REVIEW

from assoc. prof. Tsetska Grigorova Rashkova

on the dissertation “Some classes of noncommutative rings and Abelian groups‘‘,

presented for obtaining the scientific degree „doctor of sciences“

in pr. f. 4.Natural Sciences, Mathematics and Informatics, area: 4.5 Mathematics

By an Order 75/28.02.2020 of the Director of IMI-BAS I am appointed as a member of the Scientific Jury and a reviewer concerning the procedure for obtaining the scientific degree „doctor of sciences“ defending the dissertation “Some classes of noncommutative rings and Abelian groups‘‘ by d-r Peter Vassilev Danchev.

Analysis of the scientific contributions and presentation of the main of them

The submitted dissertation consists of 230 pages. A part of it, the contents of Chapter III with two sections, is a specific generalization of the investigations made in the first doctoral dissertation. An initial base are the notions of clean and exchange rings introduced by Nicholson (TAMS, 1977), nil-clean rings, considered by Diesl (Thesis, 2006; JA, 2013), uniquely clean and uniquely nil-clean rings, studied by Chen (CA, 2011), as well as the notion of clean index, introduced by Lee and Zhou (CA, 2012, 2013). However these notions are developed further on in great details, significant generalizations in a wide spectrum are made and the focus now is on noncommutative unitary rings. The new parts, the contents of Chapter IV with two sections, viewing transitive and fully transitive Abelian groups, as well as simply presented Abelian *p*-groups, show very clearly the way of developing the investigations as the apparatus used in proving already includes much more deep and far from classical algebraic results as homological algebra, formal logic and graph theory.

The main contributions of the presented dissertation could be stated as:

1. A necessary and sufficient condition is found for a ring to be weakly clean. It is stated and proved in Theorem 1.8 of the dissertation (Th. 2.4 in [D1]).
2. A full description is made (a necessary and sufficient condition is given) of exchange 2-UU rings. It is formulated in Theorem 1.30 of the dissertation (Th. 2.5 in [D3]).
3. An example is given that n-UU rings for $n>2$ do not satisfy the above description. The ring $M_2(Z_2)$, which is a 3-UU ring is considered in details in Example 1.51 of the dissertation (Example 2.6 in [D3]).
4. A necessary and sufficient condition is found for a ring to be invo-clean, formulated and proved in Theorem 1.49 of the dissertation (Th. 2.15 in [D4]).
5. Analogously to 4. a description is made of strongly invo-clean rings – Corollary 1.51 (Cor. 2.17 in [D4]).
6. A new quantitative characteristic is introduced for weakly nil-clean rings, namely weakly nil-clean index. A necessary and sufficient condition is found for it either to be 1 (Proposition 1.63 of the dissertation, Pr. 2.11 in [D5]) or 2

(Proposition 1.78 of the dissertation, Pr. 2.27 in [D5]). The indices of some matrix rings over finite fields are calculated (we point Examples 1.74, 1.75 and 1.77 of the dissertation, accordingly Examples 2.23, 2.24 and 2.26 in [D5]).

7. A very detailed characteristics of Abelian clean rings *R* with a finite exponent of the group *U(R)* is done in Theorem 1.102 of the dissertation (Th. 3.13 in [D7]).
8. A full description of the strongly *n*-periodic clean rings for odd *n* is made in Theorem 1.104 (Th.3.15 in [D7]).
9. The connection between the UU-property of a ring *R* and its group ring *R*[*G*] is investigated and as a result a necessary and sufficient condition is found for inheriting this property when *G* is Abelian *2*-group. This is Corollary 2.4 of the dissertation (Cor. 2.2 in [D6]).
10. A necessary and sufficient condition is proved for a direct sum of a divisible and a reduced group to be projectively fully transitive. It is the contents of Theorem 3.52 of the dissertation (Th. 3.4 in [D12]).
11. A necessary and sufficient condition is found for an Abelian *p*-group *G* to be a cft-group (or a scft-group), defined by the strongly commutator fully transitive action of the ring of endomorphism *E*(*G*) on the first Ulm subgroup. This is done in Lemmas 3.122 and 3.136 of the dissertation (L. 3.12 and 3.26 in [D14], respectively).
12. A detailed description of the ω-totaly Σ-cyclic groups is made in Theorem 4.6 of the dissertation (Th. 2.6 in [D10]). An equivalence of properties of proper such groups in a theoretically-set aspect is given in Theorem 4.24 of the dissertation (Th. 3.11 in [D10]).
13. A theorem of Nunke (Math.Z., 1967) is generalized for totally projective groups, as in Theorem 4.43 a necessary and sufficient condition is proved an Abelian *p*-group to be *n*-simply presented (Th. 4.4 in [D11]).

