R E V I E W

under a procedure for a defence of a dissertation
"Functions Holomorphic over Finite-Dimensional Commutative Associative Unital $\mathbb{C}$-Algebras - Some Local, Global and Universal Aspects"
for acquisition of the educational and scientific degree "Doctor"
by candidate Marin Vladimirov Genov
In the Scientific field: **4. Natural Sciences, Mathematics and Informatics,**
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Section of **"Analysis, Geometry and Topology",**
Institute of Mathematics and Informatics (IMI),
Bulgarian Academy of Sciences (BAS)

The review has been prepared by Prof. PhD Azniv Kirkor Kasparian, Department of Algebra, Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", as a member of the scientific jury for the defense of this PhD thesis according to Order № 83/14.08.2026 of the Director of the Institute of Mathematics and Informatics.

1 General characteristics of the dissertation thesis and the presented materials

The dissertation contains 227 pages and consists of an introduction, three chapters an appendix and a bibliography. The first chapter comprises some algebraic and topological preliminaries. The second chapter introduces and studies the differentiability over a finite dimensional commutative associative unital algebra $\mathcal{A}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ in representation and morphism setting. The third chapter is devoted to the integration over $\mathcal{A}$. The appendix collects some facts about the structure constants of a finite dimensional algebra, its morphisms and representations. The bibliography consists of 186 titles, from which 142 article, 28 monographs, 15 preprints and one PhD thesis. References, appeared before 2000 are 117, while the ones, published in the present century are 69. This is a testimony that Marin Genov is familiar with the classical and the contemporary research in the scientific field.

Combining important results from Complex Analysis, Commutative Algebra, Algebraic Topology and other mathematical disciplines, Marin Genov invents extremely original, complicated constructions, illustrating the interrelations of the aforementioned areas. His dissertation is a testimony for an original thinking, reach mathematical culture and in-depth understanding of a wide range of mathematical ideas. Marin Genov provides versatile comments of the obtained results and relates them successfully to the works of other authors. His thesis impresses by a refined mathematical aesthetics and an extremely fluent English. Marin Genov's dissertation is marked by his constant pursuit for perfection and beauty in doing mathematics. It certifies an enormous knowledge in the scientific field and masterful skills for doing an independent research.

2 Short CV and personal impressions of the candidate

In 2005 Marin Genov has been awarded a German Academic Scholarship for gifted pupils "Studienstiftung des Deutschen Volkes", which allowed him to study at Ludwig-Maximilian University in Munich. He graduated there as a Bachelor of Science in 2014. During his study, Marin Genov was an Assistant in Linear algebra, Real analysis, Complex analysis, Nonlinear functional analysis. He obtained his Master's degree at Sofia University "St. Kliment Ohridski" in 2018. There Marin Genov took from me with excellent grades the elective courses "Introduction to Commutative Algebra", "Introduction to Homological Algebra" and "Complex Algebraic Geometry". At the exams, he was prepared thoroughly, demonstrated a profound comprehension of the material and an original thinking. Marin Genov taught "Algebra 2" in my team. He performed perfectly his teaching by choosing appropriate problems, illustrating the theory and explaining them accessibly to the students. In such a way, he earned the respect and the high esteem of his students. As a reviewer of his Master's Thesis I was impressed by his in-depth insight, original thinking and perfect English. Marin Genov is highly respected for his honesty, qualification and the responsibility to his obligations. He is constantly pursuing excellence in research and teaching, for which he was awarded the prize "130 Years Sofia University" in 2018. From 2020 to 2024 Marin Genov is a PhD student at the Doctoral Programme "Geometry and Topology" at the Section "Analysis, Geometry and Topology" of the Institute of Mathematics and Informatics, Bulgarian academy of sciences. From 2023 to 2025 he is a "Mathematician" and from 2025 up till now is an Assistant Professor at the Institute of Mathematics and Informatics.

3 Content analysis of the scientific and applied achievements of the candidate, contained in the presented PhD thesis and the publications to it, included in the procedure

The introduction recalls how the notion of differentiability over a $\mathbb{K}$-algebra $\mathcal{A}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ arose in the attempt to generalize the complex differentiability, retaining some nice properties as Cauchy-Riemann equations, Cauchy's and Morera's Integral Theorems, Cauchy's Integral Formula and the analyticity. It outlines some approaches for solving this problem, motivates the thesis, discusses briefly the main results and lists some topics for further research and applications.

