Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

Symmetry and metric geometry in Banach spaces

Svetozar Zlatkov Stankov

Autoreferat of thesis
for conferring of academic and scientific degree **Doctor**
in professional field 4.5 Mathematics (Mathematical analysis)

Scientific advisor:
Prof. Denka Kutzarova

Sofia, 2025

During the last thirty years the theory of Banach spaces branched in many different directions. Results and techniques from this area were applied in nonlinear analysis, metric geometry, computer science, theory of operators, logic, and others. At the same time, there was also an extensive study of the structure of Banach spaces themselves, especially of separable spaces and, in particular, those with a Schauder basis. For recent developments in Banach spaces theory, see the two volumes of the Handbook on the geometry of Banach spaces, edited by W. B. Johnson and J. Lindenstrauss [23, 24].

A main line of investigation in the structure theory of Banach spaces is whether any infinite-dimensional space contains an infinite-dimensional subspace which is isomorphic to a space from a list of spaces with "nice" properties. The most natural first question was if any X contains an isomorphic copy of c_0 or ℓ_p, $1 \leq p < \infty$. All the known classical examples of Banach spaces do contain an isomorphic copy of c_0 or ℓ_p, $1 \leq p < \infty$. Additionally, there exist deep results for embedding copies of ℓ_2^n or ℓ_p^n for an arbitrary dimension n, due to Dvoretzky and Krivine, respectively. However, the case of an infinite dimensional subspace turned out to be different. In 1974 Boris Tsirelson [47], inspired by ideas in logic, constructed a reflexive space with an unconditional Schauder basis which has no subspace isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$.

Tsirelson inductively defined the unit ball B of his space as follows:

Let $K^0 = \{\pm e_n : \ n \in \mathbb{N}\}$. Given K^m, $m \geq 0$,

$$K^{m+1} = K^m \cup \left\{ \frac{1}{2}\sum_{i=1}^{d} f_i \ : \ \begin{array}{l} f_i \in K^m, i = 1, 2, \ldots, d, \quad d \in \mathbb{N}, \text{ and} \\ d \leq \min \operatorname{supp} f_1 \leq \max \operatorname{supp} f_1 < \min \operatorname{supp} f_2 \leq \\ \leq \max \operatorname{supp} f_2 < \cdots < \min \operatorname{supp} f_d \end{array} \right\}.$$

Finally, let $K = \bigcup_{m=0}^{\infty} K^m$, and the unit ball B is the closed convex hull of K.

In actuality, what is now referred to as the Tsirelson space T is the construction given by Figiel and Johnson [13], and the original space defined by Tsirelson is its dual T^*. The definition of the norm in T is given either as the limit of a sequence of norms, or, equivalently, as the implicit solution of an equation.

Consider the linear space of finitely supported real-valued sequences c_{00}. For any $x = \sum_n a_n t_n \in c_{00}$, and for any nonempty finite set E of natural numbers, we define

$$Ex = \sum_{n \in E} a_n t_n \ .$$

Here (t_n) is the canonical basis of c_{00} and (a_n) is an arbitrary sequence of real numbers. We define inductively a sequence of norms $(\| \cdot \|_j)_{j=0}^{\infty}$ on c_{00} as follows:

for any $x = \sum_n a_n t_n \in c_{00}$, let $\|x\|_0 = \max_n |a_n|$, and for $m \geq 0$,
$\|x\|_{m+1} = \max\left\{\|x\|_m,\ \frac{1}{2}\max\left[\sum_{i=1}^d \|E_i x\|_m\right],\quad d \in \mathbb{N}\right\}$,

where the inner maximum is taken over all choices of d and all choices of finite subsets $(E_i)_{i=1}^d$ of $\mathbb{N}$ as d varies such that

$$d \leq \min E_1 \leq \max E_1 < \min E_2 \leq \max E_2 < \cdots < \min E_d \ .$$

Then $\|x\|$ is defined as $\lim_{m\to\infty} \|x\|_m$, and what is now called the Tsirelson T is the completion of $(c_{00}, \|\cdot\|)$.

