Institute of Mathematics and Informatics
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APPLICATION OF STOCHASTIC AND OPTIMIZATION METHODS FOR RISK MANAGEMENT AND PRICING OF FINANCIAL INSTRUMENTS

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ABSTRACT OF THESIS

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Preface

The Market Risk holds a special rank among the financial risks since it is located at the cross-road of both exogenous and endogenous factors. It is worth mentioning the role of the supervisory entities among the exogenous factors and the interactions of various risk factors on the contemporary financial markets. An example of endogenous factor is the risk measure that the market participant applies and the interpretation given to the value of the risk measure. The market participants vary in their freedom to select the risk measure. The more the market is regulated, the less the freedom. Hence, the exogenous and the endogenous factors intertwingle.

Which risk measure to apply is a choice that depends where we put our focus. If we are interested in a single dot on the left-hand tail of the returns distribution or if we must comply with financial regulations, then we select the Value-at-Risk (VaR) as the risk measure. In case we need information how the left-hand tail behaves beyond a certain threshold and we do not need to back-test the performance of the risk measure, then we pick up the Expected Shortfall (ES). The Expectile Risk Measure (ERM) is selected when the risk measure is expected to bring information about both tails simultaneously and when we need a risk measure that is both coherent and back-testable. The Entropic VaR ($EVaR$) is the preferred risk measure when we have to identify the upper VaR bound at relatively low computations costs and the risk measure must be coherent. The ever enriching family of risk measures comprises members that are designed to capture the particular needs of the market participants.

But selecting the risk measure is not sufficient to mitigate the market risk. The market participant operates on a particular market that is marked with certain patterns. Hence, the risk management toolkit has to adequately reflect the stylized facts of that market. Here, the models for describing the returns dynamics enter the game. A multitude of stochastic processes might be leveraged to produce the necessary price evolution. Most importantly, the financial maturity supposes an optimal mix of returns model and risk measure, for example the underlying asset price follows the Heston model while the market risk is measured via the ERM.

The plan of the dissertation is as follows. The motivation of this study is presented in Chapter 1. Chapter 2 describes the main elements of the risk measures discussed and the stochastic processes applied. The models used to describe the log-returns dynamics are explored in Chapter 3. Currently the supervisors require the banks to apply two quantile-based risk measures (the VaR and the ES) that are discussed in Chapter 4. Chapter 5 presents the ERM as a risk measure that remedies the flaws of the VaR and the ES since it is both coherent and elicitable. When we use the Heston model we deal with an unobservable parameter (the initial volatility). Chapter 6 is dedicated to the way we proceed with the model when this parameter is not measurable. During the recent years we witness an increasing number of risk measures designed to meet specific requirements. The Entropic VaR has been created with the purpose to indicate the upper VaR bound at relatively low computational costs. It is presented in Chapter 7. The dissertation combines theoretical and practical aspects of the market risk measurement. In Chapter 8 we empirically challenge the theoretical topics discussed in previous chapters. The constitution of new stylized facts requires the elaboration of the supporting theoretical models. Chapter 9 discusses the Rough Volatility as a feature of the financial markets of the last decade. We conclude by Chapter 10.

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Chapter 1

Introduction

1.1 Aim of the dissertation

The dissertation aims at exploring the application of sophisticated market risk measures to asset log-returns models that capture various patterns of the market realities. At the background of the dissertation are the traditionally applied risk measures like the VaR and the ES, and the pros/cons of these measures that we witness during the last decades. The dissertation contributes to the current quest of risk measures that meet the augmenting business requirements. Recently the ERM and the $EVaR$ drew the attention of the academic community thanks to some advantages, like combining the coherence with the elicitability (in the ERM case) or representing the upper VaR bound (in the $EVaR$ case).

To achieve its goals, the dissertation introduces several theoretical innovations which are empirically challenged via S&P500 index data for a large time period. These data are used to calibrate five stochastic models.

1.2 Actuality of the topic

Risk Measures appear on the agenda of financial mathematicians mainly for two reasons: providing capital buffers against crises and optimizing portfolio via minimizing its market risk exposure. Regulatory entities issue obligatory for the credit institutions recommendation regarding the capital reserves based on regulator's acceptance set. Risk managers have to apply the risk measure that meets the regulatory requirements.

On the eve of the presidential elections in the USA by the end of 2024, the US supervisors interrupted the undergoing round of introducing new requirements. In other words, the dissertation appears in a moment when dominates the uncertainty which direction shall the supervisors take as a next step.

1.3 Methodology

The dissertation follows the below methodology:

1. The future value of a financial position is presented as a random variable.

2. We impose the requirement that a risk measure must be coherent and/or elicitable (i.e. back-testable).

3. We analyze whether the sophisticated risk measures (the ERM and the $EVaR$) outperform the currently dominating risk measures (the quantile VaR and the ES).

4. We compare the capabilities of 5 stochastic models to capture stylized facts of the financial markets. The usage of such models secures sufficient observations at the tails of the returns distributions which result can't be reached if we used empirical data only.

5. We consider the S&P500 index to be driven by a stochastic process instead of investigating its statistical properties from the empirical data.

6. We calibrate stochastic models to historical data.

7. We apply Fourier transform to the characteristic functions of the models since no probability density function is available in closed form.

8. We replace the standard Brownian Motion as the stochasticity driver of the models by a fractional Brownian Motion, to face the rough volatility as a contemporary stylized fact.

9. We run linear regression to the log-returns, to derive the value of the Heston-index and thus verify whether the financial markets nowadays experience a Rough Volatility.

10. We base the $EVaR$ on the Moment Generating Function (the MGF), we derive the MGF for the five models discussed, and we identify the diapason where the MGF is defined.

11. We treat the unobservable initial volatility with its value integrated over its stationary Gamma distribution.

1.4 Original findings

The dissertation contains several novelties:

1. The considered risk measures are examined under the assumption that the studied object (asset, stocks, commodity, index) is driven by a stochastic model.

2. the ES and the ERM are derived via the usage of truncated expectation;

3. determined is the diapason where the MGF is defined for the Heston returns averaged w.r.t. the initial volatility;

4. derived are the $EVaR$ equations and proved are the conditions for the existence and the uniqueness of the solutions;

5. performed is a detailed empirical research based on historical S&P500 index data and the results are analyzed per risk measure and per stochastic model;

6. we found that the S&P500 volatility is rough since the Hurst index value of its returns series is less than $\frac{1}{2}$, the profile of this value varies within a certain range and moves in packages.

7. we confirmed the conclusions of Gatheral[(6)] that the relationship between the scaling factor and the Hurst index value is linear. However, we found that this relationship is not linear for the STOXX50E index, the FTSE index, and the KSE index.

Chapter 2

Main results

Chapter 2 of the dissertation presents the Risk Measure and the stochastic processes used.

2.1 Risk Measure

The dissertation defines the risk measure in the following way:

Definition 2.1. *The risk measure* ρ *of position* ζ *is the mapping* $\rho(\zeta) : \mathcal{G} \to \mathbb{R}$*, where* $\mathcal{G}$ *the set of all risks.*

We use the definition of Artzner et al.[2] for the coherence of a risk measure:

Definition 2.2. *Risk measure* $\rho(\cdot)$ *that satisfies the conditions below is coherent.*

1. *Translation Invariance: If* ζ *is a random variable and* $x \in \mathbb{R}$*, then* $\rho(\zeta + x) = \rho(\zeta) - x$.