In the notations from the dissertation, Chapter 0 is a preparation for the main results. Its first section focuses on the correspondence between morphisms and representations of algebras, their standard factorisations and the interplay of the spectral point of view with the direct products. Subsection 0.1.1 associates to a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ of not necessarily commutative associative algebras over an arbitrary ring R, an $\mathcal{A}$-bimodule structure on $\mathcal{B}$, given by the compositions of φ with the regular representation and anti-representation of $\mathcal{B}$ in $\mathrm{End}_R(\mathcal{B})$. Conversely, a representation $\lambda : \mathcal{A} \to \mathrm{End}_R(M)$ gives rise to a morphism $\lambda^{\mathrm{im}} : \mathcal{A} \twoheadrightarrow \lambda(\mathcal{A})$ of R-algebras, called the image co-restriction of λ. Subsection 0.1.2 collects some known facts about decompositions ${}_{\mathcal{A}}M = \oplus_{k=1}^{m} {}_{\mathcal{A}}M_k$ into indecomposable $\mathcal{A}$-modules ${}_{\mathcal{A}}M_k$ and the complete systems of non-trivial mutually orthogonal primitive idempotents $\varepsilon_1, \ldots, \varepsilon_m \in \mathrm{End}_{\mathcal{A}}(M)$ with $M_k = \varepsilon_k(M)$, $1 \leq k \leq m$. Subsection 0.1.3 is devoted to the correspondence between the external direct products $\times_{i\in I} ({}_{\mathcal{A}_i}M_i)$ of left $\mathcal{A}_i$-modules and the internal direct products $\prod_{i\in I} {}_{\mathcal{A}}M_i$ over $\mathcal{A} := \prod_{i\in I} \mathcal{A}_i$. Subsection 0.1.4 discusses the correspondence between the external (internal) direct products of representations and the external (internal) direct products of their associated modules. In order to explain in what extent the decomposition of $\lambda(\mathcal{A})$ is incompatible with the decomposition of $\mathrm{End}_R(M)$, subsection 0.1.4 says that $\mathcal{A}$ is a sub-direct product of $\mathcal{B}_1, \ldots, \mathcal{B}_m$ if there is an inclusion $i : \mathcal{A} \hookrightarrow \mathcal{B} := \prod_{k=1}^{m} \mathcal{B}_k$ with injective $\mathrm{pr}_k^{\mathcal{B}} \circ i : \mathcal{A} \to \mathcal{B}_k$, $\forall 1 \leq k \leq m$. If ${}_{\mathcal{A}}M = \oplus_{k=1}^{m} {}_{\mathcal{A}}M_k$ for $\mathcal{A}$-modules, given by representations $\lambda : \mathcal{A} \to \mathrm{End}_R(M)$, $\lambda_k : \mathcal{A} \to \mathrm{End}_R(M_k)$ and $\mathcal{B}_k := \lambda_k(\mathcal{A})$ then $\lambda(\mathcal{A})$ is shown to be a sub-direct product of the convenient target $\mathcal{B} := \prod_{k=1}^{m} \mathcal{B}_k$ of λ and $\mathrm{pr}_k^{\mathcal{B}} : \mathcal{B} \twoheadrightarrow \mathcal{B}_k$ factor through $i^{\rm cor}|_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathrm{End}_R(M)$. As a consequence, the regular representation of an R-algebra $\mathcal{B} = \prod_{k=1}^m \mathcal{B}_k$ is related to the regular representations of its R-subalgebras $\mathcal{B}_k$ by the means of the co-restriction $i^{\rm cor} : \prod_{k=1}^m \mathrm{End}_R(\mathcal{B}_k) \to \mathrm{End}_R(\mathcal{B})$. Subsection 0.1.5 explains how an R-algebra decomposition $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$ affects a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ of R-algebras or a representation $\lambda : \mathcal{A} \to \mathrm{End}_R(M)$ with completely decomposable $\mathcal{B}$, respectively, ${}_{\mathcal{A}}M$. In the morphism case, there is a φ-adapted isomorphism $\beta : \mathcal{B} \to \prod_{k=1}^m \mathcal{B}_k$ for ideals $\mathcal{B}_k$, which are indecomposable R-algebras and a fiber-grouping index mapping $j_\varphi : [m] \to [n]$, $[m] := \{1, \ldots, m\}$, such that $\beta \circ \varphi = (\varphi_1, \ldots, \varphi_m) : \mathcal{A} \to \prod_{k=1}^m \mathcal{B}_k$ factors through $\Pi_\varphi : \mathcal{A} \to \prod_{k=1}^m \mathcal{A}_{j_\varphi(k)}$ and $(\times_{k=1}^m \overline{\varphi}_k) : \prod_{k=1}^m \mathcal{A}_{j_\varphi(k)} \to \prod_{k=1}^m \mathcal{B}_k$ for uniquely determined $\overline{\varphi}_k : \mathcal{A}_{j_\varphi(k)} \to \mathcal{B}_k$ with $\varphi_k = \overline{\varphi}_k \circ \mathrm{pr}^{\mathcal{A}}_{j_\varphi(k)}$. In the representation case, there is a decomposition ${}_{\mathcal{A}}M = \oplus_{k=1}^m {}_{\mathcal{A}}M_k$ for indecomposable ${}_{\mathcal{A}}M_k$, given by representations $\lambda_k : \mathcal{A} \to \mathrm{End}_R(M_k)$ and a fiber-grouping index mapping $j_\lambda : [m] \to [n]$, such that $(\lambda_1, \ldots, \lambda_m) : \mathcal{A} \to \prod_{k=1}^m \mathrm{End}_R(M_k)$ factors through $\Pi_\lambda : \mathcal{A} \to \prod_{k=1}^m \mathcal{A}_{j_\lambda(k)}$ and $\left(\times_{k=1}^m \overline{\lambda_k}\right) : \prod_{k=1}^m \mathcal{A}_{j_\lambda(k)} \to \prod_{k=1}^m \mathrm{End}_R(M_k)$ for uniquely determined $\overline{\lambda_k} : \mathcal{A}_{j_\lambda(k)} \to \mathrm{End}_R(M_k)$ with $\lambda_k = \overline{\lambda_k} \circ \mathrm{pr}^{\mathcal{A}}_{j_\lambda(k)}$. Subsection 0.1.6 introduces the closure $X^\varphi := \varphi^{-1}(\varphi(X))$ of a subset $X \subseteq \mathcal{A}$ with respect to a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ of R-algebras, the multiplicative submonoid $\mathcal{A}^\times_\varphi := \varphi^{-1}(\mathcal{B}) \supseteq \mathcal{A}^\times$ with respect to φ, the singular cone $\mathrm{Sing}_\varphi(\mathcal{A}) := \varphi^{-1}(\mathrm{Sing}(\mathcal{B})) \subseteq \mathrm{Sing}(\mathcal{A}) := \mathcal{A} \setminus \mathcal{A}^\times$ with respect to φ, the Jacobson radical $J_\varphi(\mathcal{A}) := \varphi^{-1}(J(\mathcal{B}))$ with respect to φ, the nilcone $\mathrm{nil}_\varphi(\mathcal{A}) := \varphi^{-1}(\mathrm{nil}(\mathcal{B})) \supseteq \mathrm{nil}(\mathcal{A})$ with respect to φ, the singular class $[a]_{\rm sing} := a + \mathrm{Sing}(\mathcal{A})$ of $a \in \mathcal{A}$, the singular class $[a]_\varphi := a + \mathrm{Sing}_\varphi(\mathcal{A}) \subseteq [a]_{\rm sing}$ of $a \in \mathcal{A}$ with respect to φ, the multiplicative class $[a]^\times := a + \mathcal{A}^\times$ of $a \in \mathcal{A}$ and the multiplicative class $[a]^\times_\varphi := a + \mathcal{A}^\times_\varphi \supseteq [a]^\times$ of $a \in \mathcal{A}$ with respect to φ. For a subset $X \subseteq \mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$, a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ with index set $I = I_\varphi = j_\varphi([m]) \subseteq [n]$ or a representation $\lambda : \mathcal{A} \to \mathrm{End}_R(M)$ with an index set $I = I_\lambda := j_\lambda([m]) \subseteq [n]$, subsection 0.1.7 studies the projection closure $\widetilde{X}_{(I)} := \cap_{j \in I} \mathrm{pr}_j^{-1}(\mathrm{pr}_j(X))$ with respect to I, the multiplicative submonoid $\mathcal{A}^\times_{(I)} := \mathrm{pr}_I^{-1}\left(\prod_{j \in I} \mathcal{A}_j^\times\right)$ with respect to I, the singular cone $\mathrm{Sing}_{(I)}(\mathcal{A}) := \mathcal{A} \setminus \mathcal{A}^\times_{(I)}$ with respect to I, the Jacobson radical $J_{(I)}(\mathcal{A}) := \mathrm{pr}_I^{-1}\left(\prod_{j \in I} J(\mathcal{A}_j)\right)$ with respect to I, the nilcone $\mathrm{nil}_{(I)}(\mathcal{A}) := \mathrm{pr}_I^{-1}\left(\prod_{j \in I} \mathrm{nil}(\mathcal{A}_j)\right)$ with respect to I, the singular class $[a]_{(I)} := a + \mathrm{Sing}_{(I)}(\mathcal{A})$ of $a \in \mathcal{A}$ with respect to I, the multiplicative class $[a]^\times_{(I)} := a + \mathcal{A}^\times_{(I)}$ of $a \in \mathcal{A}$ with respect to I. The same subsection shows that the Jacobson radical of a sub-direct product $\mathcal{A}$ of $\mathcal{B}_1, \ldots, \mathcal{B}_m$ satisfies the inclusion $J(\mathcal{A}) \subseteq J\left(\prod_{k=1}^m \mathcal{B}_k\right) = \prod_{k=1}^m J(\mathcal{B}_k)$. For an algebra $\mathcal{A}$ over a commutative ring R and a subset $X \subseteq \mathcal{A}$ with $X \cap \mathcal{A}^\times \neq \emptyset$, subsection 0.1.8 introduces the unital closure $\overline{U}_{\mathcal{A}}(X) := \langle X \cap \mathcal{A}^\times \rangle$ as the subgroup of $\mathcal{A}^\times$, generated by $X \cap \mathcal{A}^\times$. If $\overline{U}_{\mathcal{A}}(X) = X \cap \mathcal{A}^\times$ then X is said to be unitally closed in $\mathcal{A}$. The R-subalgebras $\prod_{k=1}^m \mathrm{End}_R(M_k)$, $\mathrm{End}_{\mathcal{A}}(\oplus_{k=1}^m {}_{\mathcal{A}}M_k)$ of $\mathrm{End}_R(M)$ are shown to be unitally closed in $\mathrm{End}_R(M)$ and $\mathcal{A}^\times_\lambda = \cap_{k=1}^m \mathcal{A}^\times_{\lambda_k}$. If $\varphi : (\mathcal{A}, \mathfrak{M}) \to \mathcal{B}$ is a morphism of a local R-algebra with a nil maximal ideal $\mathfrak{M}$ then $\varphi(\mathcal{A})$ is checked to be unitally closed in $\mathcal{B}$. Subsection 0.1.9 is devoted to the formal characters and the spectral closures. The maximum spectrum of a commutative algebra $\mathcal{A}$ over a commutative ring R is the set $\mathrm{Spm}(\mathcal{A})$ of the maximal ideals $\mathfrak{M}_j$ of $\mathcal{A}$. The homomorphisms $\chi_j : \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{M}_j = \kappa_j$ are called formal characters of $\mathcal{A}$. For any subset $\emptyset \neq \mathcal{I} \subseteq \mathrm{Spm}(\mathcal{A})$, let us consider the diagonal embedding $\Delta_{\mathcal{I}} : \mathcal{A} \hookrightarrow \mathcal{A}^{\mathcal{I}}$ and the R-algebra homomorphism $\chi_{\mathcal{I}} := (\times_{j \in \mathcal{I}} \chi_j) \circ \Delta_{\mathcal{I}} : \mathcal{A} \to \kappa_{\mathcal{I}} := \prod_{j \in \mathcal{I}} \kappa_j$. Subsection 0.1.9 defines and studies the uniform $\mathcal{I}$-spectral closure $\hat{X}_{\mathcal{I}} := X + \ker(\chi_{\mathcal{I}})$ of a subset $X \subseteq \mathcal{A}$, the uniform $\mathcal{I}$-spectral complement $\check{X}_{\mathcal{I}} := \left(\hat{X}_{\mathcal{I}}\right)^{\rm c}$, the full $\mathcal{I}$-spectral projection $X^{\rm sp}_{\mathcal{I}} := \prod_{j \in \mathcal{I}} \chi_j(X)$, the point-wise $\mathcal{I}$-spectral closure $\hat{\hat{X}}_{\mathcal{I}} := \chi_{\mathcal{I}}^{-1}\left(X^{\rm sp}_{\mathcal{I}}\right) = \cap_{j \in \mathcal{I}}(X + \mathfrak{M}_j)$, the point-wise $\mathcal{I}$-spectral complement $\check{\check{X}}_{\mathcal{I}} := \cap_{j \in \mathcal{I}}(X + \mathfrak{M}_j)^{\rm c}$. It discusses the interrelations among the uniform and the point-wise $\mathcal{I}$-spectral closures and complements, on one hand, and the multiplicative submonoid $\mathcal{A}^\times_{\mathcal{I}} := \mathrm{pr}_{\mathcal{I}}^{-1}\left(\prod_{j \in \mathcal{I}} \mathcal{A}_j^\times\right)$ with respect to $\mathcal{I}$, the singular cone $\mathrm{Sing}_{\mathcal{I}}(\mathcal{A}) := \cup_{j\in\mathcal{I}}\mathfrak{M}_j$ with respect to $\mathcal{I}$, the Jacobson radical $J_{\mathcal{I}}(\mathcal{A}) := \cap_{j\in\mathcal{I}}\mathfrak{M}_j$ with respect to $\mathcal{I}$, the nilradical $\mathrm{nil}_{\mathcal{I}}(\mathcal{A}) := \ker(\chi_{\mathcal{I}})$ with respect to $\mathcal{I}$, the singular class $[a]_{\mathcal{I}} := a + \mathrm{Sing}_{\mathcal{I}}(\mathcal{A})$ of $a \in \mathcal{A}$ with respect to $\mathcal{I}$ and the multiplicative class $[a]_{\mathcal{I}}^{\times} := a + \mathcal{A}_{\mathcal{I}}^{\times}$ of $a \in \mathcal{A}$ with respect to $\mathcal{I}$, on the other hand. Subsection 0.1.10 specifies some properties of the spectral closures and complements of subsets of finite direct products of local commutative R-algebras. Subsection 0.1.11 collects some properties of the commutative local rings $(\mathcal{A}, \mathfrak{M})$ and their residue fields $\kappa = \mathcal{A}/\mathfrak{M}$. A commutative local ring $(\mathcal{A}, \mathfrak{M}, \kappa)$ has coefficient field κ if the character $\chi : \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{M} = \kappa$ has such a spectral section $i_{\mathcal{A}} : \kappa \hookrightarrow \mathcal{A}$, for which $\theta_{\mathcal{A}} := \chi_{\mathcal{A}} \circ i_{\mathcal{A}}$ is an automorphism of κ. Subsection 0.1.11 studies the interrelations between $\mathrm{char}(\mathcal{A})$ and $\mathrm{char}(\kappa)$. It explains why any complete equicharacteristic $(\mathcal{A}, \mathfrak{M}, \kappa)$ has coefficient field κ and can be viewed as a κ-algebra. Subsection 0.1.12 recalls that any Artinian R-algebras is of the form $\mathcal{A} \simeq \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{M}_j, \kappa_j)$. In the special case of $\kappa_j = \mathbb{C}$, $\forall 1 \leq j \leq n$, the complex structures J_j on $\mathcal{A}_j$ are unique up to a sign, so that $J = (\pm J_1, \ldots, \pm J_n)$ are all the complex structures on $\mathcal{A}$. For a local Artinian $(\mathcal{A}, \mathfrak{M}, K)$ let $\nu(\mathcal{A})$ be the nilpotency index of $\mathfrak{M}$ and $\mu(\mathcal{A}) := \dim_K(\mathfrak{M}/\mathfrak{M}^2)$. If $\mu, \nu \in \mathbb{N}$ then $K_{\mu,\nu} := K[x_1, \ldots, x_\mu]/(x_1, \ldots, x_\mu)^\nu$ is a local Artinian K-algebra with maximal ideal $\mathfrak{M}_{\mu,\nu} := (\overline{x_1}, \ldots, \overline{x_\mu})$, which is called the jet K-algebra of type (μ, ν). Any Artinian K-algebra $(\mathcal{A}, \mathfrak{M}, K)$ is shown to be a quotient of $(K_{\mu(\mathcal{A}),\nu_o}, \mathfrak{M}_{\mu(\mathcal{A}),\mu_o}, K)$ for $\nu_o \geq \nu(\mathcal{A})$ and any such presentation of $(\mathcal{A}, \mathfrak{M}, K)$ is unique up to a K-automorphism of $K_{\mu(\mathcal{A}),\nu_o}$. Any local κ-algebra $(\mathcal{A}, \mathfrak{M}, \kappa)$ of $\dim_\kappa(\mathcal{A}) = d$ is isomorphic to a κ-algebra of lower triangular Toepliz matrices of order d. If $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{M}_j, \kappa)$ is commutative, $d_j := \dim_\kappa(\mathcal{A}_j)$ and $Z = (z_1 \oplus X_1, \ldots, z_n \oplus X_n) \in \mathcal{A}$ with $z_j \in \kappa$, $X_j \in \mathfrak{M}_j$, $1 \leq j \leq n$, then the spectrum of the matrix Z is shown to be $\sigma(Z) := \{z_1, \ldots, z_n\}$ with algebraic multiplicities d_j of z_j, $\mathrm{tr}(Z) = \sum_{j=1}^{n} d_j \chi_j(Z)$ and $\det(Z) = \prod_{j=1}^{n} \chi_j(Z)^{d_j}$. Making use of a known Inverse Function Theorem for formal power series $f \in \mathcal{A}[[Z]]$ with commutative coefficient ring $\mathcal{A}$, subsection 0.1.12 derives a Formal Implicit Function Theorem, asserting that for any $F(Z, W) \in \mathcal{A}[[Z, W]]$ with $F(Z, 0) = 0$, $\frac{\partial F}{\partial W}(0, 0) \in \mathcal{A}^\times$ there exists a unique $g \in \mathcal{A}[[Z]]$ with $F(Z, g(Z)) \equiv 0$ and $g(0) = 0$. Subsection 0.1.13 discusses the subsections 0.1.4 through 0.1.10 in the special case of a finite dimensional commutative algebra $\mathcal{A}$ over a field K. If ${}_{\mathcal{A}}M = \oplus_{k=1}^{m} {}_{\mathcal{A}}M_k$ is a complete decomposition into indecomposable $\mathcal{A}$-modules ${}_{\mathcal{A}}M_k$, then $\rho : \mathcal{A} \to \mathrm{End}_K(\oplus_{k=1}^{m} {}_{\mathcal{A}}M_k)$ is shown to factor through the convenient co-restriction $\rho^{\mathrm{mor}} = (\rho_1, \ldots, \rho_m) : \mathcal{A} \to \mathcal{B} := \prod_{k=1}^{m}(\mathcal{B}_k, \mathfrak{N}_k, \lambda_k)$, whose local factors $\mathcal{B}_k := \rho_k(\mathcal{A})$ are finite dimensional commutative K-algebras. That allows to interpret the representation setting as a special case of the morphism setting.

