Markov type inequalities and extreme zeros of orthogonal polynomials

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By $\mathcal{P}_n$ and $\mathcal{P}_n^{\mathbb{C}}$ we denote the set of real and complex algebraic polynomials of degree not exceeding n. The inequalities of the form

$$\|p'\| \le c_n \|p\|, \quad p \in \mathcal{P}_n \quad \text{or} \quad p \in \mathcal{P}_n^{\mathbb{C}}, \tag{1}$$

which hold for various norms, are called Markov–, or Markov–Bernstein–type inequalities. Andrey Andreevich Markov (1889) settled the classical case of real polynomials and the uniform norm in $[-1,1]$. Precisely, he showed that in this case the Chebyshev polynomial of the first kind, $T_n(x) = \cos n \arccos x$, $x \in [-1,1]$, is the only (up to a constant factor) extremal polynomial and the best, that is, the smallest possible constant c_n is equal to $T_n'(1) = n^2$. Vladimir Andreevich Markov (1892) extended inequality (1) to higher order derivatives, showing that the extremality of the Chebyshev polynomials persists in this case, too. For the intriguing story of the inequality of the Markov brothers and some of its proofs we refer to the survey

A. Yu. Shadrin, Twelve proofs of the Markov inequality, In: *Approximation Theory: a volume dedicated to Borislav Bojanov*, Prof. Marin Drinov Academic Publishing House, Sofia, 2004, pp. 233–298.

For more than 130 years inequalities of Markov type proved to play an important role in Approximation Theory, they have been a challenge for many outstanding mathematicians and subject to various generalizations.

A natural goal is to try finding the sharp Markov constant

$$c_n = \sup\{\|p'\|/\|p\| \,:\, p \in \mathcal{P}_n,\ p \ne 0\}\,.$$

However, this turns out to be a rather difficult task, and so far only in a few cases it has been determined explicitly. Further reasonable objectives are to:

1. Find tight two-sided bounds for the sharp Markov constant
$$\underline{c}_n < c_n < \overline{c}_n\,.$$
2. Determine the correct order γ of the sharp Markov constant,
$$c_n = \mathcal{O}(n^\gamma)\,, \quad n \to \infty\,.$$
3. Find the asymptotic Markov constant $c = \lim\limits_{n\to\infty} \dfrac{c_n}{n^\gamma}$.

When one considers the norm in a Hilbert space, the sharp Markov constant c_n admits a simple characterization: it is the largest singular value of a certain matrix. Despite this fact, even in L^2–norms induced by the weight functions of Jacobi ($w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $x \in [-1,1]$, $\alpha, \beta > -1$), Laguerre ($w_\alpha(x) = x^\alpha\, e^{-x}$, $x \in (0,\infty)$) and Hermite $w_H(x) = e^{-x^2}$, $x \in (-\infty,\infty)$), the sharp Markov constants is known only in in the Hermite case (which is a trivial one) and in the classical Laguerre case $w(x) = e^{-x}$ (P. Turán (1960)).

The talk is centered around the Markov inequalities in the L^2 norms induced by the Laguerre and Gegenbauer weight functions, and includes:

- A brief account on the known results on the Markov inequalities with the Laguerre and Gegenbauer weight functions;
- Techniques for derivation of two-sided estimates for the sharp Markov constants in L^2-norms;
- Recent results on the the Markov inequalities with the Laguerre and Gegenbauer weight functions. Our two-sided estimates identify the sharp Markov constant, roughly, within a factor of $\sqrt{6} < 2.5$ in the Laguerre case, and a factor of 2 in the Gegenbauer case;
- A class of extremal problems in a Hilbert space. A relation between the sharp Markov constant with the Laguerre and Gegenbauer weight function and the extreme zeros of orthogonal polynomials;
- The Euler-Rayleygh method as a tool for derivation of two-sided estimates for the extreme zeros of orthogonal polynomials, and thereby of the sharp Markov constants;
- A discrete weighted Markov-Bernstein inequality and the extreme zeros of orthgonal polynomials.

The results are obtained in collaboration with Alexei Shadrin (Cambridge University, UK), Dimitar K. Dimitrov (State University of Sao Paolo, Brazil), my colleague Rumen Uluchev (Sofia University) and my former PhD student Dragomir Aleksov.