S T A T E M E N T
on the competition for the acquisition
of the academic position " Associate Professor"
in Professional Direction 4.5 Mathematics
Scientific Specialty "Geometry and Topology" (Tropical Geometry)
at the Institute of Mathematics and Informatics (IMI)
of Bulgarian Academy of Sciences (BAS)
announced in St. Gaz. 106/17.12.2024 and at the Website of IMI-BAS

The statement is written by Azniv Kirkor Kasparian, Section of Algebra, Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", professional direction 4.5 Mathematics, as a member of the scientific juri for the contest, according to Order 15/17.02.2025 of the Director of the Institute of Mathematics and Informatics at Bulgarian Academy of Sciences.

The only applicant for the announced competition is Mikhail Svetoslavovich Shkolnikov, Ph.D. from IMI-BAS. He participates in the contest with 13 articles. One of the articles is standalone and twelve are joined. To the best of my knowledge, all co-authors of the joint publications have equipollent contributions. Mikhail Shkolnikov has 42 noticed citations, from which 20 are provided for the competition. Besides copies of the diplomas for acquisition of Ms.D. and Ph.D. in Mathematics from University of Geneva, the transcripts include evidences for research work experience, an author's summary of the scientific contributions, a list of noticed citations and a reference for compliance with the minimal national requirements of the Law on the Development of Academic Staff of Republic Bulgaria, the Rules on its implementation and the Rules on the Terms and Conditions for Acquisition of Academic Degrees and Occupation of Academic Positions at IMI-BAS.

Mikhail Shkolnikov held post-doc positions at the Institute of Science and Technology in Klosterneuburg, Austria and University of Geneva, Switzerland. From 2023 he works at IMI-BAS. He was a member of three contracts of Swiss National Science Foundation. Mikhail Shkolnikov participates in the competition with 13 articles, from which ten are with Impact Factor. I am very impressed from the fact that five of them are from Q1 and one is from Q2. There are also two articles from Q3, two from Q4, two with SJR and one in a refereed and indexed journal. In such a way, Mikhail Shkolnikov satisfies and even exceeds the minimal national requirements for the occupation of the academic position "Associate Professor" and the specific requirements of IMI-BAN.

Let us proceed with content analysis of the articles of Mikhail Shkolnikov for the competition. The classical phase-forgetful tropicalization $\mathrm{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}^n$, defined as $\mathrm{Log}_t(z_1, \ldots, z_n) = (\log_t(|z_1|), \ldots, \log_t(|z_n|))$ for some $t \in \mathbb{R}_{>0}$ is the quotient map by the maximal torus $\left(S^1\right)^n$ of $(\mathbb{C}^*)^n$. As its non-commutative analogue, Grigory Mikhalkin introduces the quotient map $\kappa : \mathrm{PSL}_2(\mathbb{C}) \to \mathbb{H}^3$ by $\mathrm{SO}(3)$ onto the three-dimensional real hyperbolic space $\mathbb{H}^3$ and its extension $\overline{\kappa} : \overline{\mathrm{PSL}_2(\mathbb{C})} \to \overline{\mathbb{H}^3}$ to the corresponding closures. An article of Grigory Mikhalkin and Mikhail Shkolnikov from the "Bulletin of the London Mathematical Society" reviews some results on classical unparameterized tropical curves. It studies the phase-tropical convergence of the amoebas of nodal punctured Riemann surfaces by the means of extended dual graphs and their images in $\mathbb{R}^n$. Making use of appropriate finite graphs, called $\mathbb{H}^3$-floor diagrams and their associated $\mathbb{H}^3$-tropical spherical complexes in $\overline{\mathbb{H}^3}$, the authors describe the κ-tropical convergence of scaled families of irreducible complex algebraic curves in $\overline{\mathrm{PSL}_2(\mathbb{C})} = \mathbb{P}^3(\mathbb{C})$. The article studies also the amoebas of the complex projective surfaces $S \subset \overline{\mathrm{PSL}_2(\mathbb{C})}$ by showing that $\overline{\kappa}(S)$ contains always the boundary $\partial\mathbb{H}^3$. When S is different from the quadric $Q = \partial\overline{\mathrm{PSL}_2(\mathbb{C})}$, the complement $\mathbb{H}^3 \setminus \kappa(S)$ turns to be an open, geodesically convex subset of $\mathbb{H}^3$. Surfaces S of odd degree have $\kappa(S) = \mathbb{H}^3$. The work introduces the left PSL_2-Gauss map γ_S^- of the smooth locus of $S \cap \mathrm{PSL}_2(\mathbb{C})$ in $\mathbb{P}\left(T_e^*\mathrm{PSL}_2(\mathbb{C})\right) = \mathbb{P}^2(\mathbb{C})$ and studies the behaviour of γ_N^- for planes $N \subset \overline{\mathrm{PSL}_2(\mathbb{C})}$. The smooth critical points $z \in S$ of $\kappa|_S$ are characterized as the ones with $\gamma_S^-(z) \in \mathbb{P}\left(T_e\kappa^{-1}(0)\right) \simeq \mathbb{P}^2(\mathbb{R})$.