Figiel and Johnson proved that the norm of T satisfies the following implicit equation.

$$\forall x = \sum_{n=1}^{\infty} a_n t_n \in T,$$

$$\|x\| = \max\left\{\max_n |a_n|,\ \frac{1}{2}\sup\sum_{i=1}^d \|E_i x\|\right\},$$

where the inner supremum is taken over all choices of d and all choices of finite subsets $(E_i)_{i=1}^d$ of $\mathbb{N}$ as d varies such that

$$d \leq \min E_1 \leq \max E_1 < \min E_2 \leq \max E_2 < \cdots < \min E_d \ .$$

Tsirelson space had an enormous impact on the theory of Banach spaces. Many constructions, similar in spirit, were defined to solve other important problems. Among those, we mention a construction by Tzafriri [48]. Since their introduction, the notions of type and cotype (cf. e.g. [33]) played a major role in the theory of normed spaces and their many applications. Tzafriri answered a question of Pisier by defining a space which has an equal-norm type p, but has no type p, for $1 < p < 2$. The book of Casazza and Shura [8] gathered most of the results related to Tsirelson space and its variations.

The next natural big question was whether every space contained an unconditional basic sequence, that is, a subspace with an unconditional basis. A partial result for weakly null sequences was due to Maurey and Rosenthal, cf. e.g. [32].

In 1990 E. Odell showed, in an unpublished note, that T is distortable, by constructing in every subspace sequences with two different asymptotic behaviour; this was the first example of a distortable space.

Definition 1. *Let $\lambda > 1$. A Banach space $(X, \|\cdot\|)$ is λ-distortable if there exists an equivalent norm $|\cdot|$ on X such that for every infinite dimensional subspace Y of X,*

$$\sup\left\{\frac{|y|}{|x|} : y, x \in Y, \|y\| = \|x\| = 1\right\} \geq \lambda$$

X is arbitrarily distortable if it is λ-distortable for every $\lambda > 1$.

James proved that c_0 and ℓ_1 are not distortable. Currently, there is still no example of a space which is distortable but not arbitrarily distortable, although the Tsirelson space T is a candidate.

Using a similar idea to that of Odell, Th. Schlumprecht [44] constructed in 1991 the first arbitrarily distortable space S that has an unconditional basis. Schlumprecht's space was a launching point for a renewed interest in Tsirelson type spaces and led to remarkable developments in Banach space theory.

In 1993, W. T. Gowers and B. Maurey [19], using Schlumprecht's construction, solved the famous unconditional basic sequence problem by constructing a space without such a sequence. Their space has a stronger property, namely that it is hereditarily indecomposable.

Definition 2. *A Banach space is heriditarily indecomposable (H.I.) if no subspace can be written as a topological direct sum of two infinite dimensional closed subspaces.*

In 1994, Odell and Schlumprecht [39] solved the distortion problem. In particular, by transferring sets from S, they showed that the Hilbert space ℓ_2 is arbitrarily distortable. This illustrates the impact of the Tsirelson-type spaces on the understanding of the classical Banach spaces.

Briefly jumping to the present time in order to note another example, Baudier, Lancien and Schlumprecht [49] recently used the original Tsirelson space T^* for solving a problem in metric geometry.

Gowers used the notion of hereditary incomposability to solve several other long-standing problems. He showed [15] that the H.I. property is a consequence of the absence of unconditionality, in the sense that every Banach space which does not contain any unconditional basic sequences has an H.I. subspace. Among others, he solved Banach's hyperplane problem [17] by constructing a space which is not isomorphic to any of its hyperplanes. He also [16] proved dichotomies for spaces with a basis and used them for a partial classification of Banach spaces. In 1998, Gowers was awarded a Fields medal for his outstanding work (see [18]).

Further results in this area followed, especially by Argyros and his students. Argyros and Deliyanni [4] answered a question of Gowers by constructing an asymptotic ℓ_1 herediatry indecomposable space. For that purpose, they first defined a new class of asymptotic ℓ_1 spaces with unconditional basis, namely the mixed Tsirelson spaces.

Following the properties of T, Milman and Tomczak-Jaegermann [35] defined asymptotic ℓ_p spaces with respect to a basis to be those for which, given any n, all collections of n successive normalized block-vectors supported, say, after position n of the basis are uniformly equivalent to the unit vector basis of ℓ_p^n. In the beginning, it was hoped that asymptotic ℓ_p spaces might have nicer local structure, that is to say, nicer finite-dimensional subspaces. That turned out to be incorrect as it was proved in [54] that some asymptotic ℓ_1 spaces contain uniform copies of ℓ_∞^n's. The universality of ℓ_∞ implies that these spaces contain arbitrary finite-dimensional subspaces.

