R E P O R T

on a Thesis for awarding the degree "Doctor of Sciences"

Title: Extremal Problems in Euclidean Combinatrial Geometry

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Scientific field: 4. Natural sciences, Mathematics and Informatics

Professional field: 4.5. Mathematics

Overview

This thesis is devoted to several subjects from combinatorial geometry. The goal is to present new characterization and structure results for several classes of problems. It summarizes the research by the author in the last decade or so. My general impression is that the author is very well acquainted with the state of the art and the most recent results in the problems that are in the focus of his thesis. All the treated problems are considered in the field as important theoretically and have gained considerable attention in the past decades. The author demonstrates deep knowledge of his field of research and capacity to apply his knowledge to the solution of important problems.

Description of the results

This thesis amounts 148 pages of text and consists of an introduction, six chapters, and a list of references including 128 titles. The first chapter is introductory. The author describes the main goals of his thesis, as well as the main objects under investigation. He describes brielfy the main results and presents a list of publications on which the thesis is based. The original results are contained in the next six chapters numbered 2 through 7.

In Chapter 2 the author deals with the Euclidean Steiner tree problem. This amounts to two problems:

(1) For a finite set P of points in $\mathbb{R}^d$ find a connected set $\mathcal{S}t$ of minimal length connecting P.

(2) For a given compact set $\mathcal{A} \subset \mathbb{R}^d$ find a set $\mathcal{S}t$ of minimal length such that $\mathcal{S}t \cup \mathcal{A}$ is connected.

The main result in this chapter is that the set of n-point configurations for which the solution to the planar Steiner problem is not unique has Hausdorff dimension at most $2n-1$. (Theorem 2.1.1) The author studies configurations of n points in $\mathbb{R}^d$ for which the solution is unique. For the second problem the author constructs several self-similar indecomposable examples.

Chapter 3 deals with the so-called Gilbert-Steiner problem which asks for the minimum of a special expression

$$\sum_{\{x,y\}\in E} C(|m(x,y)|)\cdot\rho(x,y),$$

called the Gilbert functional of the (μ^+,μ^-)-flow, for a special cost function $C(x) = x^p$, $p\in(0,1)$. The author proves conditions on the cost function under which all branching points in a planar solution have degree 3. Theorem 3.1.2 says that there are no branching points of degree 4 or more. The main result in this chapter is Theorem 3.3.1. According to this theorem, for any two measures μ^+,μ^- with a finite support and admissible cost function C, the eixstence of a branching point of degree at least 4 in a (μ^+,μ^-)-flow implies the existence of a (μ^+,μ^-) flow with strictly smaller value of the Gilbert functional.

In Chapter 4 the problem of minimizing the maximal distance to a given compact set is considered. More precisely, given a compact set $M\subset\mathbb{R}^d$ and a constant $l>0$, find a compact set Σ of length not exceeding l that minimzes the maximum distance from some point y from M. This chapter is the largest in the thesis and is subdivided in eight sections. The main result is Theorem 4.3.2. which describes the maximal distance minimizer for a rather large class of convex closed curves, i.e curves with a radius of the curvature at least $5r$ at every point. In this case an arbitrary minimizer for M is a union of an arc of M_r and two segments tangent to M_r at the end of the arc (this figure is called a horseshoe).

Chapter 5 is devoted to the problem of determining the independence number of certain Johnson-type graphs. These graphs, denoted by $J_\pm(d,k,t)$, have as vertices all vectors from $\{0,1,-1\}^d$ with exactly k non-zero components; two vertices are adjacent iff their dot product is 0. Related to this graphs are the graphs $K_\pm(d,k,t)$ with the same vertex set and adjacency between vertices with dot product at most t.

In this chapter the author determines the independence number (size of the largest anti-clique) of $J_\pm(d,k,t)$ for odd negative t and when d is larger than some constant which depends on k and t.