After announcing that the tropical limit of hyperbolic amoebas $\kappa(S_\alpha) \subset \mathbb{H}^3$ of surfaces $S_\alpha \subset \mathrm{PSL}_2(\mathbb{C})$, $\alpha \in A$ is the complement to an open ball, centered at $0 \in \mathbb{H}^3$, an article of Mikhail Shkolnikov and Peter Petrov from "Comptes redndus de'l Académie bulgare des Sciences" introduces a new type of a phase evaluation map VAL. More precisely, if $\mathbb{K}$ is the field of the real-power Puiseux series of one variable with complex coefficients, then $\mathrm{VAL} : \overline{\mathrm{PSL}_2(\mathbb{K})} \to \overline{\mathrm{PSL}_2(\mathbb{C})}$ relates the corresponding closures. For an arbitrary complex projective variety $V \subset \overline{\mathrm{PSL}_2(\mathbb{C})}$ without irreducible components in Q, the image of $V(\mathbb{K})$ under VAL is described by the means of the spherical coamoeba map and the circle bundle, induced by the double coamoeba map.

In 2010 S. Caracciolo, G. Paoletti and A. Sportiello introduce the notion of a sandpile and study its behaviour under waves. An article of Nikita Kalinin and Mikhail Shkolnikov from "Journal of Knot Theory and Its Ramifications" describes explicitly the relaxations of perturbations $\phi_m^P = \phi_m + \sum_{v \in P} \delta_v$ of maximal stable states ϕ_m on the vertices of a heptagonal hyperbolic lattice, whose distance to the origin is bounded above by m. For any $m \in \mathbb{N}$, the difference of the masses of ϕ_m^P and of the result under the relaxation is shown to be a positive constant. Recall that the toppling function $\mathcal{T}_\psi$ of a state ψ associates to a vertex v of the lattice the number $\mathcal{T}_\psi(v) \in \mathbb{Z}_{\geq 0}$ of the topplings at v, during the relaxation of ψ. The article obtains an explicit formula for $\mathcal{T}_{\phi_m^P}(v)$, depending only on the distances of v and P to the origin. In such a way, $\mathcal{T}_{\phi_m^P}$ turns to be independent of the cardinality of P and of the positions of the points of P.

An article of Nikita Kalinin and Mikhail Shkolnikov from " Discrete and Continuous Dynamical Systems" develops a dynamic of tropical series. For an appropriate closed convex set $\Omega \subset \mathbb{R}^2$, the authors define the Ω-tropical series as functions $f : \Omega \to \mathbb{R}_{\geq 9}$ with $f|_{\partial\Omega} \equiv 0$, which are of the form $f(x, y) = \inf_{(i,j)\in\mathcal{A}}(ix + jy + a_{ij})$ for a subset $\mathcal{A} \subseteq \mathbb{Z}^2$ and $a_{ij} \in \mathbb{R}$. The tropical curve $C(f) := P \cap \mathrm{Int}(\Omega)$ of f is the intersection of the interior of Ω with the set P of the non-smooth points of f. Let $V(\Omega, P, f)$ be the set of those Ω-tropical series $g : \Omega \to \mathbb{R}_{\geq 0}$ with non-smooth points P, for which $g|_\Omega \geq f$. The article introduces the wave operator G_P, acting by the rule $G_P(f)(x) = \inf \{g(x) \,|\, g \in V(\Omega, P, f)\}$ and studies its dynamic. The operators G_P are lifted to tropical polynomials in two variables with coefficients from a field $\mathbb{K}$ of characteristic $\mathrm{char}(\mathbb{K}) = 2$, endowed with a valuation $\mathrm{val} : \mathbb{K} \to \mathbb{R}$. The article studies the behaviour of the faces of $\Omega \setminus C(f)$ under the action of the wave operator G_P on f. The admissible Ω are approximated by a canonical family of $\mathbb{Q}$-polygons and the dynamic of nice Ω-tropical series $f : \Omega \to \mathbb{R}_{\geq 0}$ is reduced to the dynamic of tropical series on $\mathbb{Q}$-polygons, by the means of a sequence of blow ups.