Section 0.2 discusses the units group $\mathcal{A}^\times$ of a finite dimensional commutative algebra $\mathcal{A}$ over a field K. Any such $\mathcal{A} \simeq \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{M}_j, \kappa_j)$, so that the study of $\mathcal{A}^\times \simeq \prod_{j=1}^{n} \mathcal{A}_j^\times$ reduces to the description of $\mathcal{A}_j^\times$ for a local $\mathcal{A}_j$. If κ of $\mathrm{char}(\kappa) = 0$ is a coefficient field of $(\mathcal{A}, \mathfrak{M}, \kappa)$ then $\mathcal{A} \simeq \kappa \oplus_\kappa \mathfrak{M}$ and there is an isomorphism $\Phi_{\mathcal{A}} : \mathcal{A}^\times \to \kappa^\times \times (\mathfrak{M}, +)$, $\Phi_{\mathcal{A}}(z \oplus X) = \left(z, \log\left(1 \oplus \frac{X}{2}\right)\right)$, $\forall z \oplus X \in \mathcal{A}$ of algebraic groups. A finite dimensional commutative $\mathbb{R}$-algebra $\mathcal{A}$ has a connected units group $\mathcal{A}^\times$ exactly when $\mathcal{A}$ admits an almost complex structure, which holds exactly when $\chi(\mathcal{A}) \nsubseteq \mathbb{R}$ for any character $\chi : \mathcal{A} \to \mathcal{A}/\mathfrak{M}$, $\mathfrak{M} \triangleleft_{\max} \mathcal{A}$.

Section 0.3 is devoted to the normed algebras over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $(V, ||.||_V)$, $(W, ||.||_W)$ are normed $\mathbb{K}$-vector spaces then the operator norm of $\Phi \in \mathrm{Hom}_{\mathbb{K}}(V, W)$ is $||\Phi||_{\mathrm{op}} := ||\Phi||_{V,W} := \sup\left\{ \frac{||\Phi(v)||_W}{||v||_V} \,\middle|\, v \in V \setminus \{\mathcal{O}_V\} \right\}$. In particular, if $\mathbb{K}^n$ is endowed with the Euclidean norm $||.||_2$, the spectral norm $||A||_{\mathrm{sp}} := ||A||_{2,2}$ on $M_n(\mathbb{K})$ can be expressed by the spectral radius $\rho(A)$ of A, defined as the largest modulus of an eigenvalue of A. Any embedding $\mathcal{A} \hookrightarrow M_n(\mathbb{K})$ of a commutative $\mathbb{K}$-algebra defines the spectral radius $\rho_{\mathcal{A}}(a) := \rho(a)$ of $a \in \mathcal{A} \subseteq M_n(\mathbb{K})$. An algebra norm $||.||_{\mathcal{A}}$ is unital if $||1_{\mathcal{A}}||_{\mathcal{A}} = 1$. The dissertation checks that $\rho_{\mathcal{A}}$ is a unital algebra norm on $\mathcal{A}$. If $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{M}_j, \mathbb{K}_j)$, $\mathbb{K}_j \in \{\mathbb{R}, \mathbb{C}\}$ then the spectral radius of $Z = (z_1 \oplus X_1, \ldots, z_n \oplus X_n) \in \mathcal{A}$ with $z_j \in \mathbb{K}_j$, $X_j \in \mathfrak{M}_j$ is shown to be $\rho_{\mathcal{A}}(Z) = \max\{|z_j| \,|\, 1 \leq j \leq n\}$ and $\rho_{\mathcal{A}}$ is multiplicative if and only if $\mathcal{A}$ is a local ring.

For any closed ideal I in $(\mathcal{A}, ||.||_{\mathcal{A}})$ and $a \in \mathcal{A}$, $||a+I||_{\mathcal{A}/I} := \mathrm{dist}_{\mathcal{A}}(a, I) = \inf\{||a-i||_{\mathcal{A}} \,|\, i \in I\}$ is checked to be an algebra norm on $\mathcal{A}/I$. The normed left modules $(M, ||.||_M)$ over unital $\mathbb{K}$-algebras $(\mathcal{A}, ||.||_{\mathcal{A}})$ are characterized by the operator norms of the images of the continuous representations $\rho : \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$.

Let $\mathcal{A}$, $\mathcal{B}$ be finite dimensional commutative $\mathbb{K}$-algebras and M be a finite dimensional $\mathcal{A}$-module. The differentiation Chapter 1 introduces three types of differentiability of a function $f : U \to M$ or $f : U \to \mathcal{B}$ on an open $U \subseteq \mathcal{A}$ with respect to $\rho : \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ or $\varphi : \mathcal{A} \to \mathcal{B}$ and studies the properties of the corresponding derivatives. The results of Chapters 1 and 2 are announced in the article from the Proceedings of the Bulgarian Academy of Sciences, 2024.

