of the rational beings', i.e., what *humanum est* is a union of the animal properties together with the properties of rational beings.

6 Concluding Remarks

Was Leibniz's plan fulfilled? Only halfway, I think.

Leibniz's general plan to mathematize human knowledge may be divided into two parts. The first was in finding a correspondence between notions and appropriate integers, so that the composite notions would obtain the products of integers corresponding to their elements—a problem on which Leibniz wrote with zeal [Eng, p. 224]: "I think that a few selected men could finish the matter in five years"[15]. However, no such correspondence was established either in that *quinquennium* or in the next sixty *quinquennia*. And I doubt whether this problem is solvable at all. In any case, this is rather a subject of philosophy than of logic, and it will be considered in another paper.

The second part of Leibniz's plan was to replace the logical relations between notions with arithmetical relations between integers (assumming logic to coincide with syllogistic). As he showed, one possible solution is to interpret syllogistic relations $\mathcal{A}$, $\mathcal{I}$, etc., by relations between pairs of co-prime numbers (i.e., by relations based on coordinate-wise divisibility). It has been proved in the present article that two simpler interpretations can be applied using the divisibility of *single* numbers, and not of pairs. It is important to accentuate that while the interpretation in pairs of co-prime numbers (in the form given by Leibniz) is applicable only in the traditional syllogistic (using $\mathcal{A}$, $\mathcal{I}$, $\mathcal{E}$, and $\mathcal{O}$), the presented Scholastic interpretation, as well as the Leibnizian one, are applicable in the full Boolean syllogistic including term negation and term conjunction. So, if the problem was to check syllogisms by purely arithmetical means, this problem—333 years old[16]—has been solved, indeed. Moreover, it has been solved in manners suggested out by Leibniz.

It has to be noted that the transfer of syllogistic into arithmetic does not, by itself, provide our reasoning with an algorithmic procedure for checking logical inference: while syllogistic in all its variants is decidable[17], arithmetic is not. Thus, it appears that Leibniz's *Calculemus!* cannot be *ipso facto* an algorithmic panacea, and the problem of syllogistic truthfulness has been transferred from a domain where it had a solution into a domain where a solution does not exist. Of course, Leibniz could not know about decidability of syllogistic and undecidability of arithmetic. Fortunately—and maybe Leibniz's great intuition has been at work here again—the fragment of arithmetic in which he was looking for an interpretation of syllogistic, is decidable: this is arithmetic containing only multiplication. About this result G. Boolos and R. Jeffrey cite T. Scolem [BJ 89, Ch. 21].

If we glance at the literature of the last few years, it will become evident that the interest in syllogistic is not only historical. One cause for this probably is the fact that although syllogistic possesses weaker expressive abilities in comparison with the full predicate calculus, the former, unlike the later, is decidable. Another cause is the proximity of syllogistic to some topics in computer science which have arisen

[15] "Aliquot selectos homines rem intra quinquennium absolvere posse puto" ([Phi, v. VII, p. 187]; [Rus, p. 417).

[16] The birth of the idea of using *numeri characteristici* is believed to date from the period immidiately before 1666, the year of Leibniz's *Dissertatio de Arte Combinatoria*; see his own evidence [Eng, p. 222]: "I necessarily arrived at this remarkable thought, namely that a kind of alphabet of human thoughts can be worked out and that everything can be discovered and judged by a comparision of the letters of this alphabet and an analysis of the words made from them. [...] It happened that as a young man of twenty I had to prepare an academic treatise. So I wrote *Dissertation on the Art of Combinations*, which was published in book from 1666, and in which I laid my remarkable discovery before the public" ([Phi, v. VII, p. 185–186]; [Rus, p. 414]).

[17] A detail investigation of this problem see in [Pie 94].

independently of logic in its proper sense. E.g., such class of contemporary offsprings of syllogistic is connected with Pawlak's *knowledge representation systems*. D. Vakarelov thoroughly has investigated relations between objects defined on the basis of information about them; such systems he calls *property systems* [Vak 98]. His axiomatics of *informational inclusion* $\leq$ and *informational similarity* Σ is seen to be equivalent to Shepherdson's axiomatics of $(\mathcal{A}, \mathcal{I})$-syllogistic admitting empty terms [She 56]. Moreover, Vakarelov showed that $\leq$ and Σ together with *negative similarity* N, which he introduced, are sufficent to express all Boolean informational relations in property systems. Now it is seen that if *informational negation* ($'$) is introduced in the style of object properties, sNp becomes definable as $s \not\leq p'$, and Vakarelov's axiomatics of $(\leq, \Sigma, N)$ becomes expressible in A. Wedberg's axiomatics of syllogistic based on $(\mathcal{A}, ')$ [Wed 48].

A second class of revitalised syllogistics is connected with the so called *set containment inference* (see, e.g., [AP 88]) which is in use in cognitive science, database theory, knowledge representation and other fields of computer science. Finally, a third class of contemporary developments of syllogistic can be found in the theory of *generalized quantifiers* (see, e.g., [Wes 89]; [HR 93]). In such a way, our arithmetization of syllogistic may offer a new semantics for the fields just mentioned.

* * *

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Bibliographical Note. All citations of Leibniz's texts are in the language of the original, i.e., in Latin, with the exception of the fragment from *Nouveaux Essais* which original is in French. Unfortunately, the only English translation at my disposal was [Eng], and I used the material which it offers but the rest of the texts had to be translated by me. A big collection of Leibniz's logical manuscripts translated in English, obviously, is [Lei 66]; however, I had no access to it. Its contents can be seen in A. Church's review in the *Journal of Symbolic Logic*, **33** (1968), pp. 139–140; judging from the Latin titles given in the *Index of Reviews* (*ibid.*, **36** (1971, p. 779), no papers from those essentially used in our article (besides the ones cited from [Eng]) could be found there. Excellent German and Russian translations of Leibniz's logical opuscula are [Ger] and [Rus]; references are made to them. For Leibniz's editions I preferred to use more informative abbreviations: [Log] for Couturat's edition of his logical manuscripts, [Phi] for Gerhardt's edition of his philosophical papers, and [Eng], [Ger], [Rus] depending on the language of translation.

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