One of the main problems in the isomorphic theory of Banach spaces is the classification of the basic sequences of a certain type. This question is formulated in a proper way using the notion of equivalence of basic sequences. Recall that a sequence in a Banach space is a basic sequence if it is a (Schauder) basis of its closed linear span. The most important category of sequences in which this classification is studied is that of symmetric sequences.

Definition 3. *A basic sequence is called symmetric if it is equivalent to all of its permutations.*

This class of sequences includes the canonical unit vector basis of the ℓ_p and c_0 spaces. Closely related to symmetry is the notion of subsymmetry.

Definition 4. *A basic sequence is said to be subsymmetric if it is unconditional,and is equivalent to all of its subsequences.*

The question whether a symmetric basic sequence exists in every Banach space was a driving force in the development of the theory for many decades. This question was solved in the negative by the Tsirelson space.

The class of subsymmetric basic sequences is more general. In practice, the only feature that one needs about symmetric basic sequences in many situations is their subsymmetry, to the extent that when symmetric bases were introduced these two concepts were believed to be equivalent until Garling [14] provided a counterexample that disproved it.

However, subsymmetric bases, far from being just a capricious generalization of symmetric bases, played a relevant role by themselves within the general theory. Indeed, the study of Banach spaces with a subsymmetric but not symmetric basis led to the solution of major problems in the field. In example, the space S constructed by Schlumprecht has one such basis. It follows from the minimality of the Schlumprecht space and the "yardstick" construction of Kutzarova-Lin [28], which for every natural number n provides uniform copies of disjointly supported ℓ_∞^n's, that S does not contain symmetric basic sequences.

If a Banach space has a given special type of structure, it is always an important question whether that structure is unique. Vice versa, if we know that a given structure is unique in that Banach space, that leads to the interesting question what can we say about both the structure and the space itself.

Albiac, Ansorena and Wallis [3] used Garling-type spaces to provide the first example of a Banach space with a unique subsymmetric basis which is not symmetric. However, as shown in [2], that space contains a continuum of non-equivalent subsymmetric basic sequences.

Altshuler [1] (see also [32]) constructed a nontrivial space in which all symmetric basic sequences are equivalent to its symmetric basis. It was remarked in [27] that a careful look at the paper of Altshuler shows that his proof works similarly for the more general case of all subsymmetric basic sequences. In the case of the Altshuler's space the uniqueness of the subsymmetric structure implies that it is actually symmetric. This observation led to the following question, asked first in [27] and restated in [2] :

Question 5. *Does there exist a Banach space in which all subsymmetric basic sequences are equivalent to one basis, and that basis is not symmetric?*

Recently, the first example of a Banach space with a subsymmetric basis with a unique (up to equivalence) subsymmetric basic sequence which is not symmetric was given in [7]. The space under consideration was $Su(T^*)$ [8], the subsymmetric version of T^*, the original space defined by Tsirelson.

In the dissertation we give more examples of spaces with a unique, up to equivalence, subsymmetric basic sequence.

In the spirit of the Tsirelson space T, Tzafriri [48] constructed a space (with a symmetric basis) which is of equal-norm type p, but is not of type p, for $1 < p < 2$.

For convenience, we will define the following:

$$\text{Average}_{\theta_j=\pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\| = 2^{-n} \sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|$$

The important notions of type and cotype were defined by J. Hoffmann-Jørgensen in [50].

Definition 6. *A Banach space X is said to be of type p for some $1 < p \leq 2$, and respectively of cotype q for some $q \geq 2$, if there exists a constant $M < \infty$ so that for every finite set of vectors $(x_j)_{j=1}^n$ in X we have*

$$\text{Average}_{\theta_j = \pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq M \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{\frac{1}{p}}$$

and respectively

$$\text{Average}_{\theta_j = \pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\| \geq M^{-1} \left(\sum_{j=1}^{n} \|x_j\|^q \right)^{\frac{1}{q}}$$

Any constant M satisfying the first or the second inequality is called a type p, respectively cotype q, constant of X.

Equal-norm type p, and equal-norm cotype q are defined if the above inequalities are required only for vectors $(x_j)_{j=1}^n$ in X of equal norm.

The Tirilman space $Ti(p, \gamma)$, where $1 < p < \infty$ and $0 < \gamma < 1$, was introduced and studied by Casazza and Shura [8]. It is a version of the original space of Tzafriri whose Romanian surname was Tirilman.