2. *Subadditivity: If* ζ_1 *and* ζ_2 *are random variables, then* $\rho(\zeta_1 + \zeta_2) \leq \rho(\zeta_1) + \rho(\zeta_2)$.

3. *Monotonicity: If* ζ_1 *and* ζ_2 *are random variables and for all sample paths* $\zeta_1 \leq \zeta_2$*, then* $\rho(\zeta_1) \geq \rho(\zeta_2)$.

4. *Positive homogeneity: If* ζ *is a random variable and* c *is a positive constant, then* $\rho(c\zeta) = c\rho(\zeta)$.

We leverage Definition 2.2 of Föllmer and Schied[5] for the acceptance set $\mathcal{A}_\rho$ of a risk measure $\rho(\zeta)$:

Definition 2.3. *The Acceptance Set $\mathcal{A}_\rho$ for risk measure $\rho(\zeta)$ includes the set of positions for which*

$$\mathcal{A}_\rho = \{\zeta \in \mathcal{G} \mid \rho(\zeta) \leq 0\}. \tag{2.1}$$

This definition allows the risk measure to have a negative value which is not the case of the probability measure. Thus we conclude that the risk measure is different from the probability measure. The acceptance set is handy since we can derive the risk measure from it:

Proposition 2.1. *The risk measure $\rho(\zeta)$ corresponding to the acceptance set $\mathcal{A}$ is*

$$\rho(\zeta) = \inf\{m \in \mathbb{R} | (m + \zeta) \in \mathcal{A}\}, \tag{2.2}$$

where m is the minimum amount of risk-free investment (default-free zero coupon bond for example - see Zaevski et al.[15]) which if added to the position ζ makes the joint position acceptable for the supervisor.

Proof: See the proof of Proposition 4.6 of Föllmer and Schied[5]. □

The dissertation references to the Dual Representation of a coherent risk measure:

Proposition 2.2. *A functional $\rho(\zeta) : \mathcal{G} \to \mathbb{R}$ is a coherent risk measure if and only if there exists a subset $\mathcal{S} \subset \mathcal{Q}$ such that*

$$\rho(\zeta) = \sup_{\mathbb{Q} \in \mathcal{S}} \mathbb{E}_{\mathbb{Q}}[-\zeta], \zeta \in \mathcal{G}, \tag{2.3}$$

where $\mathcal{Q}$ is the set of the probability measures $\mathbb{Q}$ that are equivalent to the real world probability measure $\mathbb{P}$.
Proof: See the proof of Proposition 4.15 of Föllmer and Schied[5]. □

2.2 Models for stock log-returns

Chapter 3 presents 5 models used to describe the dynamics of the stock log-returns. These models use the following stochastic processes: Standard Brownian Motion, Fractional Brownian Motion, Lévy Process namely Poisson Process and Compound Poisson Process, and Tempered Stable Process.

2.3 Quantile-based Risk Measures

Chapter 4 studies the traditional risk measures that are based on the quantile.

2.3.1 Value at Risk

We compute the VaR for the log-returns distribution. We apply formula (6.5) of Rachev et al.[10] to formulate the VaR:

$$VaR\left(\epsilon;\zeta\right) = -\inf\left\{x \in \mathbb{R} : \mathbb{P}\left(\zeta < x\right) > \epsilon\right\} \equiv -F^{-1}\left(\epsilon\right) \equiv -q_{\zeta}\left(\epsilon\right), \quad (2.4)$$

where ζ is the log-returns, the significance level $\epsilon \in (0,1)$, F^{-1} is the inverted cumulative distribution function, $\mathbb{P}$ is the real world probability measure, and q - the quantile function. We define the VaR as the opposite of the quantile function. It has the following properties:

- coherence: The VaR is not a coherent risk measure, since it is not subadditive, i.e. it does not correctly reflect the consequences of portfolio diversification. The incoherence entails the lack of dual representation;
- elicitability: The VaR is elicitable since there is a scoring function (see Thomson[11]).
- A drawback of the VaR is the incapacity to capture the tail behaviour beyond the significance level.

2.3.2 Expected Shortfall

The ES (also known as the averaged VaR or the conditional VaR) measures the market risk by averaging the VaR beyond the significance level. The dissertation uses the common definition for the expected shortfall

$$ES(\epsilon;\zeta) = \frac{1}{\epsilon}\int\limits_0^{\epsilon} VaR(x;\zeta)dx. \tag{2.5}$$

The dissertation derives the ES through truncated expectations in the following way:

$$ES(\epsilon;\zeta) = \frac{\mathbb{E}\left[(\zeta - q(\epsilon))^-\right]}{\epsilon} - q(\epsilon). \tag{2.6}$$

The ES has the property of:

- coherence: the ES is a coherent risk measure (see Acerbi and Tasche(1) for the argumentation).
- elicitability: The dissertation presents the views of authors who claim the ES is elicitable under certain conditions, as well as references to opinions that the ES is non-elicitable.

2.4 Expectile Risk Measure

Chapter 5 introduces the expectile-based risk measure.

2.4.1 Expectiles

The expectile $\phi(\epsilon;\ \zeta)$ is presented as the solution of a square optimization problem (for details see Newey and Powell(9)):

$$\phi(\epsilon;\ \zeta) = \underset{x\in\mathbb{R}}{\arg\min}\,\mathbb{E}[\epsilon((\zeta - x)^+)^2 + (1-\epsilon)((\zeta - x)^-)^2]. \tag{2.7}$$

The following presentation gives another viewpoint on the expectile:

Proposition 2.3. *(see formula (2) of* **?***)) The expectile is the solution of the equation*

$$\epsilon\mathbb{E}\left[(\zeta - x)^+\right] = (1-\epsilon)\,\mathbb{E}\left[(\zeta - x)^-\right]. \tag{2.8}$$

2.4.2 Introducing the ERM and its properties

The $ERM\,(\epsilon;\ \zeta)$ is defined as the opposite of the expectile $\phi(\epsilon;\ \zeta)$. It is coherent (if $\epsilon \in (0, 1/2]$), and it is elicitable; something more, we consider the ERM as the only risk measure that has the both properties as long as $\epsilon \leq \frac{1}{2}$.