Section 1.1 defines the classical "limit-wise" differentiability, the strict Fréchet differentiability and the Fréchet differentiability in representation setting and morphism setting. For any open $U \subseteq \mathcal{A}$ and $k \in \mathbb{N} \cup \{\infty\}$ there holds $\mathcal{L}^k_\rho(U, M) \subseteq \mathcal{F}^{\mathrm{str,k}}_\rho(U, M) \subseteq \mathcal{F}^k_\rho(U, M) \subseteq \mathcal{F}_{\mathbb{R}}(U, M)$ for the corresponding spaces of ρ-differentiable functions $f : U \to M$. The spaces of the k-times continuously differentiable functions of any of the three types coincide and denoted by $\mathcal{C}^k_\rho(U)$, respectively, $\mathcal{C}^k_\varphi(U)$. The n-times $\mathcal{A}$-differentiable functions $U \to {}_{\mathcal{A}}M$ are those $\mathcal{A}$-differentiable ones, which are n-times $\mathbb{R}$-differentiable. Section 1.1 provides Taylor expansion formulae with Peano remainder term for $f \in \mathcal{F}^n_\rho(U)$ or integral remainder term for $f \in \mathcal{C}^n_\rho(U)$. The sheaves $\mathcal{L}_\rho$, $\mathcal{F}^{\mathrm{str}}_\rho$, $\mathcal{F}_\rho$ are acted by $\mathrm{End}_{\mathcal{A}}(M)$, while $\mathcal{L}_\varphi$, $\mathcal{F}^{\mathrm{srt}}_\varphi$, $\mathcal{F}_\varphi$ consist of $\mathcal{B}$-algebras and satisfy Leibniz rule for the derivative of a product. The chain rule is checked for compositions of some $\mathcal{A}$-differentiable functions with some $\mathcal{B}$-differentiable ones, for compositions of morphisms with $\mathcal{B}$-differentiable functions, for compositions of $\mathcal{A}$-differentiable functions with homomorphisms $\Phi : {}_{\mathcal{A}}M \to {}_{\mathcal{A}}N$, subject to $\mathrm{Ann}_{\mathcal{A}}(N) \subseteq \mathrm{Ann}_{\mathcal{A}}(M)$ in the strict Fréchet differentiable case, as well as for compositions of φ-differentiable functions with morphisms ψ, whose restrictions $\psi|_{\mathrm{im}(\varphi)}$ are injective in the strict Fréchet differential case. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism then any choices of $\mathbb{K}$-bases of $\mathcal{A} \simeq \mathcal{K}^n$, $\mathcal{B} \simeq \mathbb{K}^m$ turns the regular representation $\Upsilon_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{B})$ into a map $\Upsilon_{\mathcal{B}} : \mathcal{B} \hookrightarrow M_{m\times n}(\mathbb{K})$, transforming the derivative of a differentiable at $Z_0 \in \mathcal{A}$ function $f : \mathcal{A} \to \mathcal{B}$ into the Jacobian matrix $\Upsilon_{\mathcal{B}}(f'(Z_0)) = (J_{\mathbb{K}}f)(Z_0) \in M_{m\times n}(\mathbb{K})$ of the $\mathbb{K}$-differentiable map $f : \mathbb{K}^n \to \mathbb{K}^m$. The dissertation illustrates the above equality on three examples of commutative associative structures on $\mathbb{R}^2$ and derives Cauchy-Riemann-Scheffers equations for them. Section 1.1 ends with the Jacobian Conjecture for $\mathcal{A}$-holomorphic maps, asserting that if $\mathcal{A} = \prod_{j=1}^N (\mathcal{A}_j, \mathfrak{M}_j, \mathbb{K})$ is of $\dim_{\mathbb{K}}(\mathcal{A}) = n$ and $P : \mathbb{K}^n \to \mathbb{K}^n$ is an $\mathcal{A}$-holomorphic regular map with $\det(J_{\mathbb{K}}P) = \mathrm{const} \neq 0$ then P is biregular.

Section 1.2 studies the interaction between the regularity classes $\mathcal{L}$, $\mathcal{F}^{\mathrm{str}}$, $\mathcal{F}$ and some categorical constructions. If $\rho = \overline{\rho} \circ \pi$ or $\varphi = \overline{\varphi} \circ \pi$ for an epimorphism $\pi : \mathcal{A} \twoheadrightarrow \overline{\mathcal{A}}$ of $\mathbb{K}$-algebras and $\overline{\rho} : \overline{\mathcal{A}} \to \mathrm{End}_{\mathbb{K}}(M)$ or $\overline{\varphi} : \overline{\mathcal{A}} \to \overline{\mathcal{B}}$ then for any $f \in \mathcal{F}^{\mathrm{str,k}}_\alpha(U)$ or $f \in \mathcal{L}^k_\alpha(U)$ with $\alpha \in \{\rho, \varphi\}$ there exists a unique $\overline{f} \in \mathcal{F}^{\mathrm{str,k}}_{\overline{\alpha}}(\pi(U))$, respectively, $\overline{f} \in \mathcal{L}^k_{\overline{\alpha}}(\pi(U))$ with $f = \overline{f} \circ \pi$ and f admits an $\mathcal{F}^{\mathrm{str,k}}_\alpha$-differentiable or $\mathcal{L}^k_\alpha$-differentiable extension $f^\pi := \pi^*(\overline{f}) = \overline{f} \circ \pi$ to $U^\pi := U + \ker(\pi)$. Similar statements are true for continuously differentiable functions The aforementioned factorization properties do not hold for $\mathcal{F}_\rho$ and $\mathcal{F}_\varphi$. If $\varphi : (\mathcal{A}, \mathfrak{M}, \mathbb{K}) \to (\mathcal{B}, \mathfrak{N}, \mathbb{K})$ is a morphism of local Artinian $\mathbb{K}$-algebras then for any $f \in \mathcal{L}_{\mathcal{A}}(U, \mathcal{B})$ there is a unique $\overline{f} \in \mathcal{L}_{\mathbb{K}}(\chi_{\mathcal{A}}(U), \mathbb{K})$ with $\chi_{\mathcal{B}} \circ f = \overline{f} \circ \chi_{\mathcal{A}}$, $(\chi_{\mathcal{B}} \circ f)' = \overline{f}' \circ \chi_{\mathcal{A}}$ and $\chi_{\mathcal{B}} \circ f$ extends to a function $U + \mathfrak{M} \to {}_{\mathcal{A}}\mathbb{K}$. Section 1.2 establishes the following Theorem for Axiomatic Extension: If $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{M}_j, \mathbb{K}_j)$, ${}_{\mathcal{A}}M = \oplus_{k=1}^m {}_{\mathcal{A}}M_k$ for indecomposable ${}_{\mathcal{A}}M_k$ and $\rho = (\times_{k=1}^m \overline{\rho}_k) \circ \Pi_\rho$ for $\Pi_\rho : \mathcal{A} \to \prod_{k=1}^m \mathcal{A}_{j(k)}$, $\overline{\rho}_k : \mathcal{A}_{j(k)} \to \mathrm{End}_{\mathbb{K}}(M_k)$ then for any $f \in \mathcal{L}_\rho(U)$ there is a unique $\overline{f} = \times_{k=1}^m \overline{f}_k \in \mathcal{L}_{\overline{\rho}}\left(\prod_{k=1}^m \mathrm{pr}_{j(k)}(U)\right)$ with $f = \overline{f} \circ \Pi_\rho$ and $\mathcal{L}_\rho(U) \simeq \mathcal{L}_\rho(\widetilde{U}^\rho)$ as differential $\mathrm{End}_{\mathcal{A}}(M)$-modules for the projection closure $\widetilde{U}^\rho := \cap_{k=1}^m \mathrm{pr}^{-1}_{j(k)}(\mathrm{pr}_{j(k)}(U))$. For a morphism $\varphi : \mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{M}_j, \mathbb{K}_j) \to \mathcal{B} = \prod_{k=1}^m (\mathcal{B}_k, \mathfrak{N}_k, \mathbb{L}_k)$ there is an isomorphism of differential $\mathcal{B}$-algebras $\mathcal{L}_\varphi(U) \simeq \mathcal{L}_\varphi\left(\widetilde{U}^\varphi\right)$. The results of Section 1.2 allow to reduce the study of the $\mathcal{L}_\rho$- and $\mathcal{L}_\varphi$-differentiable functions to the cases of faithful indecomposable modules over local algebras or inclusions of local algebras.