An article of Nikita Kalinin and Mikhail Shkolnikov from "Communications in Mathematical Physics" classifies the periodic line-shaped movable states in a sandpile model and calls them solutions. After proving the existence of solutions ϕ, the authors use discrete superharmonic functions $G_n : \mathbb{Z}^2 \to \mathbb{Z}$, $\mathcal{T}_\phi - n \leq G_n \leq \mathcal{T}_\phi$, which coincide with $\mathcal{T}_\phi$ outside a finite neighbourhood of the deviation set $D(\mathcal{T}_\phi) := \left\{(x, y) \in \mathbb{Z}^2 \,|\, \Delta\mathcal{T}_\phi \neq 0\right\}$, in order to define the smoothings $S_n(\mathcal{T}_\phi) : \mathbb{Z}^2 \to \mathbb{Z}$, $n \in \mathbb{N}$ of $\mathcal{T}_\phi$. The smoothings of appropriately given explicit $\phi_{\mathrm{edge}}, \psi_{\mathrm{vertex}}, \psi_{\mathrm{node}} : \mathbb{Z}^2 \to \mathbb{Z}$ are shown to stabilize after finitely many steps. The article has an appendix, which develops a theory of sandpiles on infinite domains.

An article of Mikhail Shkolnikov from "Communications in Mathematics" shows that the relaxation on a 1-dimensional domain terminates after finitely many steps. Experiments suggest that the number of the relaxation steps is subject to a power law.

An article of Nikita Kalinin, Aldo Guzmán-Sáenz, Yulieth Pierto, Mikhail Shkolnikov, Vera Kalinina and Ernesto Lupercio from "Proceedings of the National Academy of Sciences of the United States of America" provides a tropical model with self-organized criticality, which is a sort of a scaling limit of the sandpile model. This model is continuous, i.e., not a cellular automation. The article establishes experimentally by a computer simulation that the number $N(s)$ of the avalanches of size s obeys a power law $\log(N(s)) = -1.2\log(s) + c$. Let Ω be a square in $\mathbb{R}^2$ and φ be the state with 3 grains at any point of $\Omega \cap \mathbb{Z}^2$, except on a large subset $P \subset \Omega \cap \mathbb{Z}^2$, at which the points have 4 grains. The article shows that the toppling function $\mathcal{T}_\varphi$ is the scaling limit of the pointwise minimum of concave, piecewise-linear functions with integral slopes, which are non-negative over Ω, vanish over $\partial\Omega$ and are not smooth at P.

An article of Nikita Kalinin and Mikhail Shkolnikov from "European Journal of Mathematics" studies compact convex domains Ω by associating to them the tropical curve C_Ω, defined by the lattice distance F_Ω to $\partial\Omega$. The convex polygons $\Omega_t := F_\Omega^{-1}[t,+\infty)$ can be viewed as level sets of propagation of wave front, so that to interpret the tropical curve C_Ω as its caustic. Let X be a smooth compact symplectic toric surface and $\mu : X \to \mathbb{R}^2$ be the moment map of the Hamiltonian torus action on X. Then $\Delta = \mu(X)$ is known to be a Delzant polygon, whose vertices are the fixed points of the aforementioned action. The work establishes that any irreducible divisor D of X maps to a side $s = \mu(D)$ of $\Delta = \mu(X)$. If s_1, s_2 are the edges of C_Δ, coming out of the ends of s, then s_1, s_2 and the continuation of s form a Delzant triangle if and only if $D^2 = -1$. For a side s of Δ let $\lambda_s : \Delta \to \mathbb{R}_{\geq 0}$ be the distance from the line, containing s and $F_\Delta(p)$ be the minimum of $\lambda_s(p)$ over all the sides s of Δ. Then C_Δ is the corner locus of F_Δ. If m_Δ is the maximum of the continuous, concave, piecewise-linear functions F_Δ then for any $0 < \varepsilon < m_\Delta$ there is a polygon Δ_ε with boundary $\partial\Delta_\varepsilon = F_\Delta^{-1}(\varepsilon)$, which is shown to be a Delzant polygon for $0 \leq \varepsilon < m_\Delta$. If the boundary $\partial\Omega = F_\Omega^{-1}(0)$ of a compact convex domain $\Omega \subset \mathbb{R}^2$ has no corners then the intermediate level sets $F_\Omega^{-1}(\varepsilon)$, $0 < \varepsilon < m_\Omega$ are proved to be Delzant polygons.