2.4.3 Theoretical results

We present the truncated expectation in the following way:

Proposition 2.4. *(Proposition 3.2 of Zaevski and Nedeltchev[12]) If the constants $a_1 \leq 0 \leq a_2$ satisfy the condition $\mathbb{E}\left[e^{-a_{1,2}\zeta}\right] < \infty$ and $a_1 \leq b_1 < 0 < b_2 \leq a_2$, then*

$$\mathbb{E}\left[(\zeta - x)^-\right] = -\frac{1}{2\pi}\int_{ib_2-\infty}^{ib_2+\infty}\frac{\Psi_\zeta\left(u\right)e^{-ixu}}{u^2}du = -\frac{e^{b_2 x}}{\pi}\int_0^{+\infty}\Re\left(\frac{e^{-ixv}\Psi_\zeta\left(v+ib_2\right)}{\left(b_2-iv\right)^2}\right)dv$$
$$\mathbb{E}\left[(\zeta - x)^+\right] = -\frac{1}{2\pi}\int_{ib_1-\infty}^{ib_1+\infty}\frac{\Psi_\zeta\left(u\right)e^{-ixu}}{u^2}du = -\frac{e^{b_1 x}}{\pi}\int_0^{+\infty}\Re\left(\frac{e^{-ixv}\Psi_\zeta\left(v+ib_1\right)}{\left(b_1-iv\right)^2}\right)dv. \tag{2.9}$$

where $x^- = max(-x, 0)$, $x^+ = max(0, x)$, and Ψ is the characteristic function. We apply the above formula to derive the ES and the ERM:

Theorem 2.1. *(Theorem 3.2 of Zaevski and Nedeltchev[12]) Let $0 < b_2 < a_2$ and $\epsilon < \frac{1}{2}$. Then the Expected Shortfall can be derived as*

$$\begin{aligned} ES\left(\epsilon;\zeta\right) &= -q_\zeta\left(\epsilon\right) - \frac{e^{b_2 q_\zeta(\epsilon)}}{\epsilon\pi}\int_0^{+\infty}\Re\left(\frac{e^{-iq_\zeta(v)v}\Psi_\zeta\left(v+ib_2\right)}{\left(b_2-iv\right)^2}\right)dv \\ &= \frac{\left(1-2\epsilon\right)q_\zeta\left(\epsilon\right) + i\Psi'_\zeta\left(0\right)}{2\epsilon} - \frac{1}{\epsilon\pi}\lim_{\delta\to 0}\int_\delta^{+\infty}\Re\left(\frac{e^{-iq_\zeta(\epsilon)v}\Psi_\zeta\left(v\right)}{v^2}\right)dv. \end{aligned} \tag{2.10}$$

The ϵ-expectile can be obtained as the solution of the following equation w.r.t. the variable x

$$x = \frac{2(1-2\epsilon)}{\pi} \lim_{\delta \to 0} \int_{\delta}^{+\infty} \Re\left(\frac{e^{-ixv}\Psi_{\zeta}(v)}{v^2}\right) dv. \tag{2.11}$$

2.5 Averaging over the volatility

The Heston Model describes the volatility dynamics via the Cox-Ingersoll-Ross process (CIR): $dV_t = \xi(\eta - V_t)dt + \theta\sqrt{V_t}d\tilde{B}_t$, where the Brownian motion $\tilde{B}_t$ is correlated with the Brownian motion of the returns process; ξ is the speed of reversion to the long-term value η and θ is the volatility of the volatility. Chapter 6 suggests how to deal with the initial value of the volatility V_t, i.e. v. The challenge is that v is not an observable variable for the markets; however, it appears in the formula for the characteristic function and for the moment generating function. We suggest to integrate it over the stationary Gamma distribution of the CIR process.

2.5.1 Volatility Integration

We know that the CIR process admits a Gamma stationary distribution with parameters $\alpha = \frac{2\xi}{\theta^2}$ and $\beta = \frac{2\xi\eta}{\theta^2}$. Its density is $f_\gamma(u) = \frac{\alpha^\beta}{\Gamma(\beta)} u^{\beta-1}\exp(-\alpha u) I_{u\geq 0}$, where $\Gamma(\cdot)$ is the Gamma function.

To average the initial volatility v in the characteristic function along the stationary distribution (see Dragulescu and Yakovenko[(4)]), we suggest:

Proposition 2.5. *(Proposition 3.1 of Zaevski and Nedeltchev[(13)])*
There exists a random variable $\widetilde{V}$ with characteristic function

$$\begin{aligned} \widetilde{\Psi}(u) &= \int_0^{\infty} \Psi_{Heston}(u; v) f_\gamma(v)\, dv \\ &= \int_0^{\infty} e^{C(u)+D(u)v} f_\gamma(v)\, dv \\ &= e^{C(u)} \Psi_\gamma(-iD(u)), \end{aligned} \tag{2.12}$$

where $\Psi_\gamma(\cdot)$ is the characteristic function of the Gamma distribution

$$\Psi_\gamma(u) = \left(1 - \frac{iu}{\alpha}\right)^{-\beta} \equiv \left(1 - \frac{i\theta^2 u}{2\xi}\right)^{-\frac{2\xi\eta}{\theta^2}}. \tag{2.13}$$

$$\begin{aligned} C(t,u) &= \mu uit + \frac{\xi\eta}{\theta^2}\left[(\xi - \rho\theta ui + d(u))\, t - 2\log\left(\frac{1 - g(u)\exp(d(u)t)}{1 - g(u)}\right)\right] \\ D(t,u) &= \frac{\xi - \rho\theta ui + d(u)}{\theta^2}\frac{1 - \exp(d(u)t)}{1 - g(u)\exp(d(u)t)} \\ g(u) &= \frac{\xi - \rho\theta ui + d(u)}{\xi - \rho\theta ui - d(u)} \\ d(u) &= \sqrt{(\rho\theta ui - \xi)^2 + \theta^2(ui + u^2)}. \end{aligned} \tag{2.14}$$

Next we focus on the domain $\mathcal{D} \equiv (\widetilde{x}^-, \widetilde{x}^+)$ of the MGF of the random variable $\widetilde{V}$. We prove that $\mathcal{D}$ is a subset of the MGF's domain of the original Heston's log-returns.

Lemma 2.1. *(Lemma 3.2 of Zaevski and Nedeltchev[13]) We have the inclusion* $\mathcal{D} \subset (x^-,\ x^+)$.

And the solution is as follows:

Theorem 2.2. *(Theorem 3.1 of Zaevski and Nedeltchev[13]) The abscissas of convergence* $\widetilde{x}^-$ *and* $\widetilde{x}^+$ *for the random variable* $\widetilde{V}$ *are the unique roots of the equation* $\kappa(x) = \frac{2\xi}{\theta^2}$ *in the intervals* $(x^-, 0)$ *and* $(1, x^+)$*, respectively. For every* $x \in (\widetilde{x}^-, \widetilde{x}^+)$*, the MGF is*

$$\widetilde{M}(x) = e^{\omega(x)}\left(1 - \frac{\theta^2\kappa(x)}{2\xi}\right)^{-\frac{2\xi\eta}{\theta^2}}. \tag{2.15}$$

where the functions $\omega(\cdot)$ *and* $\kappa(\cdot)$ *are defined as*

$$\begin{aligned} \omega(z) &= \frac{2\xi\eta}{\theta^2}\left((\xi - \rho\theta z)\frac{t}{2} - \ln g(z)\right) + t\mu z \\ \kappa(z) &= \left(z^2 - z\right)\frac{g_2(z)}{g(z)}. \end{aligned} \tag{2.16}$$

where function g *and* g_2 *are defined in* (2.33).