We prove that for $1 < p < \infty$ and sufficiently small $0 < \gamma < 1$, the dual space $Ti^*(p, \gamma)$, whose canonical basis is subsymmetric and not symmetric, has a unique, up to equivalence, subsymmetric basis sequence. While the normalized block bases (x_j) of the canonical basis of $Su(T^*)$, whose ℓ_∞ norm $\|x_j\|_\infty$ tends to 0 as $j \to \infty$, are asymptotic c_0, the similar block bases in $Ti^*(p, \gamma)$ are asymptotic ℓ_q basis sequences, where

$$\frac{1}{p} + \frac{1}{q} = 1$$

Theorem 7. *Let $1 < p < \infty$ and $\gamma > 0$ be sufficiently small. Every subsymmetric basic sequence of the dual space $Ti^*(p, \gamma)$ is equivalent to the subsymmetric canonical basis $(e_j^*)_{j=1}^\infty$ which is not symmetric.*

Towards the proof of this main result we also obtain the following statements that are interesting on their own:

Lemma 8. *Let (e_i) be a 1-unconditional basis of a reflexive Banach space X which is K-dominated by its normalized block bases, where $K \geq 1$. Then (e_i^*) K-dominates all normalized block bases of (e_i^*) in the dual space X^*.*

Lemma 9. *$Ti^*(p, \gamma)$ does not contain an isomorphic copy of ℓ_q (where $\frac{1}{p} + \frac{1}{q} = 1$).*

As corollaries we reach the following two results too:

Corollary 10. *Let $1 < p < \infty$ and $\gamma > 0$ be sufficiently small. Every subsymmetric basis of a quotient space of $Ti(p, \gamma)$ is equivalent to the canonical basis $(e_j)_{j=1}^{\infty}$.*

Corollary 11. *For $1 < p < \infty$ and sufficiently small γ, the basis (e_i) of $Ti(p, \gamma)$ has a continuum many non-equivalent subsymmetric block bases.*

An important hereditary property of Banach spaces, first defined by Rosenthal, is the notion of minimality.

Definition 12. *An infinite dimensional Banach space X is called minimal if every infinite dimensional subspace $Y \subset X$ contains a further subspace Z such that Z is isomorphic to X.*

Recall that

Definition 13. *Two normed spaces X and Z are called isomorphic if there exists a bounded linear operator $T : X \to Z$ that is a bijection and its inverse $T^{-1} : Z \to X$ is also a bounded linear operator.*

There are only a few examples of minimal spaces. Clearly, c_0 and ℓ_p, for $1 \leq p < \infty$, are minimal. In [51] it was proved that T^* is a nontrivial minimal space. Later, Schlumprecht [52] proved that his space S is minimal. S is also reflexive but not super-reflexive. However, based on the Schlumprecht space S, [53] give examples of super-reflexive minimal spaces. They also show that S^* is also minimal. The basis of T^* is asymptotic ℓ_∞, thus, highly non-symmetric. The bases of S and S^* are subsymmetric, but not symmetric.

An important folklore question is whether there exists a nontrivial (not a subspace of c_0 or ℓ_p) example of a minimal space with a subsymmetric basis.

As explained later, the symmetrizations of T^* and S are not minimal. The quest for such an example is one of the motivations for the consideration of the symmetrization of S^*. The theorem in Chapter **??** leaves the question of symmetric minimal space open.

The natural symmetrization $S(T)$ of the Tsirelson space T contains a subspace isomorphic to ℓ_1, while the symmetrization $S(T^*)$ of the original Tsirelson space is reflexive, so it does not have the same property [8]. In an unpublished note Schlumprecht showed that the symmetric version of his space S does contain a subspace isomorphic to ℓ_1 (a similar result can be found in [34]).

In the dissertation we use the "yardstick" construction in Schlumprecht space S to prove the following theorem.

Theorem 14. *In contrast to the case of $S(T^*)$, the symmetrization $S(S^*)$ of the dual of the Schlumprecht space does contain a subspace isomorphic to the space ℓ_1.*

Using the same type of "yardstick" construction, in the dissertation we also prove the following theorem.

Theorem 15. *Let p and γ be such that $1 < p < \infty$ and $0 < \gamma < 3^{-\frac{1}{q}}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then c_0 is finitely representable in $Ti\,(p, \gamma)$ disjointly with respect to its canonical basis.*

This is an extension of a result from [26] for a wider range of parameters.