Section 1.3 derives generalized homogeneous Cauchy-Riemann partial differential equations and Cauchy-Riemann-Scheffers equations. Let $\mathcal{A}$ be a commutative $\mathbb{K}$-algebra, $(\alpha^k_{jl})^n_{k,j,l=1}$ be the structure tensor of the multiplication in $\mathcal{A}$ with respect to a $\mathbb{K}$-basis $a_1, \ldots, a_n$, $\rho : \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ be a representation, $(\gamma^k_{jl})^n_{j=1}{}^m_{k,l=1}$ be the structure tensor of the $\mathcal{A}$-action on M with respect to the chosen $\mathbb{K}$-basis of $\mathcal{A}$ and a $\mathbb{K}$-basis $v_1, \ldots, v_m$ of M and $f = f^1 v_1 + \ldots + f^m v_m : U \to M$ be a map of an open subset $U \subseteq \mathcal{A}$ with coordinate functions $f^k : U \to \mathbb{K}$, $1 \leq k \leq m$. Then f is ρ-differentiable at $Z_0 = Z^1_0 a_1 + \ldots + Z^n_0 a_n$ if and only if f is totally $\mathbb{K}$-differentiable at Z_0 and there exists an M-valued function $g := g^1 v_1 + \ldots + g^m v_m$, defined at Z_0, such that $\frac{\partial f^k}{\partial z^j}(Z_0) = \sum_{s=1}^m \gamma^k_{js} g^s(Z_0)$, $\forall 1 \leq j \leq n$, $\forall 1 \leq k \leq m$ and $f'(Z_0) = g(Z_0)$. Moreover, if $\mathcal{A}$ has a unit $1_{\mathcal{A}} = \varepsilon_1 a_1 + \ldots + \varepsilon_n a_n$ then the generalized Cauchy-Riemann equations reduce to $\frac{\partial f^k}{\partial Z^j}(Z_0) = \sum_{r=1}^m \sum_{s=1}^m \varepsilon_r \gamma^k_{js} \frac{\partial f^s}{\partial Z^r}(Z_0)$, $\forall 1 \leq j \leq n$, $\forall 1 \leq k \leq m$ and $f'(Z_0) = \sum_{r=1}^m \varepsilon_r \frac{\partial f}{\partial Z^r}(Z_0)$. The number of the generalized Cauchy-Riemann equations is minimized for the choice of $a_1 = 1_{\mathcal{A}}$, which allows to express all partial derivatives $\frac{\partial f^k}{\partial Z^j}(Z_0) = \sum_{s=1}^m \gamma^k_{js} \frac{\partial f^s}{\partial Z^1}(Z_0)$, $\forall 2 \leq j \leq n$, $\forall 1 \leq k \leq m$ by the "free parameters" $\frac{\partial}{\partial Z^1}$ and $f'(Z) = \frac{\partial f}{\partial Z^1}(Z_0)$. The generalized Cauchy-Riemann-Scheffers equations assert that f is ρ-differentiable at Z_0 if and only if f is totally $\mathbb{K}$-differentiable at Z_0 and $\left[\left(\sum_{r=1}^m \alpha^r_{ij} \frac{\partial}{\partial Z^r}\right) f^k\right](Z_0) = \frac{\partial}{\partial Z^j}\left(\sum_{s=1}^m \gamma^k_{is} f^s\right) Z_0)$, $\forall 1 \leq i, j \leq n$, $\forall 1 \leq k \leq m$. Section 1.3 shows that a totally $\mathbb{K}$-differentiable function $f : U \to M$ belongs to $\mathcal{F}_\rho(U)$ if and only if $f \in \cap_{1 \leq i < j \leq n} \ker(d_{ij})$ for the generalized Wirtinger operators $d_{ij}(f) := a_i \frac{\partial f}{\partial Z^j} - a_j \frac{\partial f}{\partial Z^i}$, $1 \leq i, j \leq n$. In the special case of $a_1 = 1_{\mathcal{A}}$, $f \in \mathcal{F}_\rho(U)$ if and only if $f \in \cap_{j=2}^n \ker(d_{1j})$ and the operators d_{1j} are derived also by fixing $Z_1 = Z^1 a_1 + \ldots + Z^n a_n$ and choosing such $Z_2, \ldots, Z_n$ that $\frac{\partial Z_i}{\partial Z_j} = \delta_{ij}$, $1 \leq i, j \leq n$.

Section 1.4 associates to $f \in \mathcal{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ the differential 1-form $\omega := \rho(dZ) f(Z) \in \Omega^1(U, {}_{\mathcal{A}}M)$ and serves as a prelude to the next integration chapter. By definition, $f : U \to {}_{\mathcal{A}}M$ is $\mathcal{L}_\rho$-, $\mathcal{F}^{\mathrm{str}}_\rho$-, $\mathcal{F}_\rho$- or $\mathcal{C}^1_\rho$-integrable if there exists $F : U \to {}_{\mathcal{A}}M$ from the corresponding differentiability class with $F' = f$. For any loop $\gamma \in \mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, U)$, Section 1.4 shows that $\oint_\gamma \rho(dZ) f(Z) = 0$ whenever $f \in \mathcal{C}^0(U, {}_{\mathcal{A}}M)$ and $\left|\left|\oint_\gamma \rho(dZ) f(Z)\right|\right|_M \leq ||f||_{\gamma,M} L_\rho(\gamma) \leq ||f||_{\gamma,M} L_{\mathcal{A}}(\gamma)$ for $f \in \mathcal{C}^0(\gamma, {}_{\mathcal{A}}M)$, $L_\rho(\gamma) := \oint ||\rho(\gamma'(t))||_{M,M}\, dt$, $L_{\mathcal{A}}(\gamma) := \oint_{\mathbb{S}^1} ||\gamma'(t)||_{\mathcal{A}} dt$. The dissertation provides two pre-Morera ρ-integrability criteria for $f \in \mathcal{C}^0(U, {}_{\mathcal{A}}M)$. The first one assumes that $U \subseteq \mathcal{A}$ is a connected open subset and $\oint \rho(dZ) f(Z) = 0$, $\forall \gamma \in \mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, U)$. The second one requires $U \subseteq \mathcal{A}$ to be a star-convex open subset with center $u_0 \in U$ and $\oint_{\partial \triangle} \rho(dZ) f(Z) = 0$ for all triangles $\triangle \subseteq U$ with vertex u_0. The dissertation reproves a result of Waterhouse from 1992, asserting that $f \in \mathcal{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ if and only if $f \in \mathcal{F}_{\mathbb{K}}(U, {}_{\mathcal{A}}M)$ and $d_{\mathbb{K}}(\rho(dZ) f(Z)) = 0$ for $d_{\mathbb{R}} = d$ or $d_{\mathbb{C}} = \partial$. Section 1.4 establishes Cauchy-Goursat Integral Theorem $\oint_{\partial \triangle} \rho(dZ) f(Z) = 0$ for any connected open subset $U \subseteq \mathcal{A}$, any $Z_1, \ldots, Z_s \in U$, any $f \in \mathcal{F}_{\mathcal{A}}(U \setminus \{Z_1, \ldots, Z_s\}), {}_{\mathcal{A}}M)$ and any triangle $\triangle \subseteq U$. Similarly, the homological Cauchy Integral Theorem claims that $\int_\Gamma \rho(dZ) f(Z) = 0$ for any $f \in \mathcal{C}^1_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ and any 1-cycle $\Gamma \in Z_1(U, \mathbb{Z})$, homologous to $[\Gamma] = 0 \in H_1(U, \mathbb{Z})$. The homotopic Cauchy Integral Theorem establishes that $\oint_\gamma \rho(dZ) f(Z) = 0$ for all $f \in \mathcal{C}^1_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ and all $\gamma \in \mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, U)$. The functions $f : U \to {}_{\mathcal{A}}M$, integrable infinitely many times are shown to form $\mathcal{F}_\rho(U, {}_{\mathcal{A}}M)$ for a star-convex open subset $U \subseteq \mathcal{A}$. The continuous $f : U \to {}_{\mathcal{A}}M$, integrable infinitely many times constitute $\mathcal{C}^1_\rho(U, {}_{\mathcal{A}}M)$ for an open homologically 1-connected $U \subseteq \mathcal{A}$.

Chapter 2 of the dissertation introduces the generalized index of a loop or an 1-cycle, in order to derive various Generalized Cauchy Integral Formulae. The proofs from the first three sections appeared in an article from Serdica Mathematical Journal 2026 in the special case of a local $(\mathcal{A}, \mathfrak{M}, \mathbb{C})$, without the additional sophistication by the means of representations.