The main results of this chapter are contained in Theorems 5.1.7–5.1.9 from [15] and Theorems 5.1.10–5.1.12 from [23].

The first group of theorems are exact values on the independence number $\alpha[J_\pm(d,k,-1)]$, $\alpha[J_\pm(d,k,t)]$, d - odd, negative, and $\alpha[J_\pm(d,k,0)]$ for large values of d. Theorem 5.1.7 claims that for $d>k2^{k+1}$ the independence number of $J_\pm(d,k,-1)$ is given by the binomial coefficient $\binom{d}{k}$. Theorem 5.1.8 generalizes this result replacing -1 by an arbitrary odd negative t:

$$\alpha[J_\pm(d,k,t)] = S(k,|t|-1)\binom{d}{k},$$

where $S(d,D)$ is a certain sum of binomial coefficients. Finally, Theorem 5.1.9 states that for $d>\frac{9}{2}k^3 2^k$

$$\alpha[J_\pm(d,k,0)] = 2\binom{d-1}{k-1}$$

The proofs of these results rely on the so-called Katona averaging method.

The second group of results deals with the chromatic number of the graphs $J_\pm(d,2,-1)$, $J_\pm(d,3,-1)$, $J_\pm(d,3,-2)$. Theorems 5.1.9. and 5.1.10 give the asymptotic bounds:

$$\log_2 d \le \chi(J_\pm[d,2,-1]) \le \log_2 d + \left(\frac{1}{2} + o(1)\right) \log 2 \log_2 d,$$

and

$$c \log_2 d \le \chi(J_\pm[d,3,-1]) \le C \log_2 d,$$

where c and C are constants. Theorem 5.1.12 gives a lower and a upper bound on the chromatic number of the graphs $J_\pm(d,3,-2)$:

$$\lceil \log_2 \lceil \log_2 d \rceil \rceil \le \chi(J_\pm[d,3,-2]) \le 4\lceil \log_2 \lceil \log_2 d \rceil \rceil + 6.$$

Chapter 6 is devoted to the problem of determining the chromatic number of the 2-dimensional spheres $S^2(r)$. A coloring of such a sphere is considered to be proper if there is no monochromatic pair of points at distance 1. The interesting cases arise when $r > 1/2$. It is relatively easy to see that in this case $\chi(S^2(r)) \ge 3$. To see this one has simply to take a cicle on circle of radius $r_1 < r$ that is of odd length and that consists of points at distance 1. In an earlier paper G. Simmons proved that for $r \ge \sqrt{3}/3$ a better estimate holds, namely $\chi(S^2(r)) \ge 4$, and conjectured that this is true for all spheres of radius $r > 1/2$.

The main result in this chapter is the proof of Simmons' conjecture: for every coloring of a 2-dimensional sphere of radius $> 1/2$ in three colors there exists a pair of points at distance 1 that are of the same color. The chapter contains also the nice corollary that for $1/2 < r \le \sqrt{3 - \sqrt{3}}3$ the chromatic number $\chi(S^2(r))$ is exactly 4.

In chapter 7 the author studies the chromatic numbers of the so-called 3-dimensional slices $\mathrm{Slice}(3,6,\varepsilon)$, where $\mathrm{Slice}(d,k,\varepsilon)$ is defined as $\mathbb{R}^d \times [0,\varepsilon]^k$. These chromatic numbers might be useful in determining the chromatic number of the real d-space becuse of the obvious inequalities:

$$\chi(\mathbb{R}^d) \le \chi(\mathrm{Slice}(d,k,\varepsilon)) \le \chi(\mathbb{R}^{d+k}).$$

The main results here are contained in Theorems 7.1.2 and 7.1.3. In the first, it is proved that there exists a postive ε_0 such that for every $\varepsilon < \varepsilon_0$

$$10 \le \chi(\mathbb{R}^3 \times [0,\varepsilon]^6).$$

The second theorem states that every proper coloring of $\mathbb{R}^d$ in a finite number of colors contains a point from a regular simplex $T \subset \mathbb{R}^d$ with edge length $a = \sqrt{2d(d+1)}$ belonging to the closures of at least $d+1$ colors.