2.5.2 Positions of the convergence abscissas

Let us denote the function

$$\begin{aligned} p(x) &= (\xi - \rho\theta x)^2 + \theta^2 \left(x - x^2\right) \\ &\equiv -x^2\theta^2 \left(1 - \rho^2\right) + x\theta (\theta - 2\rho\xi) + \xi^2. \end{aligned} \tag{2.17}$$

It has two real roots, possibly infinitely small (large), $-\infty \le x_1 < 0 < x_2 \le \infty$, $p(x) > 0$ when $x \in (x_1, x_2)$, and $p(x) < 0$ if $x < x_1$ or $x > x_2$:

$$x_{1,2} = \frac{\theta - 2\rho\xi \pm \sqrt{\theta^2 - 4\rho\theta\xi + 4\xi^2}}{2\theta (1 - \rho^2)}. \tag{2.18}$$

Next we discuss the position of $\widetilde{x}^-$ and $\widetilde{x}^+$ w.r.t. x_1 and x_2, where x_1 and x_2 are the roots of function (2.17) and are given by formulas (2.18). We consider the case when $|\rho| < 1$ and when $|\rho| = 1$.

We define the constants $y_{1,2}$ and $t_{1,2}$ as

$$\begin{aligned} y_{1,2} &= \pm \frac{2\rho^2 - 1}{\rho} \\ t_{1,2} &= \frac{4\xi}{\theta^2 \left(x_{1,2}^2 - x_{1,2}\right) - 2\xi (\xi - \theta\rho x_{1,2})}. \end{aligned} \tag{2.19}$$

For the $\widetilde{x}^-$ position w.r.t. x_1 and x_2 when $|\rho| < 1$ we conclude that

Theorem 2.3. *(Theorem 3.2 of Zaevski and Nedeltchev[13])*

We have $\widetilde{x}^- \in (x^-, x_1)$ in the following cases: $\left\{\rho \le \frac{1}{2}\right\}$, $\left\{\rho > \frac{1}{2},\ \frac{\theta}{\xi} \ge y_1 + 2\rho\right\}$, and $\left\{\rho > \frac{1}{2},\ \frac{\theta}{\xi} < y_1 + 2\rho, t < t_1\right\}$. If $\left\{\rho > \frac{1}{2},\ \frac{\theta}{\xi} < y_1 + 2\rho, t > t_1\right\}$, then $\widetilde{x}^- \in (x_1, 0)$. The equality $\widetilde{x}^- = x_1$ holds when $\left\{\rho > \frac{1}{2},\ \frac{\theta}{\xi} < y_1 + 2\rho, t = t_1\right\}$.

Next we look at the position of $\widetilde{x}^+$ w.r.t. x_1 and x_2 for the $|\rho| < 1$:

Theorem 2.4. *(Theorem 3.3 of Zaevski and Nedeltchev[13]) The following statements characterize the position of $\widetilde{x}^+$ w.r.t. x_2.*

1. *If $\xi \ge \rho\theta$, then*

   (a) *$\widetilde{x}^+ \in (x_2, x^+)$ when*

- $\{\rho \geq 0\}$,
- $\left\{\rho \in (-0.5, 0), \frac{\theta}{\xi} \leq 2\rho - y_2\right\}$,
- $\left\{\rho \in (-0.5, 0), \frac{\theta}{\xi} > 2\rho - y_2, t < t_2\right\}$, *or*
- $\{\rho \leq -0.5, t < t_2\}$.

(b) $\widetilde{x}^+ \in (1, x_2)$ *when* $\left\{\rho \in (-0.5, 0), \frac{\theta}{\xi} > 2\rho - y_2, t > t_2\right\}$ *or* $\{\rho \leq -0.5, t > t_2\}$.

(c) $\widetilde{x}^+ = x_2$ *when* $\left\{\rho \in (-0.5, 0), \frac{\theta}{\xi} > 2\rho - y_2, t = t_2\right\}$ *or* $\{\rho \leq -0.5, t = t_2\}$.

2. *Suppose that* $\xi < \rho\theta$. *Note that* $t_2 < \bar{t}$ ($\bar{t}$ *is defined by formula* $\bar{t} = \frac{2}{\rho\theta x_2 - \xi}$) *because the function* $\kappa(x_2; t)$ *increases w.r.t.* t *and it is infinity for* $\bar{t}$.

   (a) *If* $t < t_2$, *then* $\widetilde{x}^+ \in (x_2, x^+)$.

   (b) *If* $t = t_2$, *then* $\widetilde{x} = x_2$.

   (c) *If* $t_2 < t$, *then* $1 < \widetilde{x}^+ < x_2 < x^+$.

Further we present the $|\rho| = 1$ case, i.e., the Brownian motion of the returns is perfectly correlated with the Brownian motion of the volatility. The results for the case $2\xi = \theta$ are provided in the following theorem:

Theorem 2.5. *(Theorem 3.4 of Zaevski and Nedeltchev(13))*
If $\rho = 1$ *and* $2\xi = \theta$,[1] *then*

$$\widetilde{x}^{-,+} = \mp\sqrt{\frac{1}{2}\left(\coth\left(\frac{\xi t}{2}\right) + 1\right)}. \tag{2.20}$$

When $\rho = -1$ the root of function (2.17) is $-\frac{\xi^2}{\theta(2\xi+\theta)}$. We note it is negative and thus this is the smaller root x_1. Also, $x_2 = +\infty$. We can easily check Theorem 2.3 and 2.4 having in mind $\kappa(x_1) < \frac{2\xi}{\theta^2}$ and $t_2 = 0$.

2.6 Entropy-based Value at Risk

Chapter 7 introduces and discusses the entropy-based VaR and its properties.

[1] Let us mention that $x_1 = -\infty$ and $x_2 = \infty$ in this case.

2.6.1 Entropic VaR for stock returns

Definition 2.4. *The Entropic VaR at level* $\epsilon \in (0,1)$ *of random variable* ζ *is defined as*

$$EVaR\left(\epsilon;\zeta\right)=\inf_{z>0}d\left(\epsilon,z;\zeta\right)\equiv\inf_{z>0}\frac{1}{z}\ln\left(\frac{M\left(-z;\zeta\right)}{\epsilon}\right). \tag{2.21}$$

We obtain the acceptance set for the $EVaR$ in the following proposition:

Proposition 2.6. *(Proposition 2.9 of Nedeltchev and Zaevski[8]) Let* $\epsilon^*\left(\zeta\right)$ *be defined for a random variable* ζ *as*

$$\epsilon^*\left(\zeta\right)=\inf_{0\leq z\leq c_\zeta}M\left(-z;\zeta\right), \tag{2.22}$$

where the convergence abscissa $c_\zeta=\arg\sup_{z>0}M\left(-z;\zeta\right)<\infty$.
The acceptance set for the $EVaR$ *at the level* ϵ *is* $\mathcal{A}_\epsilon=\{\zeta:\epsilon^*\left(\zeta\right)\leq\epsilon\}$.