Section 2.1 describes the admissible points and singular 1-cycles, needed for the existence of a generalized Cauchy integration kernel. It prepares the spectral homotopy argument, used in the next section for an explicit calculation of the generalized relative index of a loop. Let $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{M}_j, \mathbb{C})$ be a complete decomposition of a $\mathbb{C}$-algebra $\mathcal{A}$ of $\dim_{\mathbb{C}}(\mathcal{A}) < \infty$ into local factors, ${}_{\mathcal{A}}M = \oplus_{k=1}^{m} {}_{\mathcal{A}}M_k$ be a complete decomposition into indecomposable ${}_{\mathcal{A}}M_k$, $(\rho_1, \ldots, \rho_m) = (\times_{k=1}^{m}\overline{\rho_k}) \circ \Pi_\rho$ for $\Pi_\rho : \mathcal{A} \to \prod_{k=1}^{m} \mathcal{A}_{j_\rho(k)}$, $\overline{\rho_k} : \mathcal{A}_{j_\rho(k)} \to \mathrm{End}_{\mathbb{C}}(M_k)$, $\rho_k = \overline{\rho_k} \circ \mathrm{pr}_{j_\rho(k)}^{\mathcal{A}}$, $I_\rho := \rho([m]) \subseteq [n]$. The spectral closure $\hat{U} := \cap_{k=1}^{m}\chi_{j_\rho(k)}^{-1}\left(\chi_{j_\rho(k)}(U)\right) = \cap_{k=1}^{m} U + \mathfrak{M}_{j_\rho(k)}$ of an open subset $U \subseteq \mathcal{A}$ is a strong deformation retract of the full spectral projection $U^{\mathrm{sp}} := \prod_{k=1}^{m} \mathrm{pr}_{j_\rho(k)}(U)$, so that $H_1\left(\hat{U}, \mathbb{Z}\right) \simeq H_1\left(U^{\mathrm{sp}}, \mathbb{Z}\right) \simeq \oplus_{k=1}^{m} H_1\left(\chi_{j_\rho(k)}(U), \mathbb{Z}\right)$. The I_ρ-forbidden zone of $\gamma \in \mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, U)$ is $\mathcal{F}_\rho(\gamma) := \{Z \in \mathcal{A} \,|\, \exists j \in I_\rho : \chi_j(Z) \in \chi_j(\gamma)\}$ for $\chi_j : \mathcal{A} \twoheadrightarrow \mathcal{A}_j/\mathfrak{M}_j = \mathbb{C}$ and the I_ρ-admissible zone is $\mathrm{Adm}_\rho(\gamma) := \mathcal{F}_\rho(\gamma)^{\mathrm{c}}$. For $\Gamma := \sum_{l=1}^{p} g_l \gamma_l \in Z_1(U, G)$ one defines $\mathcal{F}_\rho(\Gamma) := \cup_{l=1}^{p} \mathcal{F}_\rho(\gamma_l)$ and $\mathrm{Adm}_\rho(\Gamma) := \mathcal{F}_\rho(\Gamma)^{\mathrm{c}}$. Loops $\gamma, \delta \in \mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, U)$ are spectrally homotopic if $\chi_{j_\rho(k)}(\gamma)$ and $\chi_{j_\rho(k)}(\delta)$ are homotopic in $\chi_{j_\rho(k)}(U) \subseteq \mathbb{C}$ for all $1 \le k \le m$. The spectral fundamental group $\pi_1^{\mathrm{sp}}(U)$ is the quotient of $\mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, U)$ by the equivalence relation of the spectral homotopy. Note that $\chi_j : U \to \chi_j(U) \subseteq \mathbb{C}$, $j \in I_\rho$ induce maps $(\chi_j)_\sharp : B_1(U, G) \to B_1(\chi_j(U), G)$ of the 1-boundaries and allow to define the space $B_1^{\mathrm{sp}}(U, G) := \cap_{j \in I_\rho} (\chi_j)_\sharp^{-1}(B_1(\chi_j(U), G))$ of the spectral 1-boundaries, whereas the first spectral homology group $H_1^{\mathrm{sp}}(U, G) := Z_1(U, G)/B_1^{\mathrm{sp}}(U, G)$.

Section 2.2 studies the generalized index $\mathrm{Ind}_\rho^{\mathbb{Z}}(\gamma, Z_0) := \frac{1}{2\pi i}\oint_\gamma \frac{\rho(dZ)}{\rho(Z - Z_0)} \in \mathrm{End}_{\mathbb{C}}(M)$, $Z_0 \in \mathrm{Adm}_\rho(\gamma)$, respectively, $\mathrm{Ind}_\varphi^{\mathbb{Z}}(\gamma, Z_0) := \frac{1}{2\pi i}\oint_\gamma \frac{\varphi(dZ)}{\varphi(Z - Z_0)} \in \mathcal{B}$, $Z_0 \in \mathrm{Adm}_\varphi(\gamma)$. These definitions extend to 1-cycles over $\mathbb{Z}$ and 1-cycles over an arbitrary abelian group G. For $[\Gamma_1] = [\Gamma_2] \in H_1\left([Z_0]_\rho^\times, \mathbb{Z}\right)$, $[Z_0]_\rho^\times := Z_0 + \rho^{-1}(\mathrm{GL}_{\mathcal{A}}(M))$ there holds $\mathrm{Ind}_\rho^{\mathbb{Z}}(\Gamma_1, Z_0) = \mathrm{Ind}_\rho^{\mathbb{Z}}(\Gamma_2, Z_0)$. If $\Gamma \in Z_1(U, \mathbb{Z})$ is $\mathcal{C}^1_{\mathrm{pw}}$-regular and $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ are morphisms then $\mathrm{Ind}_{\psi\circ\varphi}^{\mathbb{Z}}(\Gamma, Z_0) = \mathrm{Ind}_\psi^{\mathbb{Z}}\left(\varphi_\sharp(\Gamma), \varphi(Z_0)\right) = \psi\left(\mathrm{Ind}_\varphi^{\mathbb{Z}}(\Gamma, Z_0)\right)$, $\forall Z_0 \in \mathrm{Adm}_{\psi\circ\varphi}(\Gamma)$. Whenever $\varphi = \left(\times_{j=1}^{n}\varphi_j\right) : \mathcal{A} = \prod_{j=1}^{n}\mathcal{A}_j \to \prod_{j=1}^{n}\mathcal{B}_j = \mathcal{B}$ corresponds to a complete system of idempotents $\{e_j^{\mathcal{B}} \,|\, 1 \le j \le n\}$, there holds $\mathrm{Ind}_\varphi^{\mathbb{Z}}(\Gamma, Z_0) = \sum_{j=1}^{n} e_j^{\mathcal{B}} \mathrm{Ind}_{\varphi_j}^{\mathbb{Z}}(\mathrm{pr}_j(\Gamma), \mathrm{pr}_j(Z_0))$, $\forall Z_0 \in \prod_{j=1}^{n}\mathrm{Adm}_{\varphi_j}(\mathrm{pr}_j(\Gamma))$. If $(\mathcal{A}, \mathfrak{M}, \mathbb{C})$ is a local $\mathbb{C}$-algebra then $\mathrm{Ind}_\varphi^{\mathbb{Z}}(\Gamma, Z_0) = \mathrm{ind}(\chi(\Gamma), \chi(Z_0))$ is the classical index in $\chi(\mathcal{A}) = \mathcal{A}/\mathfrak{M} = \mathbb{C}$ and the application of this property to an endomorphism $\varphi : (\mathcal{A}, \mathfrak{M}, \mathbb{C}) \to (\mathcal{A}, \mathfrak{M}, \mathbb{C})$ yields the d-closedness of $X^k dX \in \Omega^1_{\mathrm{hol}}(\mathfrak{M}, \mathfrak{M})$, $1 \le k \le \nu - 2$ for the nilpotency index ν of $\mathfrak{M}$. For a local $(\mathcal{B}, \mathfrak{N}, \mathbb{C})$ there follows $\mathrm{Ind}_\varphi^{\mathbb{Z}}(\Gamma, Z_0) \in \mathbb{Z}$, $\forall Z_0 \in \mathrm{Adm}_\varphi(\Gamma)$. If $\varphi = (\varphi_1, \ldots, \varphi_m) = (\times_{k=1}^{m}\overline{\varphi_k}) \circ \Pi_\varphi : \mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{M}_j, \mathbb{C}) \to \prod_{k=1}^{m}(\mathcal{B}_k, \mathfrak{N}_k, \mathbb{C}) = \mathcal{B}$ for $\Pi_\varphi : \mathcal{A} \to \prod_{k=1}^{n}\mathcal{A}_{j_\varphi(k)}$, $\overline{\varphi_k} : \mathcal{A}_{j_\varphi(k)} \to \mathcal{B}_k$, $\varphi_k = \overline{\varphi_k} \circ \mathrm{pr}_{j_\varphi(k)}^{\mathcal{A}}$ then $\mathrm{Ind}_\varphi^{\mathbb{Z}}(\Gamma, Z) = \sum_{k=1}^{m} e_k^{\mathcal{B}} \mathrm{ind}\left(\chi_{j_\varphi(k)}(\Gamma), \chi_{j_\varphi(k)}(Z_0)\right) \in \mathbb{Z}^m \subset \mathbb{C}^m \hookrightarrow \mathcal{B}$, $\forall Z_0 \in \mathrm{Adm}_\varphi(\Gamma)$. Since the index $\mathrm{Ind}_\varphi^G(\Gamma, Z_0) = \mathrm{Ind}_\varphi^G(\Gamma, Z_0 + X_0)$ is invariant under translations by $X_0 \in \mathrm{nil}_\varphi(\mathcal{A}) := \varphi^{-1}(\mathrm{nil}(\mathcal{B}))$, $\mathrm{Ind}_\varphi^G(\Gamma, \ )$ induces a map $\mathrm{Adm}_\varphi(\Gamma)/\mathrm{nil}_\varphi(\mathcal{A}) \to G^m$. For arbitrary $\varphi, \psi : (\mathcal{A}, \mathfrak{M}, \mathbb{C}) \to (\mathcal{B}, \mathfrak{N}, \mathbb{C})$, $\Gamma \in Z_1(U, G)$, $Z_0 \in \mathrm{Adm}_{\mathcal{A}}(\Gamma)$ there holds $\mathrm{Ind}_{\mathcal{B}}^G\left(\varphi_\sharp(\Gamma), \psi(Z_0)\right) = \mathrm{Ind}_{\mathcal{A}}^G(\Gamma, Z_0) = \mathrm{ind}_{\mathbb{C}}^G\left(\chi(\mathcal{A}), \chi_{\mathcal{A}}(Z_0)\right)$. The locally constant $\mathrm{Ind}_\rho^G(\Gamma, \ ) : \mathrm{Adm}_\rho(\Gamma) \to G$ has level sets of the form $\prod_{j=1}^{n}\mathcal{A}_j'$, where $\mathcal{A}_j' = C_j \times \mathfrak{M}_j$ for $j \in I_\rho$ and a level set C_j of the locally constant function $\mathrm{ind}(\chi_j(\Gamma), \ )$ or $\mathcal{A}_j' = \mathcal{A}_j$ for $j \in [n] \setminus I_\rho$. The set $\mathcal{C}_\rho(\Gamma) := \{Z \in \mathrm{Adm}_\rho(\Gamma) \,|\, \mathrm{Ind}_\rho^{\mathbb{Z}}(\Gamma, Z) \in \mathrm{GL}_{\mathbb{C}}(M)\}$ is identified with $\prod_{j=1}^{n}\mathcal{A}_j'(\chi_j(\Gamma))$, where $\mathcal{A}_j'(\chi_j(\Gamma)) = C_j(\chi_j(\Gamma)) \times \mathfrak{M}_j$ for $j \in I_\rho$, $C_j(\chi_j(\Gamma) := \{z_j \in \mathbb{C} \setminus \chi_j(\Gamma) \,|\, \mathrm{ind}(\chi_j(\Gamma), z_j) \ne 0\}$ or $\mathcal{A}_j'(\chi_j(\Gamma) = \mathcal{A}_j$ for $j \in [n] \setminus I_\rho$. If $[\gamma] = [\delta] \in \pi_1\left([Z_0]_\rho^\times\right)$ then $\mathrm{Ind}_\rho^{\mathbb{Z}}(\gamma, Z_0) = \mathrm{Ind}_\rho^{\mathbb{Z}}(\delta, Z_0)$, which holds exactly when $[\gamma] = [\delta] \in \pi_1^{\mathrm{sp}}\left([Z_0]_\rho^\times\right)$. Similarly, $\mathrm{Ind}_\rho^{\mathbb{Z}}(\Gamma, Z_0) = \mathrm{Ind}_\rho^{\mathbb{Z}}(\Delta, Z_0)$ for $[\Gamma] = [\Delta] \in H_1^{\mathrm{sp}}(U, \mathbb{Z})$, $\forall Z_0 \in \check{U} := \cap_{j \in I_\rho}(U + \mathfrak{M}_j)^{\mathrm{c}}$.

