Review

by Prof. Dr. Tony Pantev

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on the applications submitted for the position of Associate Professor for the needs of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences announced in State Newspaper No. 106 on December 17, 2024.

Area of higher education: *4. Natural Sciences, Mathematics and Informatics*,

Professional field: *4.5 Mathematics*,

Scientific speciality: *"Geometry and Topology" (Tropical geometry)*

I am an external member of the scientific panel for this search. The panel was convened by order No 15/17.02.2025 of the Director of the Institute of Mathematics and Informatics Corresponding Member of BAS, Prof. D.Sc. Peter Boyvalenkov. The only application for the position was submitted by Dr. Mikhail Shkolnikov, a Staff Researcher at IMI-BAS. As a member of the scientific panel, I have received from Dr. Shkolnikov all the administrative and scientific documents required by the Act on the Development of the Academic Staff in the Republic of Bulgaria (ADASRB), the Rules for its implementation, and the Rules on the terms and conditions for awarding of academic degrees and occupying of academic positions at the Bulgarian Academy of Sciences.

1 Biographical data

Dr. Mikhail Shkolnikov was born on October 4, 1991. From 2008 to 2012 he was enrolled in the bachelor program of the St Peterburg State University majoring in Mathematics. From 2012 to 2013 he was enrolled in the Pure Math Masters Program at the University of Geneva. In 2013 he graduated with an M.Sc. from the University of Geneva and in 2013 he was accepted as doctoral student at the Department of Mathematics of the University of

Geneva. He defended his Ph.D. in 2017. From 2017 to 2019 he was a postdoctoral fellow at ISTA, Vienna, Austria, and from 2019 to 2020 he was a postdoctoral fellow at the University of Geneva. Since 2023 he is a staff researcher in ICMS of IMI-BAS.

2 Scientific work

The research of Dr. Shkolnikov is organized in two main streams - work on quantum link and knot invariants, and research in tropical geometry. Both research directions are vast subjects in their own right, and comprise some of the the most interesting open problems in algebraic topology and algebraic geometry that are currently been investigated by the mathematical community. On the quantum topology side, Shkolnikov's work is well known and recognized as one of the best sources of sharp counterexamples and explicit formulas in knot theory. On the tropical side, Shkolnikov is one of the prime movers in the field of tropical geometry, with original works contributing equally to the conceptual development and applicational utility of the theory. A distinguishing feature of his research program here is that it mainly targets the most challenging and unexplored ideas of the tropicalization toolbox. This constant pushing of the boundaries of the subject, frequently leads Shkolnikov to unexpected discoveries and allows him to further develop tropicalization as a modelling method. Shkolnikov is one of the principal architects of two fascinating new fields in this framework - tropical optics and non-abelian tropicalization - and has made significant advances in the study of tropical and abelian sandpiles. His research papers are always on the forefront of their focus topics, and are geared toward applications in a variety of subjects including representation theory, enumerative geometry, quantum knot invariants, discrete and complex dynamics, non-commutative harmonic analysis and the geometry of special functions.

The works submitted to the competition fall roughly into four general streams, which I will briefly describe next:

(i) *Sandpile models.* A major portion of Shkolnikov's research program advances the analysis of the behavior of cellular automata. In the landmark paper [8] Lang and Shkolnikov study the shape of the possible scaling limits of abelian sandpile models and propose a powerful novel framework for codifying the sandpile evolution under harmonic fields. The new

idea here is to enhance the sandpile model by allowing real number of particles at boundary vertices. This framework is similar and formally dual to affine $\mathbb{R}$-buildings associated to non-archimedean Lie groups and indeed Lang and Shkolnikov show that the harmonic fields become closed geodesics in a connected Lie group associated with the enhanced sandpile and that they provide intrinsic coordinatization which allows for direct comparison of sandpile configurations in different domains. This gives a way of viewing classical abelian sandpiles as discretizations of enhanced ones and allows the authors to show that the usual sandpile group is renormalizable. In a followup project [10] Lang and Shkolnikov investigate the one dimensional consequences of the functorial behavior of their enhanced sandpile groups. They discover that the discretization process produces monomorphisms between tiling groups for usual sandpiles which are encoded in a new type of tiling of convex polyforms in the plane. This yields a new elegant combinatorial model of limits of usual sandpile groups.

In a different direction, Shkolnikov and collaborators constructed in [4] the only known continuous tropical sandpile model with self-organized criticality. The model is extremely appealing as it arises as a scaling limit of the classical sandpile model, and provides a new universal way to recognize an describe critical phenomena and pattern formation across a number of fields including algebraic geometry, algebraic topology, quantum field theory, and quantum statistical mechanics.