One of the most natural way to grasp the geometry of a metric space is to understand in which metric spaces, in particular which Banach spaces, it does, or it does not, bi-Lipschitzly embed.

Definition 16. *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is called a bi-Lipschitz embedding if there exist $s > 0$ and $D \geq 1$ such that for all $x, y \in X$,*

$$s \cdot d(x, y) \leq d_Y(f(x), f(y)) \leq D \cdot s \cdot d(x, y). \tag{1}$$

As usual,

$$c_Y(X) := \inf\{D \geq 1 \mid \text{equation (1) holds for some embedding } f\}$$

denotes the Y-distortion of X. If there is no bi-Lipschitz embedding from X into Y then we set $c_Y(X) = \infty$. A sequence $(X_k)_{k\in\mathbb{N}}$ of metric spaces is said to equi-bi-Lipschitzly embed into a metric space Y if $\sup_{k\in\mathbb{N}} c_Y(X_k) < \infty$.

If two normed spaces are uniformly homeomorphic then all finite-dimensional subspaces of one of the spaces are uniformly embeddable in the other by means of a linear mapping.

Ribe's rigidity theorem [41], namely that if two normed spaces are uniformly homeomorphic then all finite-dimensional subspaces of one of the spaces are uniformly embeddable in the other by means of a linear mapping, suggests that it is reasonable to believe that local geometric properties of Banach spaces could be characterized in purely metric terms. This result spawned what later came to be known as "the Ribe programme".

James [21] introduced the important property of super-reflexivity: a Banach space X is super-reflexive if every Banach space Y which is finitely representable in X is reflexive. Enflo [12] showed that super-reflexivity of X is equivalent to X having an equivalent uniformly convex norm.

The first successful step in the Ribe programme was obtained by Bourgain [6] when he showed that the sequence $(B_k)_{k\in\mathbb{N}}$ of binary trees of height k is a uniformly characterizing sequence for super-reflexivity, i.e. that spaces which are not super-reflexive as those for which the binary trees B_n of depth n embed with uniformly bounded distortion.

We refer to [5] and [36] for a thorough description of the Ribe programme and its successful achievements.

Subsequently, Johnson and Schechtman [25] found two new uniformly characterizing sequences for super-reflexivity, namely the sequence $(D_k^2)_{k\in\mathbb{N}}$ of (2-branching) diamond graphs and the sequence $(L_k^2)_{k\in\mathbb{N}}$ of (2-branching) Laakso graphs. The best known estimate in the literature for the distortion of embeddings of D_n into spaces which are not super-reflexive, due to Pisier [40], is $2+\varepsilon$ for every $\varepsilon > 0$, while the best known estimate for the distortion of embeddings of D_n into $L_1[0,1]$, due to Lee and Rhagavendra [31], is $4/3$.

In the dissertation we construct embeddings of $\mathcal{L}_n$ into arbitrary Banach spaces which are not super-reflexive space with disortion $2+\varepsilon$.

Theorem 17. *Suppose X is not super-reflexive. Then, for each $\varepsilon > 0$ and $n \geq 1$, there exists a mapping $f_n\colon \mathcal{L}_n \to X$ such that, for all $a, b \in \mathcal{L}_n$,*

$$\frac{1}{2}d(a,b) - \varepsilon \leq \|f_n(a) - f_n(b)\| \leq d(a,b). \tag{2}$$

The embeddings of $\mathcal{L}_n$ which we define depend on the following characterization of not being super-reflexive. Its negation is the characterization of super-reflexivity known as J-convexity.

Theorem 18. *[22, 43] X is not super-reflexive if and only if, for each $m \geq 1$ and $\varepsilon > 0$, there exist $e_1, \dots, e_m$ in the unit ball of X such that, for each $1 \leq j \leq m$, we have*

$$\|e_1 + \cdots + e_j - e_{j+1} - \cdots - e_m\| \geq m - \varepsilon. \tag{3}$$

We then prove a stronger result for $X = L_1[0,1]$.

Theorem 19. *For each $n \geq 1$, there exists a mapping $f_n\colon \mathcal{L}_n \to L_1[0,1]$ such that, for all $a, b \in \mathcal{L}_n$,*

$$\frac{3}{4}d(a,b) \leq \|f_n(a) - f_n(b)\|_1 \leq d(a,b). \tag{4}$$

The analogue of Theorem 19 for D_n is proved in [31, Theorem 5.1] with the same distortion of $4/3$. Moreover, it is remarked without proof that $4/3$ is the best constant for the distortion of embeddings of D_n as $n \to \infty$ [31, p. 359]. In fact, we do not know of any embedding of D_2 into $L_1[0,1]$ with distortion smaller than $4/3$.