This chapter contains also the interesting corollary $\chi_\varepsilon(\mathbb{R}^3) \ge 10$, where $\chi_\varepsilon(\mathbb{R}^d)$ for the minimal number of colors for which there is a coloring of $\mathbb{R}^d$ without a pair of vertices at distance which is in the interval $[1, 1+\varepsilon]$.

Remarks and comments

I have the following remarks, questions and comments related to this thesis:

(1) The thesis is nicely and carefully written. The exposition is lucid and clear. The text is accompanied with more than 50 pictures that clarify immensly the argumentation in the proofs.

(2) The numbering of the statements – Lemmata, Theorems, Propositions, Remarks – is a bit confusing. For instance, in the text we have Lemma 2.4.1, Definition 2.4.1, Theorem 2.4.1, and Remark 2.4.1. Of course, this is a matter of taste, but I personally would favor a unified numbering of the statements within each chapter.

(3) It appears to me that the lower bound in Theorem 5.1.10. (p.101) is $\log_2 d$. (This inconsistency is probably due to the change of notation.)

(4) All chapters close up with ea list of open problems that mark possible directions for future research.

All the above remarks are insignificant mathematically and do not change my very good impression of the deep research carried out by the author.

Publications related to the thesis

The results of this thesis are published in 10 papers. Eight of the the papers are published or accepted for publication in mathematical journals while two of them are submitted and are expected to appear in near future. Five of the published papers are in journals with an impact factor, two of them are in a journal with an SJR, and one in a journal refereed in ZBl MATH, as follows:

- Optimization and Calculus of Variations (Q2. IF 1.295) – 1 paper
- Fractal and Fractional (Q1, IF 5.4) - 1 paper
- Bulletin of the Brazilean Math. Society (Q3, IF 0.7) – 1 paper
- Discrete and Computational Geometry (Q3, IF 0.8) – 1 paper
- Zapiski Nauchnyh Seminarov POMI (SJR 0.314, Q3) – 2 papers
- International Mathematics Research Notices (Q2, IF 1.000) – 1 paper
- St. Petersburg Math. Journal – 1 paper (submitted)
- Pure and Applied Fuctional Analysis – 1 paper (to appear)
- Serdica Math. Journal – 1 paper (submitted)

In one publication Danila Cherkashin is the only author. In six papers he has one coauthor, and in three – three coauthors. I accept that in all joint publications the contribution of Dr. Cherkashin is equal to that of the other coauthors.

The candidate presents a list of 24 citations of his results in papers that are published in refereed journals. I accept that his results are well-known and highly valued in the professional community.

The presented publications and citations exceed the minimal national requirements as described in the corresponding documents. No plagiarism has been detected in the scientific papers submitted for the competetion.

Author's summary

The author's summary is made according to the regulations and reflects properly the main results and contributions of this thesis.

Conclusion

This thesis is focused on problems from combinatorial geometry. The obtained results mark a considerable progress in these field of mathematics.

I am deeply convinced that the presented thesis "Extremal Problems in Euclidean Combinatrial Geometry" by Danila Dmitrievich Cherkashin contains results that are an original contribution to the field of combinatorial geometry. The candidate demonstrates deep knowledge of his field and the capacity to develop it in a new and important way. With this, he meets the national requirements prescribed by the law and the specific regulations of the BAS and the IMI-BAS for the professional field 4.5 Mathematics. I assess **positively** the presented thesis and recommend that this panel awards **Danila Cherkashin** the scientific degree "Doctor of Sciences" in the scientific field 4. Natural sciences, mathematics and informatics, professional field 4.5 "Mathematics".

Sofia, 28.04.2024 Member of the Scientific Panel:

(Prof. Ivan Landjev)