Shkolnikov also has a number of important works written jointly with Kalinin. These include the very interesting paper [9] in which they prove the existence of simple junctions of sandpile solitons, the fundamental paper [5] developing the wave dynamics of tropical series, and the paper [3] describing the stable states on the heptagonal tiling of hyperbolic space. In addition Shkolnikov also has a very recent and important solo paper [12] in which he proposes a power law that should be obeyed by the number of steps in the deepest tropical hypersurface degenerations.

(ii) *Tropical optics.* Another major theme of Shkolnikov's current research activity is the study of wavefronts and caustics in tropical geometry. This is a new area which was introduced and shaped into its current state by Shkolnikov and his collaborators, and is one of the most exciting and original modern developments in tropical geometry. Among the highlights of Shkolnikov's contributions to this subject is his remarkable joint paper with Kailinin [7] which gives explicit algorithms for summing tropical series over convex domains via the study of tropical caustics of Delzant polytopes. Another somewhat technical but

conceptually illuminating work is Shkolnikov's recent joint paper with Mikhalkin on on two dimensional tropical optics.

(iii) *Non-abelian tropicalization.* Perhaps the most challenging part of Dr. Shkolnikov's research program has to do with constructing and understanding the mechanics of tropicalization and phase tropicalization for affine subvarieties of complex reductive groups and symmetric spaces. The foundations of this theory and the first classes of non-trivial examples are contained in the recent paper of Mikhalkin and Shkolnikov [11] in which they classify tropical limits of hyperbolic amoebas of curves in terms of $\mathbb{H}^3$-spherical complexes, and the very recent work of Shkolnikov and Petrov [13] in which the simplest version of non-abelian phase tropicalization is studied i a fascinating $\mathbb{P}SL_n(\mathbb{C})$-model. It is clear that these results and constructions are just the beginning of a long reaching and very rich research program.

(iv) *Quantum invariants in low dimensional topology and representation theory.* A very fruitful research direction pursued by Dr. Shkolnikov is his spectacular work on quantum knot invariants and his construction of quantum groups as symmetries of impurities of states in an interacting quantum system.

Shkolnikov's papers [1] and [2], written jointly with Duzhin, prove a cornerstone results in lower dimensional topology. In [1] a necessary algebraic condition was proven for a knot to admit a diagram with paired matching crossings. Using this condition Duzhin and Shkolnikov were able to construct several examples of knots that do not admit a matching diagram, solving one of the Kirby problems which was previously open for 25 years. In [2] the authors give a succinct explicit formula for the HOMFLY polynomial of a rational link in terms of a continuous fraction expansion of the rational number that defines the link. The formula is elegant and very easy to work with especially when compared to a previous formula due to Nakabo, which is quite unwieldy and messy.

Finally, in the paper [6], written jointly with Yakaboylu and Lemeshko, Shkolnikov develops a very original approach to the study of quantum many-body systems, in which a quantum group arises as symmetries of impurities and allows one to effectively compute both strong and weak coupling limits of the theory.

3 Publications and impact

Dr. Shkolnikov has an impressive publication record and has submitted to the competition a collection of thirteen publications of extremely high scientific quality and interdisciplinary impact. Shkolnikov's papers are carefully designed, perfectly executed, and are a model for mathematical rigor and conceptual clarity. His results have been published in high profile professional journals including the Proceedings of the National Academy of Sciences of the US, the European Journal of Mathematics, Communications of Mathematical Physics, Journal of Knot Theory and its Ramifications, Bulletin of the London Mathematical Society, Fundamenta Mathematicae, and Comptes Rendus Mathématique.

4 Conclusion

The papers submitted by Dr. Mikhail Shkolnikov to this competition unequivocally demonstrate that he is a true leader in tropical geometry and quantum topology. His work has had and will continue to have a trasformative impact on representation theory, enumerative geometry, low dimensional topology, singularity theory, and mathematical physics. Shkolnikov's scientific achievements are more than sufficient to satisfy the requirements for an appointment as an Associate Professor at any high profile academic institution.

Based on all this, I strongly recommend to the scientific panel to approve the candidacy of Dr. Shkolnikov and to propose to the Scientific Council of IMI-BAS to appoint Dr. Mikhail Stanislavovich Shkolnikov as an Associate Professor in the **area of higher education:** 4. Natural sciences, mathematics and informatics; **professional field:** 4.5 Mathematics; **scientific specialty:** Geometry and Topology (Tropical geometry).