Our next result shows that $\mathcal{L}_2$ does not embed into $L_1[0,1]$ with distortion smaller than $9/8$.

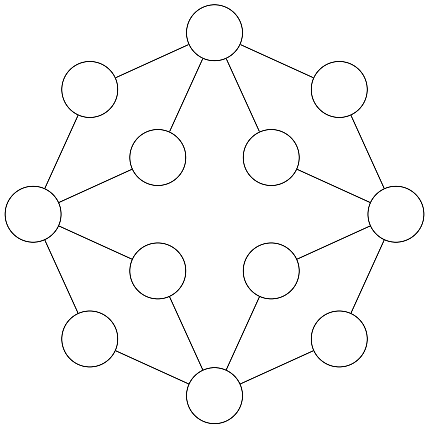

Figure 1: Diamond graph D_2.

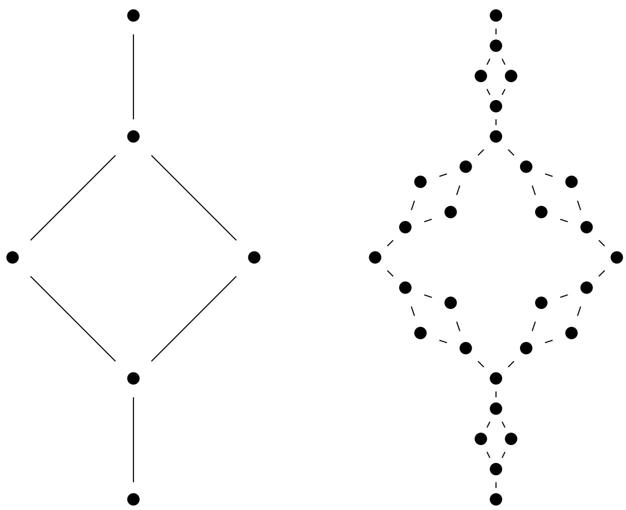

Figure 2: Laakso graphs $\mathcal{L}_1$ and $\mathcal{L}_2$

Theorem 20. *Let* $f\colon \mathcal{L}_2 \to L_1[0,1]$ *satisfy*

$$d(a,b) \le \|f(a) - f(b)\|_1 \le c d(a,b).$$

Then $c \ge 9/8$.

The proof uses the following characterization of isometric embeddability into $L_1[0,1]$.

Theorem 21. *[9, Theorem 6.2.2] Let* (M,ρ) *be a finite metric space. Then* (M,ρ) *is isometric to a subset of* $\ell_2^2 := (\ell_2, \|\cdot\|_2^2)$ *if and only if, for all* $k_i \in \mathbb{Z}$ $(1 \le i \le n)$ *such that* $\sum_{i=1}^n k_i = 0$, *we have*

$$\sum_{1\le i<j\le n} k_i k_j \rho(x_i, x_j) \le 0,$$

where $x_1, \dots, x_n$ *are the distinct elements of* M.

In a similar way we can estimate the distortion of metric embeddings of the diamond graph D_2 into $L_1[0,1]$.

Theorem 22. *Let* $f\colon D_2 \to L_1[0,1]$ *satisfy*

$$d(a,b) \le \|f(a) - f(b)\|_1 \le c d(a,b).$$

Then $c \ge 5/4$.

We shall remark that embedding into $L_1[0,1]$ and the respective estimations for the distortions provide a very useful tool for a number of problems in computer science.

Main contributions

1. We prove Theorem 7: Let $1 < p < \infty$ and $\gamma > 0$ be sufficiently small. Every subsymmetric basic sequence in the dual space $Ti^*(p,\gamma)$ is equivalent to the subsymmetric canonical basis $(e_j^*)_{j=1}^{\infty}$ which is not symmetric. In other words, these spaces whose canonical basis is subsymmetric have, up to equivalence, a unique subsymmetric basic sequence which is not symmetric.

Towards the proof of this main result we also obtain the following statements that are interesting on their own:

Lemma 8: Let (e_i) be a 1-unconditional basis of a reflexive Banach space X which is K-dominated by its normalized block bases, where $K \geq 1$. Then (e_i^*) K-dominates all normalized block bases of (e_i^*) in the dual space X^*.