2.6.2 Deriving the Entropic VaR

We derive the $EVaR$ in the following way:

Theorem 2.6. *(Theorem 3.3 of Nedeltchev and Zaevski[8]) Let the functions* $\phi\left(\cdot\right)$ *and* $h\left(\cdot\right)$ *be defined through the MGF as*

$$\begin{aligned}\phi\left(z;\zeta\right)&=\ln M\left(-z;\zeta\right)\\ h\left(z,\epsilon;\zeta\right)&=\phi'\left(z\right)z-\phi\left(z\right)+\ln\epsilon.\end{aligned} \tag{2.23}$$

If $h\left(c_\zeta,\epsilon;\zeta\right)\leq 0$*, then*

$$EVaR\left(\epsilon;\zeta\right)=\frac{1}{c_\zeta}d\left(-c_\zeta,\epsilon;\zeta\right)\equiv\frac{1}{c_\zeta}\left(\phi\left(c_\zeta;\zeta\right)-\ln\epsilon\right) \tag{2.24}$$

where a_ζ *is the value at which the minimum of* $EVaR\left(\epsilon;\zeta\right)$*occurs, and* $c_\zeta=\arg\sup_{z>0}M\left(-z;\zeta\right)<\infty$*. Otherwise, if* $h\left(c_\zeta,\epsilon;\zeta\right)>0$*, then the equation* $h\left(z,\epsilon;\zeta\right)=0$ *has a unique solution in the interval* $\left(0,c_\zeta\right)$ *which is* a_ζ *and thus*

$$EVaR\left(\epsilon;\zeta\right)=d\left(-a_\zeta,\epsilon;\zeta\right)\equiv\frac{1}{a_\zeta}\left(\phi\left(a_\zeta;\zeta\right)-\ln\epsilon\right). \tag{2.25}$$

2.6.3 The EVaR of the considered stochastic models

- Black-Scholes model

  The Gaussian MGF is always defined and it is

$$M(z) = e^{t\left(\left(\mu - \frac{\sigma^2}{2}\right)z + \frac{\sigma^2 z^2}{2}\right)}. \tag{2.26}$$

  Hence, $c_{BS} = +\infty$. Thus functions (2.23) turn into

$$\begin{aligned} \phi(z;\zeta) &= t\left(-\left(\mu - \frac{\sigma^2}{2}\right)z + \frac{\sigma^2 z^2}{2}\right) \\ h(z,\epsilon;\zeta) &= \phi'(z)\,z - \phi(z) + \ln\epsilon. \end{aligned} \tag{2.27}$$

  We conclude that the values of minimizer a_{BS} and the $EVaR$ are

$$\begin{aligned} a_{BS} &= \frac{1}{\sigma}\sqrt{-\frac{2\ln\epsilon}{t}} \\ EVaR(\epsilon;BS) &= \left(\frac{\sigma^2}{2} - \mu\right)t + \sigma\sqrt{-2t\ln\epsilon}. \end{aligned} \tag{2.28}$$

- Exponential tempered stable model

  The MGF is defined in the interval $(-\lambda_1, \lambda_2)$ and thus $c_{ETS} = \lambda_1$. It can be written as

$$M_{TS}(z) = e^{t(\mu z + \psi_1(z) + \psi_2(z))}, \tag{2.29}$$

  where

$$\begin{aligned} \psi_1(z) &= \begin{cases} \Gamma(-\alpha_1)\,\lambda_1^{\alpha_1} c_1 \left(\left(1 + \frac{z}{\lambda_1}\right)^{\alpha_1} - 1 - \frac{z\alpha_1}{\lambda_1}\right), & if \quad \alpha_1 \neq 1 \\ -zc_1 + c_1(\lambda_1 + z)\ln\left(1 + \frac{z}{\lambda_1}\right), & if \quad \alpha_1 = 1 \end{cases} \\ \psi_2(z) &= \begin{cases} \Gamma(-\alpha_2)\,\lambda_2^{\alpha_2} c_2 \left(\left(1 - \frac{z}{\lambda_2}\right)^{\alpha_2} - 1 + \frac{z\alpha_2}{\lambda_2}\right), & if \quad \alpha_2 \neq 1 \\ zc_2 + c_2(\lambda_2 - z)\ln\left(1 - \frac{z}{\lambda_2}\right), & if \quad \alpha_2 = 1. \end{cases} \end{aligned} \tag{2.30}$$

Hence, the function $\phi(\cdot; ETS)$ from (2.23) is $\phi(z; ETS) = t(-\mu z + \psi_1(-z) + \psi_2(-z))$. Therefore, $\phi'(z; ETS) = -t(\mu + \psi_1'(-z) + \psi_2'(-z))$, where

$$\psi_1{}'(z) = \begin{cases} \Gamma(-\alpha_1)\lambda_1^{\alpha_1-1}c_1\alpha_1\left(\left(1+\frac{z}{\lambda_1}\right)^{\alpha_1-1}-1\right), & if \quad \alpha_1 \neq 1 \\ c_1 \ln\left(1+\frac{z}{\lambda_1}\right), & if \quad \alpha_1 = 1 \end{cases}$$
$$\psi_2'(z) = \begin{cases} \Gamma(-\alpha_2)\lambda_2^{\alpha_2-1}c_2\alpha_2\left(\left(1-\frac{z}{\lambda_2}\right)^{\alpha_2-1}+1\right), & if \quad \alpha_2 \neq 1 \\ -c_2 \ln\left(1-\frac{z}{\lambda_2}\right), & if \quad \alpha_2 = 1. \end{cases} \tag{2.31}$$

We introduce the following Proposition:

Proposition 2.7. *(Proposition 3.4 of Nedeltchev and Zaevski[8])*
Let $D(x)$ be the digamma function and $D^{-1}(x)$ be its inversion in the interval $(-2,-1)$. Note that it exists because the digamma function increases between minus and plus infinity in this interval. Let the constant b be defined as $b = c_1\Gamma\left(d^{-1}(\ln\lambda_1)\right)\lambda_1^{-D^{-1}(\ln\lambda_1)}$. The following statements hold:

1. *If $b - \psi_2'(-\lambda_1)\lambda_1 - \psi_2(-\lambda_1) + \frac{\ln\epsilon}{t} > 0$, then $a_{ETS} < c_{ETS}$.*
2. *If $b - \psi_2'(-\lambda_1)\lambda_1 - \psi_2(-\lambda_1) + \frac{\ln\epsilon}{t} \leq 0$, then there exist two constants b_1 and b_2, $1 < b_1 < -d^{-1}(\ln\lambda_1) < b_2 < 2$, for which $a_{ETS} < c_{ETS}$ when $\alpha_1 \in (0, b_1) \cup (b_2, 2)$ and $a_{ETS} = c_{ETS}$ when $\alpha_1 \in [b_1, b_2]$.*

The $EVaR$ is obtained through Theorem 2.6.