An article of Moritz Lang and Mikhail Shkolnikov from "Procedings of the National Academy of Sciences of the United States of America" analyzes the evolution of the sandpile identity under harmonic fields of order at most 4. A stable configuration in the sandpile model is recurrent if it appears infinitely many times in the Markov process of dropping a particle at any vertex with positive probability and relaxing the resulting states. The set of the recurrent configurations forms the abelian sandpile group G with respect to vertex-wise addition of particles, followed by a relaxation. In order to show that the sandpile dynamic possesses scaling limits, the authors introduce the extended sandpile model by allowing the boundary vertices of a convex domain Γ to have a real number of grains. The recurrent configurations for the extended sandpile model constitute a Lie group $\widetilde{G}$, on which the harmonic fields define closed geodesics. The usual sandpile group G is a subgroup of $\widetilde{G}$, which can be viewed as a discretization of $\widetilde{G}$. The quotient $\widetilde{G}/G \simeq (\mathbb{R}/\mathbb{Z})^{\partial\Gamma} \simeq \left(S^1\right)^{\partial\Gamma}$ is a torus. Let $\mathcal{H}$ be the space of the real-valued harmonic functions on $\mathbb{Z}^2$ and $\mathcal{H}_\mathbb{Z}$ be its subspace of the integral-valued harmonic functions. For an arbitrary exhausting injective family of domains $\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_n \subset \Gamma_{n+1} \subset \ldots \subset \mathbb{Z}^2$, $\cup_{n\in\mathbb{N}}\Gamma_n = \mathbb{Z}^2$, the extended sandpile groups $\widetilde{G}_n$ of Γ_n have canonical epimorphisms $\widetilde{G}_{n+1} \to \widetilde{G}_n$, whose projective limit $\widetilde{G}_\infty := \lim_{n\to\infty} \widetilde{G}_n$ is shown to be independent on the choice of $\{\Gamma_n\}_{n\in\mathbb{N}}$. The natural inclusion $\mathcal{H}/\mathcal{H}_\mathbb{Z} \subset \widetilde{G}_\infty$ provides universal coordinates and induces renormalizations of Γ_n.

Another article of Moritz Lang and Mikhail Shkolnikov from "Comptes Rendus Mathématique" associates to a directed colored tiling of a polyform P_2 by a polyform P_1 an injective homomorphism $G_{P_1} \to G_{P_2}$ of the corresponding sandpile groups. More precisely, if M is the unique tiling of $\mathbb{R}^2$ by isoceles triangles with base of length 1 and height of length $\frac{1}{2}$ then an M-polyform P is a finite connected subset of triangles of M. Let P_1^{DC} be an M-polyform, whose edges are directed and colored in such a way that each edge has a different color. Then a directed colored tiling of an M-polyform P_2 by P_1 is such a tiling of P_2 by copies of P_1^{DC}, for which any common edge of two adjacent tiles has the same color and direction in these tiles. By appropriate examples, the authors illustrate that the correspondence, associating to a directed colored tiling of P_2 by P_1^{DC} a monomorphism $G_{P_1} \to G_{p_2}$ is neither injective, nor surjective. For an arbitrary convex open subset $P \subseteq \mathbb{R}^2$ (not necessarily an M-polyform), the intersection $\Gamma = P \cap \mathbb{Z}^2$ is called a convex domain. Let Δ_Γ be the reduced graph Laplacian and $\mathcal{H}_R^\Gamma$ with $R = \mathbb{Z}$ or $R = \mathbb{Q}$ be the R-module of the harmonic functions $\Gamma \to R$. If $\mathcal{H}_G^\Gamma$ is the quotient of $\left\{H \in \mathcal{H}_\mathbb{Q}^\Gamma \mid \Delta_\Gamma H|_{\partial\Gamma} \in \mathbb{Z}^{\partial\Gamma}\right\}$ by $\mathcal{H}_\mathbb{Z}^\Gamma$ then $-\Delta_\Gamma : \mathcal{H}_G^\Gamma \to G_\Gamma$ is an isomorphism with the sandpile group G_Γ, so that G_Γ embeds in $\mathcal{H}_\mathbb{Q}^\Gamma/\mathcal{H}_\mathbb{Z}^\Gamma$ as a subgroup, whose quotient is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\partial\Gamma}$. For any finite convex domain $\Gamma \subset \mathbb{Z}^2$, the order of the sandpile group G_Γ is shown to be the absolute value of the determinant of the potential matrix of a basis of $\mathcal{H}_\mathbb{Z}^\Gamma$.