Lemma 9: $Ti^*(p,\gamma)$ does not contain an isomorphic copy of ℓ_q (where $\frac{1}{p} + \frac{1}{q} = 1$).

As applications of the main theorem we get:

Corollary 10: Let $1 < p < \infty$ and $\gamma > 0$ be sufficiently small. Every subsymmetric basis of a quotient space of $Ti(p,\gamma)$ is equivalent to the canonical basis $(e_j)_{j=1}^{\infty}$.

Corollary 11: For $1 < p < \infty$ and sufficiently small γ, the basis (e_i) of $Ti(p,\gamma)$ has a continuum many non-equivalent subsymmetric block bases.

2. We construct "yardsticks" in Tirilman spaces. More precisely, we prove Theorem 15: Let p and γ be such that $1 < p < \infty$ and $0 < \gamma < 3^{-\frac{1}{q}}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then c_0 is finitely representable in $Ti\,(p,\gamma)$ disjointly with respect to its canonical basis. This provides an alternative proof that the canonical basis of a Tirilman space is not symmetric.

$(\ell_\infty^n)_{n=1}^{\infty}$ are universal for all finite dimensional Banach spaces, that is to say, for every finite dimensional space X there exists an n such that X can be isometrically embedded in ℓ_∞^n. Thus, the finite dimensional structure of $Ti(p,\gamma)$ for $1 < p < \infty$, $\gamma < 3^{-\frac{1}{q}}$ is as rich as possible in the sense that every finite dimensional space is $1+\epsilon$ embeddable in $Ti(p,\gamma)$ for arbitrary $\epsilon > 0$. The last follows from the theorem of James that c_0 is not distortable, and its finite dimensional version.

3. We prove Theorem 14: The symmetrized version of the dual of Schlumprecht's space, $S(S^*)$, contains a subspace isomorphic to ℓ_1.

4. We study bilipschitz embeddings of Laakso graphs into Banach spaces. We estimate the distortion of such embeddings. More precisely, we prove

Theorem 17: Suppose X is not super-reflexive. Then, for each $\varepsilon > 0$ and $n \geq 1$, there exists a mapping $f_n \colon \mathcal{L}_n \to X$ such that, for all $a, b \in \mathcal{L}_n$,

$$\frac{1}{2}d(a,b) - \varepsilon \leq \|f_n(a) - f_n(b)\| \leq d(a,b).$$

We prove a stronger result in the particular case when the Banach space is $L[0,1]$, namely Theorem 19: For each $n \geq 1$, there exists a mapping $f_n \colon \mathcal{L}_n \to L_1[0,1]$ such that, for all $a, b \in \mathcal{L}_n$,

$$\frac{3}{4}d(a,b) \leq \|f_n(a) - f_n(b)\|_1 \leq d(a,b)$$

Bilipschitz embedding into $L[0,1]$ are important in Computer Science.

5. We create a computer program to find a lower bound of the distortion of embedding Laakso and diamond graphs into $L[0,1]$. In this way we obtain:

Theorem 20: Let $f \colon \mathcal{L}_2 \to L_1[0,1]$ satisfy

$$d(a,b) \leq \|f(a) - f(b)\|_1 \leq cd(a,b).$$

Then $c \geq 9/8$.

Theorem 22: Let $f \colon D_2 \to L_1[0,1]$ satisfy

$$d(a,b) \leq \|f(a) - f(b)\|_1 \leq cd(a,b).$$

Then $c \geq 5/4$.

Since the proofs of Theorems 20 and 22 used only negative type inequalities, by [9, Theorem 6.2.2] they remain valid if $L_1[0,1]$ is replaced by $(\ell_2, \|\cdot\|_2^2)$. This is a stronger result as $L_1[0,1]$ is isometric to a subset of $(\ell_2, \|\cdot\|_2^2)$.

