R E P O R T

by Prof. D.Sc. Johann Todorov Davidov
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on a competition for occupying the academic position of **Associate Professor**

area of higher education: *4. Natural Sciences, Mathematics and Informatics*,
professional field *4.5 Mathematics*
, scientific speciality *Geometry and Topology*
(Tropical geometry)

for the needs of the Institute of Mathematics and Informatics,
Bulgarian Academy of Scinecs

announced in State Gazette No. 106 of December 17, 2024 and the websites of IMI and BAS

I am a member of the scientific panel for this procedure according to order No. 15/17.02.2025 of the Director of the Institute of Mathematics and Informatics Prof. D.Sc. Peter Boyvalenkov. Documents for participation in the announced competition have been submitted only by Dr. Mikhail Stanislavovich Shkolnikov. As a member of the scientific panel, I have received from Dr. Shkolnikov all the administrative and scientific documents required by the Act on the Development of the Academic Staff in the Republic of Bulgaria (ADASRB), the Rules for its implementation and the Rules on the terms and conditions for awarding of academic degrees and occupying of academic positions at Bulgarian Academy of Sciences.

Personal data

Dr. Mikhail Shkolnikov was born in 1991. In the period 2008 - 2012, he was a student at the Saint Petersburg State University. In 2013, he graduated from a Master of Science in Mathematics Program of the University of Geneva. Then he was a Ph.D. student at this university. In 2017, he received Ph.D. degree in Mathematics from the University of Geneva. He had postdoctoral positions at the Institute of Science and Technology, Klosterneuburg, Austria, 2017 - 2019, and at the University of Geneva, 2019 - 2020. Since 2023, he has held a research position at the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences.

General characterization of the scientific works of the applicant.

The candidate M. Shkoolnikov has submitted 13 scientific papers for participation in the competition. These papers meet the minimal requirements according to ADASRB. Shkolnikov is the only author of one of them. He has 1 co-author in ten papers, 2 co-authors in one paper, 5-co-authors in one paper. I assume that each author contributes equally to the joint papers. The 13 papers submitted by M. Shkolnikov for the competition have been quoted 20 times by other authors. The list of all publications by the candidate includes 16 papers quoted 42 times (without self-citations).

The papers presented by M. Shkolnikov have not been used for awarding the scientific degree ”doctor” or occupying an academic position.

I would like also to mention that I have not discovered any plagiarism.

Analysis of the scientific achievements of the applicant

In this brief review of the candidate's results, we will follow the numbering of the papers in his list of publications for the competition.

Papers Nos. 1 and 2 deal with the knot theory. In my opinion, the most impressive result is the proof in paper No. 1 of a conjecture by J. Prezytycki that has been opened for 24 year. In paper No. 2, an explicit formula for the HOMFLY polynomial (named after J. Hoste, A. Ocneanu, K. Millet, P.J. Freyd, W.B. R. Lickorish, D.N. Yetter) of rational knots and links is obtained.

Papers Nos. 3, 4, 5, 8, 9, 10, 12 explore different aspects of the sandpile model dynamics and its relations with tropical geometry.

Recall that in tropical geometry the addition and multiplication of real numbers are replaced by the operations $x + y = max\{x, y\}$ and $xy = x + y$ for $x, y \in \mathbb{R} \cup \{-\infty\}$. A tropical polynomial is a function of the form $(*)$ $F(x, y) = max\{a_{ij} + ix + jy : \ (i, j) \in \mathcal{A}\}$ where $\mathcal{A}$ is a finite subset of $\mathbb{Z}^2$ and $a_{ij} \in \mathbb{R}$. The set of points (x, y) where the graph of the function F is not smooth is called a tropical curve, the reason for this name being that such a curve is a "tropicalization" of an algebraic curve. In fact, a tropical curve is a graph of a special type. If the set $\mathcal{A}$ in $(*)$ is infinite and the corresponding F is locally a tropical polynomial, the function F is called a tropical series. The tropical curve associated with a tropical series is a locally finite graph.