- Heston model

  We leverage

$$\begin{aligned} d(\epsilon, -z; \zeta) &= \frac{1}{z}\left(\phi(z;\zeta) - \ln\epsilon\right) \\ \frac{\partial}{\partial z}\left(d(\epsilon, -z; \zeta)\right) &= \frac{h(z,\epsilon;\zeta)}{z^2}, \end{aligned} \tag{2.32}$$

to introduce the following Proposition:

Proposition 2.8. *(Proposition 3.5 of Nedeltchev and Zaevski*(8)*) Let* x^* *be the highest root of* $g(\cdot)$ *which is less than* x_1*. The abscissa of* $EVaR$ *is* $c_{Heston} = -x^*$*.*[2] *Functions* (2.23) *related to the MGF of the log-price in its domain can be written as*

$$
\begin{aligned}
\phi(z;\xi) &= -x_0 z - \mu t z + \frac{2\xi\eta}{\theta^2}\left((\rho\theta z + \xi)\frac{t}{2} - \ln g(-z)\right) + v_0 \frac{z+z^2}{g(-z)} g_2(-z) \\
h(z,\epsilon;\xi) &= \phi'(z) z - \phi(z) + \ln\epsilon \\
&= \ln\epsilon + \frac{2\xi\eta}{\theta^2}\left(-\frac{\xi t}{2} + \frac{g'(-z)}{g(-z)} z + \ln g(-z)\right) \\
&\quad + v z^2 \left(\frac{g(-z) + (1+z) g'(-z)}{g^2(-z)} g_2(-z) - \frac{1+z}{f(-z)} g_2'(-z)\right) \\
g'(x) &= g_1'(x) - \rho\theta g_2(x) + (\xi - \rho\theta x) g_2'(x) \\
g_1'(x) &= \frac{t}{4} g_2(x) p'(x) \\
g_2'(x) &= \operatorname{sgn}(p(x)) \frac{p'(x)}{2P^2(x)}\left[\frac{t}{2} g_1(x) - g_2(x)\right] \\
p'(x) &= -2x\theta^2\left(1-\rho^2\right) + \theta(\theta - 2\rho\xi).
\end{aligned}
\tag{2.33}
$$

We obtain the minimizer that determines the value of $EVaR$ *(*a_ξ *or* c_ξ*) through Theorem 2.6, more precisely via the sign of* $h(c_\xi, \epsilon; \xi)$*.*

- Stochastic volatility jump models

  The MGF of the jump part the Bates model is always defined and it is written as $M(z) = e^{t\psi_{CP}(z)}$, where $\psi_{CP}(u)$ is

$$
\psi_{CP}(z) = \lambda\left(e^{\frac{\beta^2 z^2}{2} + \alpha z} - 1\right). \tag{2.34}
$$

  The jumps of the stochastic volatility jump model (see Zaevski et al.(14)) are presented by a tempered stable process. We formulate the following results:

[2] In fact, x^* is the left abscissa of convergence of the MGF of the log-price in the Heston model. The right one is more variable – the complete exposition is provided in Theorem 3.1, points 1-3 of del Baño Rollin et al.(3).

Proposition 2.9. *(Proposition 3.6 of Nedeltchev and Zaevski[8]) The domain of the MGF of the the log-returns of a stochastic volatility jump model is the intersection between the domains of the continuous and jump parts. Particularly, the support for the Bates model coincides with those for the Heston's one. Thus, the abscissas* $c_{Bates} = c_{Heston}$ *and* $c_{SVTS} = \min\{c_{Heston}, \lambda_1\}$*, where the subscript* $SVTS$ *stands for the stochastic volatility model with tempered stable jumps.*

Based on Proposition 2.9, we derive the minimizer for the $EVaR$ as well as its own value through Theorem 2.6. Functions (2.23) turn into

$$\begin{aligned} \phi(z;) &= \ln M_{Heston}(-z) + \ln M_J(-z;\xi) \\ h(z,\epsilon) &= \left[\phi'_{Heston}(z) + \phi'_J(z)\right] z - \phi_{Heston}(z) - \phi_J(z) + \ln\epsilon. \end{aligned} \tag{2.35}$$

The subscript J stands for the jump part. The terms for the Bates model are

$$\begin{aligned} \phi_J(z) &= t\lambda\left(e^{-\left(\frac{\delta^2 z^2}{2}+\left(k-\frac{\delta^2}{2}\right)z\right)} - 1\right) \\ h(z,\epsilon;\xi) &= \left[\phi'_{Heston}(z) + \phi'_J(z)\right] z - \phi_{Heston}(z) - \phi_J(z) + \ln\epsilon \\ \phi'_J(z) &= -t\lambda e^{-\left(\frac{\delta^2 z^2}{2}+\left(k-\frac{\delta^2}{2}\right)z\right)}\left(\delta^2 z + k - \frac{\delta^2}{2}\right). \end{aligned} \tag{2.36}$$

2.6.4 Averaging w.r.t. the volatility

This section follows the approach applied and the conclusions drawn in Section 2.5 of the dissertation. Leveraging Theorem 2.2, we derive the left abscissa of convergence c_{av} of the MGF :

Proposition 2.10. *(Proposition 3.7 of Nedeltchev and Zaevski[8]) For a fixed* t*, the equation* $\kappa(t,x) = \frac{2\xi}{\theta^2}$ *has a unique root in the interval* $(-c_{Heston}, 0)$*. The boundary* c_{av} *is the opposite value of this root.*

The function $h(z,\epsilon;\zeta)$ that describes the minimizer of the $EVaR$ turns into

$$\begin{aligned}
h(z,\epsilon;\xi) &= \phi'(z)\,z - \phi(z) + \ln\epsilon \\
\phi(z,\epsilon;\xi) &= \omega(-z) - \frac{2\xi\eta}{\theta^2}\ln\left(1 - \frac{\theta^2\kappa(-z)}{2\xi}\right) \\
\phi'(z,\epsilon;\xi) &= -\omega'(-z) - \frac{2\xi\eta}{2\xi - \theta^2\kappa(-z)}\kappa'(-z) \\
\omega'(x) &= -\frac{2\xi\eta}{\theta^2}\left(\frac{\rho\theta t}{2} + \frac{g'(x)}{g(x)}\right) + t\mu \\
\kappa'(x) &= \frac{(2x-1)\,g_2(x) + (x^2 - x)\,g_2'(x)}{g(x)} - \left(x^2 - x\right)\frac{g_2(x)\,g'(x)}{g^2(x)}.
\end{aligned} \tag{2.37}$$

The derivatives $g'(x)$ and $g_2'(x)$ can be found in formulas (2.33).

2.7 Computations and empirical results analysis

In Chapter 8 we challenge the theoretical aspects from the preceding chapters by applying them to real-life cases and drawing conclusions therefrom.

2.7.1 Historical data

We use historical data for the S&P500 index for the period between January 2, 2003 and February 19, 2025. The period has been chosen to include both tranquil times and financial turmoils (like the Global Financial Crisis and the COVID19 pandemia). We deal with the daily Adjusted Close values and we apply a moving window with 1000 days of length.

2.7.2 Calibration methodology

We denote by S_1, S_2,...,S_N, $i = 1, 2, ..., N-1$ the S&P500 index values for a particular sample. We derive the log-returns in the usual manner $r_i = \ln(S_{i+1}) - \ln(S_i)$. We derive the empirical PDF by dividing the interval $(-0.25, 0.25)$ into $M = 250$ bins with length $\Delta = 0.002$. The interval is chosen this way because the higher deviation in the S&P500 index log-returns is -0.2290 at the Black Monday, October 19, 1987. We denote by n_i, $i = 1, 2, ..., 250$ the number of log-returns belonging to each bin. The empirical PDF in the center of the i-th bin is approximated by $\frac{n_i}{N\Delta}$.

To calibrate the models, we also need the theoretical PDF for the different models. Except for the Black-Scholes model, the PDFs in closed form are not available for the considered models, hence we must derive them through the characteristic functions. We invert the characteristic function using the Fast Fourier transform (FFT) algorithm. To find a relatively good approximation, we derive the PDFs at 500 points uniformly taken at the interval $(-0.5, 0.5)$ – the length of the bins is again $\Delta = 0.002$. Finally, we derive the PDF using the cubic spline approach.