Section 2.3 is on some versions of Cauchy's Integral formula and consequences of the Cauchy

Transform. The Local Cauchy Integral formula states that $\mathrm{Ind}^{\mathbb{Z}}_{\rho}(\gamma, Z)f(Z) = \frac{1}{2\pi i}\oint_{\gamma}\frac{\rho(dW)}{\rho(W-Z)}f(W)$ for a polydisc $\mathcal{P} \subseteq \mathcal{A}$, $f \in \mathcal{O}_{\rho}(\mathcal{P}, {}_{\mathcal{A}}M)$, $\gamma \in \mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, \mathcal{P})$, $Z \in \mathcal{P} \cap \mathrm{Adm}_{\rho}(\gamma)$. Towards the Nil-Radical Cauchy Integral Formula, let $V := \prod_{j=1}^{n} V_j$ with arbitrary open $V_j \subseteq \mathcal{A}_j$ for $j \in [n] \setminus I_{\rho}$ or $V_j = \{c \in \mathbb{C} \,|\, |c| < r_j\} \times \mathfrak{M}_j$ for $j \in I_{\rho}$ and some $r_j \in \mathbb{R}^{>0}$. Then for any $f \in \mathcal{O}_{\rho}(V, {}_{\mathcal{A}}M)$, $\gamma \in \mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, V)$ with $0 \in \mathrm{Adm}_{\rho}(\gamma)$, $\mathrm{Ind}^{\mathbb{Z}}_{\rho}(\gamma, 0) = 1$ and $X \in \mathrm{nil}_{\rho}(\mathcal{A}) := \rho^{-1}(\mathrm{nil}\,(\mathrm{End}_{\mathbb{C}}(M)))$ there holds $f(X) = \frac{1}{2\pi i}\oint_{\gamma}\frac{\rho(dW)}{\rho(W-X)}f(W)$. That results in a Local Automatic Analytic Continuation $\widehat{f}(Z + X) := \frac{1}{2\pi i \mathrm{Ind}^{\mathbb{Z}}_{\rho}(\gamma,Z)}\oint_{\gamma}\frac{\rho(dW)}{\rho(W-Z-X)}f(W)$ to the spectral closure $\hat{\mathcal{P}} := \cap_{j \in I_{\rho}}(\mathcal{P} + \mathfrak{M}_j)$ of a polydisc $\mathcal{P} \subseteq \mathcal{A}$ for any $f \in \mathcal{O}_{\rho}(\mathcal{P}, {}_{\mathcal{A}}M)$, $\gamma \in \mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, \mathcal{P})$, $Z \in \mathcal{P} \cap \mathcal{C}_{\rho}(\gamma)$, $X \in \mathrm{nil}_{\rho}(\mathcal{A})$. The corresponding Global Automatic Analytic Continuation is stated as an isomorphism $\mathcal{O}_{\rho}(U) \simeq \mathcal{O}_{\rho}(\hat{U})$ of $\mathcal{A}$-modules or an isomorphism $\mathcal{O}_{\varphi}(U) \simeq \mathcal{O}_{\varphi}(\hat{U})$ of $\mathcal{B}$-algebras. Thus, the domains of $\mathcal{A}$-holomorphy are of the form $D \times \mathfrak{M}$ for a domain $D \subseteq \mathbb{C}$. For a connected open $U \subseteq \mathcal{A}$, $f \in \mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M)$, $[\Gamma] = [0] \in H_1^{\mathrm{sp}}(U, \mathbb{Z})$ and $Z \in U \cap \mathrm{Adm}_{\rho}(\Gamma)$ there holds the Spectrally Null-Homological Generalized Cauchy Formula $\int_{\Gamma}\rho(dW)f(W) = 0$. That implies the Spectrally Null-Homotopical Cauchy Integral Formula $\mathrm{Ind}^{\mathbb{Z}}_{\rho}(\gamma, Z)f(Z) = \frac{1}{2\pi i}\int_{\gamma}\frac{\rho(dW)}{\rho(W-Z)}f(W)$ for a connected open $U \subseteq \mathcal{A}$, $f \in \mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M)$, $[\gamma] = [0] \in \pi_1^{\mathrm{sp}}(U)$, $Z \in U \cap \mathrm{Adm}_{\rho}(\gamma)$. For any connected open $U \subseteq \mathcal{A}$ the Spectrally Null-Homological Cauchy Differentiation Formula asserts that $\mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M) = \mathcal{O}^{\infty}_{\rho}(U, {}_{\mathcal{A}}M)$ and $\mathrm{Ind}^{\mathbb{Z}}_{\rho}(\Gamma, Z)f^{(k)}(Z) = \frac{k!}{2\pi i}\int_{\Gamma}\frac{\rho(dW)}{\rho(W-Z)^{k+1}}f(W)$ for $[\Gamma] = [0] \in H_1^{\mathrm{sp}}(U, \mathbb{Z})$, $Z \in U \cap \mathrm{Adm}_{\rho}(\Gamma)$. Combining with the Automatic Analytic Continuation, one concludes that for a local $(\mathcal{A}, \mathfrak{M}, \mathbb{C})$, $f \in \mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M)$, $Z \in \hat{U}$, $X \in \mathfrak{M}$ and the nilpotency index ν of $\mathfrak{M}$ there holds $\widehat{f}(Z + X) = \sum_{k=0}^{\nu-1}\frac{\rho(X)^k}{k!}f^{(k)}(Z)$. For a connected open $U \subseteq \mathcal{A}$, $f \in \mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M)$, $\Gamma \in B_1^{\mathrm{sp}}(U, \mathbb{Z})$, $Z_0 \in U \cap \mathcal{C}_{\rho}(\Gamma)$ the Global Cauchy Inequality over $\mathcal{A}$ asserts that $||f^{(k)}(Z_0)||_2 \leq \frac{k!}{2\pi}\left|\left|\frac{1}{\mathrm{Ind}^{\mathbb{Z}}_{\rho}(\Gamma,Z_0)}\right|\right|_{\mathrm{sp}} L_{\rho}(\Gamma)\left|\left|\frac{1}{\rho(Z-Z_0)^{k+1}}\right|\right|_{\mathrm{sp},\Gamma}||f||_{2,\Gamma}$, where ${}_{\mathcal{A}}M$ is endowed with the Euclidean norm $||.||_2$. The Central Cauchy Inequality over $\mathcal{A}$ is $||f^{(k)}(Z_0)||_M \leq \frac{k!}{r^k}||f||_{M,\gamma_r}$ for a sufficiently small $r \in \mathbb{R}^{>0}$ and $\gamma_r(t) := Z_0 + re^{2\pi i t}$, $\forall t \in [0, 1]$. The aforementioned two types of Cauchy's Inequality over $\mathcal{A}$ imply a Liouville Theorem over $\mathcal{A}$ and a Scalar Maximum Modulus Principle over $\mathcal{A}$. Section 2.3 provides also two Weierstrass Convergence Theorems for $\{f_n\}_{n=1}^{\infty} \subset \mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M)$ on a connected open $U \subseteq \mathcal{A}$. If $f_n \to f$ locally uniformly then $f \in \mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M)$ and $f'_n \to f'$ locally uniformly. If $\sum_{k=1}^{n} f_k \to f$ locally normally then $f \in \mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M)$ and $\sum_{k=1}^{n} f'_k \to f'$ locally normally. For a local Artinian $\mathbb{C}$-algebra $(\mathcal{A}, \mathfrak{M}, \mathbb{C})$, a finite dimensional ${}_{\mathcal{A}}M$, a finite measure $\mu : S \to M$, a measurable function $\gamma := g \oplus G : S \to \mathbb{C} \oplus \mathfrak{M} = \mathcal{A}$ and such an open $U \subseteq \mathcal{A}$ that $\inf_{s \in S}(|g(s) - z| - ||\rho(G(s)) - X||_{\mathrm{End}}) > 0$, $\forall Z = z \oplus X \in U \subseteq \mathbb{C} \oplus \mathfrak{M} = \mathcal{A}$, the Cauchy Transform over $\mathcal{A}$ provides a ρ-analytic function $f(Z) := \int_S \rho(\gamma(s) - Z)^{-1}d\mu(s)$ on U with $\frac{f^{(k)}(Z)}{k!} = \int_S \rho(\gamma(s) - Z)^{-(k+1)}d\mu(s)$. In particular all $f \in \mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M)$ are ρ-analytic in U. There holds Morera Theorem $\oint_{\gamma}\rho(dZ)f(Z) = 0$ for $f \in \mathcal{O}_{\rho}(U, {}_{\mathcal{A}}M) \cap \mathcal{C}^0(U, {}_{\mathcal{A}}M)$, $\gamma \in \mathcal{C}^1_{\mathrm{pw}}(\mathbb{S}^1, U)$.