As a conclusion the following could be said: Studying two different at first glance topics in modern algebra the author has succeeded in finding a close relationship among them and the above stated contributions are a successful result reached by demonstrating a new insight of ideas and methods and both developing and applying a modern technology for the investigations made.

General description of the papers on which the dissertation is based

On the dissertation 16 scientific publications are cited, 5 of them are individual ([D1]-[D4], [D16]) and the rest 11 have one co-author each.

In [D1] the notions of clean and exchange rings defined by Nicholson (TAMS,1977) are generalized. The definitions of weakly clean and weakly exchange rings in the noncommutative case are given. We repeat that a ring *R* (associative with unity) is weakly clean when the presentation $r=u+e$ or $r=u-e$ for any $r\in R$ holds and the elements *u* and *e* are an invertible element and an idempotent in *R*, respectively. A characteristics of the two types of rings is done by the factor ring *R/J*(*R*) and the lifting modulo *J*(*R*) (i.e. for $r\in R$: $r-r^2\in J(R)$ there exists $e\in Id(R)$: $e-r\in J(R)$, where *Id*(*R*) is the set of the idempotents of the ring *R* and *J*(*R*) is its Jacobson radical). It is proved (Theorem 2.4) that for $2 \in J(R)$ the characterization made appears to be a necessary and sufficient condition the ring to be weakly clean. An analogous result for weakly exchange rings is Theorem 2.2. For $2 \notin J(R)$ the notions of clean and weakly clean ring coincide (Proposition 2.6), the same is valid for exchange and weakly exchange rings (Proposition 2.5).

In [D2] the notion of a ring R with Jacobson units only is introduced and it is defined as an JU ring (i.e. $U(R)=1+J(R)$). The properties of such rings are studied and the additional property $U(R)=1+Nil(R)$ (which is the definition of an UU ring) is discussed as well. The notations $U(R)$ and $Nil(R)$ are the standard ones for the set of the invertible elements (i.e. the units in R) and the nilpotents of R, respectively. We point three of the related properties proved in [D2]:

- the finite UU rings are JU;

- the finite JU rings are UU;

- there exist UU rings which are not JU and vice versa.

The exchange JU rings are described, applications of the results obtained are discussed for commutative group rings. Due to Pr. 5.2 in [D2] if R is commutative and G is Abelian group, then if $R(G)$ is UU (or JU), then R is UU too (or JU).

In [D3] four equivalent conditions are found characterizing 2-UU rings (Theorem 2.5). In Example 2.6 the ring $M_2(Z_2)$ is considered in details and it is shown that for $n>2$ the cited Theorem 2.5 is no more valid.

[D4] is devoted to invo-clean rings. Their definition is given ($U(R)=1+Inv(R)$ for $Inv(R)$ being the set of the involutions of R) and its full characterization is obtained (Theorem 2.15). An analogous approach is applied to strongly invo-clen rings (the condition is $ve=ev$, $v \in Inv(R)$) and the result is Corollary 2.17. The algebraic structure of the strongly invo-clean rings with uniqueness of the idempotent in the corresponding presentation is studied as well (Theorem 2.19).

In [D5] a new quantitative characteristics is introduced for weakly nil-clean rings, namely weakly nil-clean index. Conditions are given when it is 1 (Proposition 2.11) and when it is 2 (Proposition 2.27). Its value is calculated for some matrix rings over finite fields (we point Examples 2.23, 2.24, 2.26).