An article of Sergei Duzhin and Mikhail Shkolnikov from "Arnold Mathematical Journal" provides two explicit formulae for the HOMFLY-polynomial P_n of a rational link, associated with a continued fraction with n even denominators. They are derived from a recurrent relation on P_n, P_{n-1} and P_{n-2}.

Another article of Sergei Duzhin and Mikhail Shkolnikov from "Fundamenta Mathematicae" establishes the existence of knots, which are not bipartite, i.e., have not matched diagrams. To this end, the Alexander matrix of a bipartite knot K is shown to be of the form $I_n + \left(t + t^{-1} - 2\right) B$ for some $B \in M_{n\times n}(\mathbb{Z})$ and if an Alexander ideal $I_m(K) \neq \mathbb{Z}\left[t, t^{-1}\right]$ is non-trivial then $1+t \notin I_m(K)$. Thus, if a knot K has $1 + t \in I_2(K) \neq \mathbb{Z}\left[t, t^{-1}\right]$ then K is not bipartite. The aforementioned result provides a series of examples of knots, which do not have matched diagrams and are not bipartite.

An article of Enderalp Yakaboylu, Mikhail Shkolnikov and Mikhail Lemeshko from "Physical Review Letters" applies the quantum group approach for studying the interaction of a linear rigid rotor with a bath of bosons. The symmetry group of an isolated rotor is SO(3). After immersing in a many-particle environment, the resulting object is described by the quantum group $\mathrm{SO}_q(3)$. Many-particle interactions result in "dressings" of the rotor by a cloud of bosonic excitations, which induce renormalization of the rotor's rotational constant B to $B^* < B$. The quantum group $\mathrm{SO}_q(3)$ is a non-commutative deformation of the universal enveloping algebra of the Lie algebra of SO(3). If $q = \mathrm{e}^{\sqrt{-1}\tau}$ for $\tau \in \mathbb{R} \cup \sqrt{-1}\mathbb{R}$ then $B^* = B\cos(3\tau)$ and the main result of the article provides an explicit formula for the deformation parameter τ . Though derived for weak interactions, the announced formula for τ is valid for arbitrary values of the impurity-bath coupling strength. The article compares the quantum group approach with a variational method, as well as with the diagrammatic Monte Carlo technique.

After getting acquainted with the materials and the scientific works, presented for the competition, and based upon the aforementioned analysis of their scientific significance and applicability, I confirm that the scientific contributions comply with the Law on Development of the Academic Staff of Republic Bulgaria, the Rules on its Implementation and the Rules on the Terms and Conditions for Acquisition of Academic Degrees and Occupation of Academic Positions at the Institute of Mathematics and Informatics of Bulgarian Academy of Sciences, for occupation by the applicant of the academic position " Associate Professor" in the scientific field and the professional direction of the contest. In particular, the applicant satisfies the minimal national requirements in the professional direction and no plagiarism was found in the presented scientific works. That is why,

I evaluate positively the application of Mikhail Svetoslavovich Shkolnikov.

Based upon the aforementioned, **I strongly recommend** the Scientific Juri to propose the appropriate election authority of the Institute of Mathematics and Informatics at Bulgarian Academy of Sciences to elect

Mikhail Svetoslavovich Shkolnikov
as an " Associate Professor" in Professional Direction 4.5 Mathematics
Scientific Specialty "Geometry and Topology" (Tropical Geometry)
at the Institute of Mathematics and Informatics of Bulgarian Academy of Sciences.

March 20, 2025

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