Section 2.4 is devoted to a Generalized Cauchy-Pompeiu Integral Formula over a compact connected orientable surface $\Sigma \subset \mathcal{A}$ in a general position with respect to the radical $\mathfrak{M}$ of a local $\mathbb{C}$-algebra $(\mathcal{A}, \mathfrak{M}, \mathbb{C})$ of $\dim_{\mathbb{C}}(\mathcal{A}) < \infty$. More precisely, let ν be the nilpotency index of $\mathfrak{M}$ and Σ be a $\mathcal{C}^{\nu+1}$-surface with $\mathcal{C}^1_{\mathrm{pw}}$-boundary $\partial\Sigma$. A point $W_0 \in \Sigma \setminus \partial\Sigma$ is $\mathcal{A}$-transverse when the intersection $(W_0 + \mathfrak{M}) \cap \Sigma$ is transverse at W_0. Any $\mathcal{A}$-transverse point W_0 has an $\mathcal{A}$-transverse open neighbourhood $S \ni W_0$ in Σ, on which $\chi_{\Sigma} := \chi|_{\Sigma} : S \to \chi_{\Sigma}(S)$ has a $\mathcal{C}^{\nu+1}$-diffeomorphic section $W : \chi_{\Sigma}(S) \to S \subseteq \Sigma \subseteq \mathcal{A}$, $W(w) = w \oplus Y(w) \in \mathbb{C} \oplus \mathfrak{M} = \mathcal{A}$, called a spectral local parametrization. If $f \in \mathcal{C}^{k+1}(U, {}_{\mathcal{A}}M)$ and $W \in \mathcal{C}^{k+2}(\chi_{\Sigma}(S), \mathcal{A})$ then $\lim_{\varepsilon \to 0}\frac{1}{2\pi i}\int_{\partial\mathbb{D}_{\Sigma}(W(w_0),\varepsilon)}\frac{dZ}{(z-w_0)^{k+1}}f(Z) = \frac{1}{k!}\frac{\partial^k}{\partial w^k}|_{w_0}\left(\frac{\partial W}{\partial w}(w)f(W(w))\right)$. If $W \in \Sigma \setminus \partial\Sigma$ and $(W + \mathfrak{M}) \cap \Sigma = \{p_l \,|\, 1 \leq l \leq m\}$ is transverse then any p_l has a local $\mathcal{A}$-transverse neighbourhood S_l with a spectral local parametrization $W_l : \chi_{\Sigma}(S_l) \to S_l$.

One can choose all S_l to have one and a same $S^{\rm sp} := \chi_\Sigma(S_l)$ and to view $W_l(w)$ as a spectral local parametrization of the $\mathcal{A}$-transverse sheet S_l of the m-sheeted covering $\chi_\Sigma : \coprod_{l=1}^m S_l \to S^{\rm sp}$. Then $\varepsilon_l := \deg(\chi_\Sigma|_{S_l}) \in \{\pm 1\}$, depending on whether χ_Σ preserves or reverses the orientation on S_l. The Global "Spectral" Cauchy-Pompeiu Formula asserts that $\sum_{l=1}^m \varepsilon_l \frac{\partial W_l}{\partial w}(w) f(W_l(w)) = \frac{1}{2\pi i}\int_{\partial\Sigma} \frac{dZ}{z-w} f(Z) + \frac{1}{2\pi i}\int_\Sigma \left[\frac{dz\wedge dX}{(z-w)^2} f(Z) + \sum_{j=1}^n \left(\frac{dZ\wedge dz_j}{z-w}\frac{\partial f}{\partial z_j}(Z) + \frac{dZ\wedge d\overline{z_j}}{z-w}\frac{\partial f}{\partial \overline{z_j}}(Z)\right)\right]$, $\forall w \in S^{\rm sp}$. The Generalized Cauchy-Pompeiu Formula over $\mathcal{A}$ is $\sum_{l=1}^m \varepsilon_l f(W_l(w)) = \frac{1}{2\pi i}\int_{\partial\Sigma}\frac{dZ}{Z-W} f(Z) + \frac{1}{2\pi i}\sum_{j=2}^n \int_\Sigma \frac{dZ\wedge dZ_j}{Z-W}\frac{\partial f}{\partial Z_j}(Z) + \frac{1}{2\pi i}\sum_{j=1}^n \int_\Sigma \frac{dZ\wedge d\overline{Z_j}}{Z-W}\frac{\partial f}{\partial \overline{Z_j}}(Z)$ at all $w \in S^{\rm sp}$.

The appendix collects some facts about the structure tensors of finite dimensional algebras, their morphisms and representations. Section A1 provides several necessary and sufficient conditions for a finite dimensional algebra $\mathcal{A}$ over a field K to be unital, associative or commutative, in terms of its structure tensor. The same section gives a necessary and sufficient condition for a monomorphism $\lambda : \mathcal{A} \hookrightarrow M_{n\times n}(K)$ to be unital and shows that the commutativity of $\lambda(\mathcal{A})$ suffices for its associativity. Section A2 introduces the representing matrix $\Phi \in M_{m\times n}(K)$ of a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ with respect to a K-basis $a_1, \ldots, a_n$ of $\mathcal{A}$ and a K-basis $b_1, \ldots, b_m$ of $\mathcal{B}$, as well as the matrices $\Gamma_j \in M_{m\times m}(K)$, respectively, $\widehat{\Gamma_j} \in M_{m\times m}(K)$, representing the left, respectively, right multiplications by $\varphi(a_j)$ on $\mathcal{B}$. The matrices $\Gamma_j, \widehat{\Gamma_j}$, $1 \leq j \leq n$ are expressed by the structure constants of $\mathcal{B}$ and by Φ. The compatibility of $\varphi : \mathcal{A} \to \mathcal{B}$ with the multiplication, $\varphi(1_\mathcal{A}) = 1_\mathcal{B}$, the associativity of $\mathcal{B}$ and the commutativity of $\mathcal{B}$ are formulated in terms of the structure tensors of $\mathcal{A}$, $\mathcal{B}$ and by Φ, Γ_j, $\widehat{\Gamma_j}$, $1 \leq j \leq n$. Special attention is paid to the case of $a_1 = 1_\mathcal{A}$, $b_1 = 1_\mathcal{B}$, with or without an assumption on injectiveness or surjectiveness of φ. Section A2 provides also the transformation laws of Φ, Γ_j and the structure tensor of $\mathcal{A}$ under changes of the K-basis of $\mathcal{A}$ and the K-basis of $\mathcal{B}$. Section A3 expresses the associativity of $\mathcal{A}$ by the structure constants of $\mathcal{A}$ and of a representation $\lambda : \mathcal{A} \to {\rm End}_K(M)$.

4 Approbation of the results

The results of the dissertation are published in two articles. One of them appears in the Proceedings of the Bulgarian Academy of Sciences in 2024 with IF from the Q4 and earns 24 points, according to Web of Science. The other one is published in Serdica Mathematical Journal in 2026, which is with SJR and is estimated by 20 points. In such a way, Marin Genov has 44 points and not only fulfils but exceeds considerably the minimal national requirements of 30 points under Art. 2b, paragraphs 2 and 3 of the Act on Development of the Academic Staff in the Republic Bulgaria, as well as the additional requirements of the Bulgarian Academy of Sciences for acquisition of the educational and scientific degree "Doctor" in the scientific area and the professional field of the procedure. The results of the dissertation have been reported at 3 international conferences and 10 national scientific forums.

The results, presented by the candidate in the dissertation work and the scientific works to it do not repeat such from previous procedures for acquisition of a scientific degree or academic position. There is no plagiarism in the submitted dissertation work and the scientific papers under this procedure, proven in the legally established order.

5 Qualities of the abstract

The abstract in Bulgarian reflects truthfully the results and the content of the dissertation. It formulates all the statements from the thesis and the main contributions. The abstract provides also the approbation of the results, a declaration on originality and a part of the bibliography.

6 Conclusion

Having become acquainted with the PhD thesis presented in the procedure and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific and applied contributions contained therein, **I confirm** that the presented PhD thesis and the scientific publications to it, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the Act on Development of the Academic Staff in the Republic of Bulgaria, the Rules for its Implementation and the corresponding Rules of the Bulgarian Academy of Sciences for acquisition by the candidate of educational and scientific degree "Doctor" in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics. In particular, the candidate meets the minimal national requirements in the professional field and no plagiarism has been detected in the scientific papers submitted for the competition.

Based on the above, **I strongly recommend** the scientific jury to award

Marin Vladimirov Genov

the educational and scientific degree "Doctor" in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.5. Mathematics, Doctoral program "Geometry and Topology".

September 14, 2026

Reviewer:

(Prof. PhD Azniv Kasparian)