In [D6] the connection in the UU-property for a ring R and its group ring $R[G]$ is investigated. Corollary 2.2 is a necessary and sufficient condition for both the rings R and $R[\mathbf{G}]$ to be UU.

In [D7] the strongly n-periodic clean rings are investigated in details (the additional conditions are $u^n=1$ and the commutative property of the elements e and u). In Theorem 3.4 it is proved that these rings satisfy a polynomial identity of degree $2n$ and its Jacobson radical is nil with bounded by n index. Theorem 3.13 describes the strongly n-periodic Abelian rings. The case of odd n is included in Theorem 3.15.

In [D8] in connection with the general problem of classifying the fully invariant subgroups of a reduced Abelian p-group, the socles of fully invariant subgroups are considered, namely the class of the socle-regular groups, which includes the class of the fully transitive groups. Basic properties of such groups are discussed in two theorems. In Theorem 1.2 it is proved that a direct sum of a separable group with any group *G* is socle-regular if and only if *G* is socle-regular. Furthermore *G* is socle-regular if and only if its direct powers are socle-regular groups as well (Theorem 1.4).

In [D9] the connection between the different notions of transitivity for Abelian *p*-groups is considered. In Corollary 3.2 it is proved that socle-regular groups are exactly direct sums in strongly socle-regular groups.

In [D10] a description is made of the class of the prime Abelian groups with separable subgroups being direct sums of cyclic groups (i.e. of the ω-total Σ-cyclic groups) by several equivalent conditions (Theorem 2.6). The proper $\omega+n$-total $p^{\omega+n}$--projective groups are investigated as well (Theorem 3.11).

[D11] includes many properties of the (strongly) *n*-simply presented groups as an important class of Abelian *p*-groups. In the paper the theorem of Nunke (Math.Z., 1967), being a necessary and sufficient condition a group to be totally projective, is generalized for *n*-simply presented groups (Theorem 4.4).The apparatus applied includes homological algebra and uses notions as strongly *n*-balanced exact sequences, balanced projective group resolution, valuated groups and valuated vector spaces.

On the base of the definition of full transitivity given by Kaplansky (PNAS, 1952), in [D12] two new notions are introduced which though more restrictive (Pr. 3.5) give the possibility for more fruitful investigations of the new objects. Using them it is proved that a direct sum of a divisible and a reducible group is projectively fully transitive exactly when the reduced group has the same property (Th. 3.4). An analogue of the last theorem for strongly fully projective groups is Th. 4.3.

In [D13] commutator socle-regular groups are considered. Theorem 2.12. is a detailed description of them. It is proved that the commutator socle regularity is inherited by a direct summand with the same property if the additional summand is a separable group (Theorem 3.8). Of interest is Example 3.1 proving that there is transitive group which is neither commutator socle-regular nor projectively socle-regular and additionally the subring of the full ring of automorphisms for the considered group is calculated in details.

[D14] considers two problems connected with the ring of endomorphisms of an Abelian group: when it is generated by commutators and when it is additively generated by them. For this purpose the notions of commutator full transitivity and strong commutator full transitivity are introduced. For the corresponding Abelian groups the notations of a cft-group and a scft-group are used. Properties of the first type groups are investigated in Lemma 3.5 and Proposition 3.6, while of those of the second type - in Corollary 3.22. Example 3.30 is an interesting one giving a ring *S*, for which there exists a *S*-commutator fully transitive group not being a *S*-strongly commutator fully transitive group.

In [D15] Abelian groups with trivial fully invariant subgroups are investigated, as well as such that all its nontrivial fully invariant subgroups are isomorphic. We point that due to Proposition 2.14 if *A* is torsion free group of the first type, has a finite rank and all its endomorphisms are monomorphisms this is equivalent to the property the ring $E(A)$ of its endomorphisms to be a commutative one.