We calibrate via the least square errors (LSqE) approach. We denote by Υ the set of all possible values of the parameters of some theoretical model and by υ some particular realization, $\upsilon \in \Upsilon$. Regarding the ERM we apply the following LSqE minimization criterion

$$\min_{\upsilon \in \Upsilon} \left\{ \sum_{i=1}^{250} \left(\ln\left(p_{th,i} + 1\right) - \ln\left(p_{emp,i} + 1\right) \right)^2 \right\}, \tag{2.38}$$

where $p_{th,i}$ and $p_{emp,i}$ are the values of the theoretical PDF based on the parameter's set υ and the empirical PDF respectively. We use the logarithm to give relatively equal impact of the terms in the distribution's center and the tails. The unit is added to the PDFs because some of the empirical values may be equal to zero.

We calibrate the $EVAR$ via the minimization problem

$$\min_{\upsilon \in \Upsilon} \left\{ \sum_{n=1}^{230} \left(p_{th,n} - p_{emp,n} \right)^2 + 10^3 \sqrt{\sum_{n=1}^{6} \left(EVaR_{th}\left(\epsilon_n\right) - EVaR_{emp}\left(\epsilon_n\right) \right)^2} \right\}, \tag{2.39}$$

The second term of the cost function is related to the deviations from the empirical $EVaR$s produced by the theoretical models. For a relative equalization of the weights of both components, we multiply the $EVaR$-part by 10^3.

In the ERM case we capture the tails only, while in the $EVaR$ case we deal with the whole distribution.

2.7.3 Validation and comparison of the fits

We validate the goodness of fit by an approach that resembles the maximum likelihood method but is applied to forecasting. We denote by $p\,(x;i)$ the calibrated PDF for some model at the i-th date. We use the following forecast function

$$\sum_{i=1}^{N-j} \ln\left(p\left(r_{i+j}, i\right)\right). \tag{2.40}$$

This way we may evaluate the forecast of the fit at the date i for the value at the date $i+j$. We calculate the values which function (2.40) generates for $j=1$ (one day), 5 (one week), and 20 (approximately one month). We see that all models (except the Black-Scholes one) produce relatively equal forecasts. Also, we observe that the tempered stable models issue a better forecast regardless of the presence or not of the stochastic volatility.

2.7.4 Analyzing Heston Model Calibration

We focus on the behaviour of the Heston Model. The most important observation regarding μ is that it changes its sign; it reaches the lowest values during the COVID19 pandemia and has record high values from the end-2012 to the end-2016. The speed of the mean reversion ξ is a very volatile parameter. The parameter η is of very low value until the break of the GFC 2008; also, from August 2015 till the break of the COVID19; its value decreases from end of 2012 to the mid-2015 and increases steadily during the remaining periods. We observe that the profile of ξ and θ are surprisingly identical although of different scale. The correlation coefficient ρ has a very stable behavior through the calm market periods and is significantly volatile during financial turbulence.

Last but not least, we compare the market risk measured by the expectile riks measure and the entropic VaR. We see a good match for the time drift μ, including during the GFC and the COVID19. We note that ξ and θ are of very different scale. The variance volatility θ is the same from the second half of 2015 to the begin of the COVID19. During the rest of the period, we see opposite values. Furthermore, we see that the correlation coefficient for the entropic VaR is rather stable whereas it is more volatile for the expectile risk measure.

2.8 Rough Volatility: A New Stylized Fact

Chapter 9 introduces the Rough Volatility as a characteristic for the contemporary financial markets.

We model the volatility during the COVID19 period as $\sigma_t = c \exp\left(vB_t^H\right)$, where c, v are positive constants and B_t^H is a fBM.

We use the data set available online from the Oxford-Man library (https://oxford-man.ox.ac.uk/research/realized-library/). It includes 31 indexes that cover the Americas (the USA, and a couple of Emerging Markets), Europe, and Asia (incl. some Emerging Markets). The time ranges from January 2000 to March 2021. We are talking of high-frequency series for 5-min realized variance (RV), 10-min RV. Various kernels (Tukey-Hanning, Two-Scale/Bartlett, and Non-Flat Parzen) are applied to integrate the RV.

We calculated the moments of log-volatility differences:

$$m\left(q, \Delta\right) = \frac{1}{N} \sum_{k=1}^{N} |\ln \sigma_{k\Delta} - \ln \sigma_{(k-1)\Delta}|^q, \tag{2.41}$$

where m is the moment of order q, Δ is the time lag (days), and σ is the volatility. Given the fBM monofractal property for various q, we anticipate to observe $m\left(q, \Delta\right) \propto \Delta^{\zeta_q}$, where ζ_q is the scaling factor. Hence, we can derive the H index value by running regression

$$\zeta_q \approx q\hat{H}. \tag{2.42}$$

We confirmed the monofractal property which is compatible with Gatheral et al.[7]. Also, we questioned the link between ζ_q and the empirically derived Hurst index value $\hat{H}$ for the S&P500, the STOXX50E, the FTSE, and the KSE. We find that the link between ζ_q and $\hat{H}$ for the latter 3 indexes is far from being linear. Hence, we are not able to estimate the H-index by running the *linear* regression in formula (2.42). This is why we run the regression $\ln\left(1 - \kappa\left(\Delta\right)\right) = a + 2H \ln \Delta$ for sufficiently small $\frac{\Delta}{t}$, where $\kappa\left(\Delta\right) = \frac{1}{2}\left\{1 + \left(1 + \frac{\Delta}{t}\right)^{2H} - \left(\frac{\Delta}{t}\right)^{2H}\right\}$. Next, we checked whether the same observations apply to other indexes; we find that the H-index series for $STOXX50E$ (for example) follow the same patterns. We double check this result by drawing the two series S&P500 vs. STOXX50E. We would like to exclude any role of the integrating kernel; hence, we confirmed the above results with two different kernels.

Chapter 3

Concluding remarks and further works

The dissertation tackles the ERM as a risk measure that is the opposite to the expectile which on its turn is found via a square optimization. It has the advantage to be the only risk measure that is both coherent and elicitable when certain constraints on the significance level are met. We suggest to compute the truncated expectation via the characteristic function and later apply them to derive the ES as well as the ERM.

The dissertation's computations include S&P500 index data set of the daily Adjusted Close price ranging over the most recent 23 years. We applied a moving window of 1000 observations. We select the start date and the window width in a way to use the calibration results from previous studies.

We found a satisfactory good match between the theoretical and the empirical values of the VaR, ES, and ERM. The theoretical - empirical comparison leverages a coefficient that corrects the VaR/ES ratio and the VaR/ERM ratio. This innovation better preserves the information available at the returns distribution when changing the measurement level. We suggest value of 0.6 for the former coefficient and 1.25 for the latter one. We observe that all models (except the Black-Scholes one) match the empirical VaR and the tempered stable jumps model reaches the best match.

Because of the lack of PDFs in closed form for the considered models (except for the Black-Scholes model), we derived the PDF by applying Fast Fourier Transform to the characteristic functions of the models. The theoretical VaR values were computed as a quantile of the CDF; on its turn, the CDF was computed as the inverse value of cubic splines applied to the cumulative sum of the theoretical PDFs. The theoretical ES values were obtained by integrating the VaR values.