In paper No. 3, sandpile models on graphs embedded in the hyperbolic plane are studied. In paper No. 4, it is discussed a sandpile model that has a nice interpretation in terms of tropical geometry. In paper No. 5, the theory of tropical series and associated tropical curves is developed and explored in various aspects. Paper No. 8 deals with the sandpile dynamics induced by a harmonic field. In paper No. 9, a smoothing process is defined to prove existence of a sandpile model with rational slopes. A fact well-known only experimentally is also proved in this paper. In paper No. 12, a sandpile model defined by a multi-dimensional tropical series is considered and, in dimension 1, it is shown that a relaxation is achieved after a finite number of applications of a naturally defined state-deforming operator (in general, the relaxation process is only known to converge).

In papers Nos. 8 and 10, sandpile groups are studied. Such a group is an Abelian group that encodes certain combinatorial properties of the sandpile model. In paper No. 8, the sandpile group of the harmonic model discussed in the paper is considered. The main result of paper No. 10 gives a sufficient condition for existence of monomorphisms between sandpile groups defined on certain domains of $\mathbb{Z}^d$.

It is shown in paper No. 6 that quantum groups can be used to simplify quantum many-particle problems.

In paper No. 7, the method of exhaustion for finding areas is put in the framework of tropical and toric geometry.

The notion of an amoeba of an algebraic variety V in $(C \setminus \{0\})^n$ has been introduced by Gelfand, Kapranov, Zelevinsky. The amoeba of V is the image of V under the map $Log : (C \setminus \{0\})^n \to \mathbb{R}^n$ defined by $Log(z_1, ..., z_n) = (\ln |z_1|, ..., \ln |z_n|)$.

This map "forgets" the phase part $e^{i\varphi_k} \in S^1$ of each complex number $z_k = |z_k|e^{i\varphi_k}$, $k = 1, ..., n$, and can be considered as a map $(C \setminus \{0\})^n \to (C \setminus \{0\})^n/(S^1)^n$. It is an idea of G. Mikhalkin to replace $(C \setminus \{0\})^n$ by a complex group G (possibly non-commutative) and $(S^1)^n$ by a maximal compact subgroup of G. In paper No. 11, this idea is applied for $G = PSL_2(\mathbb{C})$ and its subgroup $SO(3)$, the quotient being the hyperbolic space $\mathbb{H}^3$. The images of subvarieties of G are kind of non-commutative amoebas. They are studied in paper No. 11 where interesting results about tropical limits of such amoebas of curves are obtained. Consider $\mathbb{H}^3$ as the space of 2×2-Hermitian positive definite matrices with determinant 1. In paper No. 13, an amoeba type map $\varkappa : PSL_2(\mathbb{C}) \to \mathbb{H}^3$ is defined by $\varkappa(A) = AA^*$ and tropical limits of amoebas of this type are obtained.

CONCLUSION

The documents and materials presented by Dr. Mikhail Shkolnikov meet the requirements of the Act on the Development of the Academic Staff in the Republic of Bulgaria , the Rules for its implementation and the Rules on the terms and conditions for awarding of academic degrees and occupying academic positions at Bulgarian Academy of Sciences.

The results obtained by M.Shkolnikov are a serious contribution to the field of knot theory, tropical geometry and (discrete) dynamical systems. They go beyond the usual standards for holding the academic position of associate professor.

Based on the comments above, I give a positive assessment of the scientific work of Dr. Mikhail Shkolnikov and recommend to the scientific jury to advise the Scientific Council of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences to appoint Dr. Mikhail Stanislavovich Shkolnikov as an Associate Professor in area of higher education 4. Natural Sciences, Mathematics and Informatics, professional field 4.5 Mathematics, scientific speciality "Geometry and Topology" (Tropical geometry).

16.03.2025 Reviewer:

(Prof. D.Sc. Johann Davidov)