Paper [D16] is in some sense a continuation of [D10] and [D11]. It gives a series of properties of groups being a generalization of those in [D10] and [D11] (Theorems 3.1 and 3.13) , as well as such in the spirit of Nunke Theorem (Math.Z., 1967) (Theorem 4.4).

Reflection of the results in the papers of other authors

The papers on which the dissertation is based have a summary impact factor

IF=4,035 and **SJR=0,672**. The individual papers [D16] and [D4] have IF=0,365 and SJR=0,227, respectively.

The general number of citations is 31 in papers of 31 authors in common. Paper [D11] is cited in a monograph of Springer P.H. This data shows undoubtfully the significance of the scientific results obtained by the author and their reflection on world scientific investigations in similar areas.

Contribution of the author in team papers

Not having an additional information I consider of equal value the contribution

of the authors in the team papers (11 of those on which the dissertation is based).

Critical remarks and recommendations

The possibility a dissertation to be submitted in English obliges the author to edit it in a proper way. To my opinion it is not done for the presented work.

Some examples are the following:

1. Some definitions are written in Italic only, others are numerated, while thirds are repeatedly given on different places in the text (for example Definition 1.1 and Definition 1.33, Definition 1.2 and Definition 1.23), an element of a ring *R* is denoted in a different way – once by „*a*“, once by „*x*“ or by „*r*“.

On some places in the text the sequence of numeration is not followed, for example on pages. 78, 80, 82, 92, 111, 122, 127.

2. In the proof of some basic theorems there are references to papers without concrete numbered theorems or propositions (for example in Theorem 1.16 are cited [93] and [96] in general, in Theorem 1.30 is cited [93], in Corollary 1.50 - [41] , in Corollary 1.51 is cited [68]). While looking in [41] one could find the cited proposition which is exactly Proposition 3.16. This is not the same however for [68], where it is very difficult, even impossible to reach the concrete cited proposition, though this is in a corollary given by the author as one of the basic contributions in his dissertation. These are only some of the examples.

3. In the formulations of Lemma 1.38 the equality 24=0 is a bit confusing without other clarification and it could be dropped as further on the result used is that 6 is nilpotent. To my opinion the fact that *3* is an element of the ring needs to be explained.

4. The paragraph before Definition 1.53 is a needless burden of the text.

5. The notion of „strongly weakly nil-clean ring“ after Definition 1.35 is difficult for pronunciation even. A much better option is „strongly“ to be substituted by the condition itself i.e. *qe=eq*.

To my opinion the great amount of definitions in the dissertation does not help its reading with understanding. The author's ambition to include in the dissertation not a small part of his really enormous and varied scientific production is understandable but its use for information and particularly for qualification is very difficult.

Qualities of the Autoreview (the variant in Bulgarian)

The Autoreview of the dissertation includes 40 pages. It reveals properly the most significant part of the dissertation. The scientific contributions of the author are formulated clearly as well as the open problems important to be attacked in future. The papers on the dissertation are cited at the end.

I make two remarks and give two recommendations:

1. The process "copy-paste" needs obligatory the final result to be checked. The title given in section General information is of the first dissertation not the one submitted for "Doctor of Science".
2. The role of the different members of the Scientific Jury is not completely correct maybe due to some decisions made at the last moment.
Remark: These comments rely only on the variant sent to me by e-mail. The one on the site of the Seminar in IMI – BAS does not contain the section discussed.
3. The first and the last paragraph in section Perspectives for development and basic unsolved problems have one and the same meaning. The first one is more clearly written and it is the one which has to remain only and to be at the end of the section.
4. The References have to include only papers cited in the Autoreview (their number is 31) not the full variant of 110 given at the end of the dissertation.

Conclusion: The submitted dissertation and the accompanying documents fulfill all the requirements both on a governmental level and in IMI-BAS concerning the documentation for the procedure of obtaining scientific degrees.

The scientific achievements presented in the submitted dissertation have values of high level and **my opinion for them is positive**.

This gives me the confidence **to recommend the conferring the degree "doctor of sciences" to d-r Peter Vassilev Danchev**.

June 8-th, 2020 Signature:

Tsetska Rashkova