We verified how well the PDF values are calibrated. To this end, we applied an approach that resembles the maximum likelihood estimation. We found a relatively good fit of all models; the best fit is achieved by the tempered stable model, while the Black-Scholes model produced the worst fit. The check we undertook confirmed the leverage effect as a stylized fact. Also, we found that the Bates model produces results of a relatively large intensity and a low jump size. We see that the calibrated results are marked with relatively large deviations in the distribution center and a strong fit in the tails.

The dissertation transforms the initial way of presenting the $EVaR$ from the right-hand distribution tail (i.e., the profit) to the left-hand one (i.e., the losses) which is of interest for the finance. This risk measure is defined as a minimization of a MGF[-related function which requires us to determine the diapason where the MGF is defined, since contrary to the CDF, the MGF is not defined everywhere.

The dissertation presents the $EVaR$ as a coherent risk measure. Hence, its dual representation is described. A contribution of the dissertation is the derivation of the $EVaR$'s acceptance set.

The $EVaR$ is derived via an MGF-related minimizer. The dissertation elaborates the MGF for the 5 models considered and the value of the MGF-related minimizer. Also, the text indicates the interval where the MGF is defined.

In the case of the Heston model and models that incorporate it (like the Bates model or the Stochastic Volatility Tempered Stable) we deal with the initial volatility as an unobservable parameter of the model and hence enters the MGF formula. To solve this issue, we take advantage of the fact that the volatility is described by a Cox-Ingersoll-Ross process that has a Gamma stationary distribution. We average the volatility over the Gamma distribution and we define the diapason of the averaged MGF. This approach reduces the number of the model's parameters but shrinks the diapason where the MGF is defined.

The dissertation computes the $EVaR$ values for 6 confidence levels and we see discrepancies between the theoretical and the empirical values. We dive deeper by calculating the $EVaR$ via the calibration approach used for ERM purposes. Again we observe gaps that we attribute to over-fitting and this is why we add a $EVaR$-related part to the cost function. After calibrating the models, we compared the generated PDFs with the empirical ones; the gaps we see stem from over-fitting. We find satisfactory ability of the $EVaR$ to capture the risk profile of the S&P500 index: the zones of higher market risk are well visible while the Black-Scholes model is the worst performing among the models.

Next, the dissertation summarizes the capacity of the calibrated models to generate reliable values of the VaR, ES, ERM, and $EVaR$. The results confirm that the applied approach produces satisfactorily good results.

Last but not least, we analyze the time series of the parameters of the calibrated Heston model. We see that the parameters change qualitatively its behaviour during turmoils (the Global Financial Crisis 2008 and the COVID19 period). We see an identical profile for ξ and θ. We note that the volatility of the parameter ρ increases during financial crises.

The dissertation compares the calibrated results for the ERM and the $EVaR$ by drawing a joint plot of the parameters. We conclude that the $EVaR$ better reflects the financial crises.

The theoretical aspects discussed in the dissertation and the empirical computations done allow us to perform the following comparison of the ERM and the $EVaR$:

1. The ERM is both coherent (under the condition we deal with a significance level below 0.5) and elicitable, while for the $EVaR$ is coherent;

2. the ERM is well defined only for random variables with finite second moments;

3. we can apply the whole quantile-related toolkit to the expectile;

4. once computed, the truncated expectation value can be used for calculating both the $EVaR$ and the ES;

5. the $EVaR$ value is calculated by minimizing a MGF-related function. Hence, we have to define the diapason where the MGF is defined and keep in mind that averaging the volatility reduces this diapason;

6. the $EVaR$ better reacts to financial crises.

Regarding the Rough Volatility, the dissertation was launched from the ubiquitous position where the dominant way to estimate volatility (as the standard deviation of asset log-returns) complicates the derivation of volatility-related stylized facts. We are facing lack of commonly adopted algorithm to reveal volatility-related stylized facts nor short-list of them. Stylized facts for the historic/realized volatility are better researched than the features of the implied volatility.

Based on a representative data set for an extensive time period, we observe scaling property of the log-volatility for a large pool of market indexes. Also, we confirmed there is a non-linear link between the scaling factor and the Hurst index value for several indexes. Our calculations read a material increase of the H-index values during the COVID19 period; we do not spot similar increase during other turmoils, like the Global Financial Crisis 2008 for example.

The dissertation reveals a couple of new stylized volatility-related facts. We find that the log-volatility is rough since the Hurst index value is $H < \frac{1}{2}$. Additionally, the H-index value varies within a range and moves in packages. We noted that the log-volatility becomes smoother during market misbehavior. We observe that the choice of the kernel type determines the H-index value but preserves the above conclusions.

We suggest further researches to create eco-system of the volatility-related stylized facts that would unleash new findings of academic and practical merits. Additional efforts are necessary to separate the volatility studies from the returns series and hence a new generation of volatility-estimation methods is needed.

Chapter 4

Scientific Contributions

- The main contribution is the application of stochastic models (the Black-Scholes model, the Heston model, the Bates model, the exponential tempered stable model, the model with stochastic volatility and tempered stable process) to various risk measures (the Value at Risk, the Expected Shortfall, the Expectile Risk Measure, the Entropic VaR).

- The Truncated Expectation is derived through the characteristic function of the process (Proposition 5.4). On this basis, the Expected Shortfall and the Expectile Risk Measure are calculated via the Truncated Expectation (Theorem 5.2).

- The logarithm of the Heston returns are presented as random variables (Proposition 6.2) by averaging on the stationary distribution of the volatility process. Their convergence abscissas are identified as a subset of the abscissas of the initial process (Lemma 6.2 and Theorem 6.1). All possible positioning of the abscissas are discussed (Lemma 6.4, Theorem 6.2, Theorem 6.3, and Theorem 6.4). The same approach is applied also to models that upgrade the Heston model, like the Bates model and the model with stochastic volatility / tempered stable process.

- The dissertation derives the Acceptance Set of the Entropic VaR (Proposition 7.2). It proves a theorem regarding the calculation of the value of the EVaR by minimizing an MGF-related function (Theorem 7.1). The text provides formulas for the EVaR of all the 5 models: the Black-Scholes model (formulas (7.18) and (7.19)), the Heston model

(Proposition 7.5), the Bates model (Proposition 7.6), the exponential tempered stable mode (Proposition 7.4), model with stochastic volatility / modified stable process (Proposition 7.7).

- The dissertation empirically investigates the 4 risk measures applied to the 5 stochastic models for the S&P500 index for the last 23 years that include periods of normal market functioning as well as financial crises. The main conclusion in this part of the dissertation is that the EVaR reacts more adequately to market crises; the models with stochastic volatility and jumps outperform the rest of the models.

- The value of the Hurst index is calculated for 4 leading indexes (S&P500, STOXX50E, FTSE, and KSE). The calculations read a linear relationship between the scaling coefficient and the value of the Hurst index for the S&P500 index. For the rest of the indexes the relationship is non-linear. The results show that the Hurst index is less than 0.5 for the last 21 years, i.e. the volatility is rough. The value of the Hurst index varies within certain borders and evolves in packages. During financial turbulences the value of the Hurst index increases, i.e. the volatility gets smoother.

Chapter 5

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