Publications related to the thesis

1. Stephen J. Dilworth, Denka Kutzarova, Bünyamin Sarı, Svetozar Stankov, Duals of Tirilman spaces have unique subsymmetric basic sequences, Bulletin of the London Mathematical Society Volume 56, 150-158, https://doi.org/10.1112/blms.12920

2. S. J. Dilworth, Denka Kutzarova, Svetozar Stankov, Metric embeddings of Laakso graphs into Banach spaces, Banach Journal of Mathematical Analysis 16 (2022), no. 4, Paper No. 60, 14 pp., http://doi.org/10.1007/s43037-022-00212-7

3. Svetozar Stankov, On the symmetrized dual of Schlumprecht's space, C. R. Acad. Bulg. Sci., 78, No 1, 2025, https://doi.org/10.7546/CRABS.2025.01.02

Approbation of the thesis

The results from the thesis have been presented in the following talks:

1. Stephen J. Dilworth, Denka Kutzarova, Svetozar Stankov, Metric embeddings of Laakso graphs into Banach spaces, Week of Mathematics and Informatics 2024, September 23-27, 2024, Duni Royal Resort, Bulgaria, https://www.fmi.uni-sofia.bg/bg/wmi-2024-program

2. Stephen. J. Dilworth, Denka Kutzarova, Svetozar Stankov, Metric embeddings of Laakso graphs into Banach spaces, Annual Scientific Session of Analysis, Geometry and Topology Department, December 5, 2024, IMI-BAS, https://math.bas.bg/event/%d0%be%d1%82%d1%87%d0%b5%d1%82%d0%bd%d0%b0-%d1%81%d0%b5%d1%81%d0%b8%d1%8f-%d0%bd%d0%b0-%d1%81%d0%b5%d0%ba%d1%86%d0%b8%d1%8f-%d0%b0%d0%bd%d0%b0%d0%bb%d0%b8%d0%b7-%d0%b3%d0%b5%d0%be%d0%bc%d0%b5%d1%82/

Declaration of originality

The author declares that the thesis contains original results obtained by him in cooperation with the scientific advisor. The usage of results of other scientists is accompanied by suitable citations.

Acknowledgements

First I would like to express my sincere gratitude to the Analysis, Geometry and Topology Department of the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences for the significant help I have gotten from them over the years - not only during my PhD program but ever since high school. Of course, I owe my thanks to the entire Institute of Mathematics and Informatics as a whole. The Operations Research, Probability and Statistics Department bears specific mention, not only for helping me with starting my PhD program but for all the guidance over the years.

I would also like to thank Professor Mikhail Krastanov not only for organizing my exams and helping me with the communication with the staff, but also for all the mathematical guidance ever since my first year in university.

I have the utmost gratitude for Academician Stanimir Troyanski for sparking my interest in the field of Banach spaces and for all the help I got from him for my master's thesis.

I would also like to thank the entire Sofia University Faculty of Mathematics and Informatics for all the opportunities and knowledge I got during my time there.

Professor Stephen J. Dilworth has my absolute appreciation for all the suggestions, discussions and joint mathematical work over the years. This would certainly not have been possible without him.

Finally, saving the most important part for the end, I would like to most sincerely express my gratitude to Professor Denka Kutzarova for her invaluable help, motivation, patience, and support. Professor Kutzarova, thank you for being my advisor and teaching me so much!

Bibliography

[1] Z. Altshuler, A Banach space with a symmetric basis which contains no ℓ_p or c_0, and all its symmetric basic sequences are equivalent, Compositio Math. 35 (1977), no 2, 189–195.

[2] F. Albiac, J. L. Ansorena, S. J. Dilworth, and D. Kutzarova, A dichotomy for subsymmetric basic sequences with applications to Garling spaces, Trans. Amer. Math. Soc. 374 (2021), no. 3, 2079–2106.

[3] F. Albiac, J. L. Ansorena, and B. Wallis, Garling sequence spaces, J. London Math. Soc. (2) 98 (2018), no. 1, 204–222.

[4] Argyros, S. A.; Deliyanni, I. Examples of asymptotic ℓ_1 Banach spaces. Trans. Amer. Math. Soc. 349 (1997), no. 3, 973вЂ“995.

[5] K. Ball, The Ribe programme, Astérisque 352 (2013), Exp. No. 1047, viii, 147 – 159. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043 – 1058.

[6] J. Bourgain, The metrical characterization of superreflexivity in Banach spaces, Israel J. Math. 56 (1986), 222–230.

[7] P. Casazza, S. J. Dilworth, D. Kutzarova and P. Motakis, On uniqueness and plentitude of subsymmetric sequences, Israel J. Math, to appear.

[8] P. G. Casazza and T. Shura, Tsirelson's space, Lecture Notes in Mathematics 1363, Springer-Verlag, Berlin, 1989.

[9] M. M. Deza and M. Laurent, Geometry of Cuts and Metrics, Algorithms and Combinatorics 15, Springer-Verlag, Berlin, 1997.

[10] Stephen J. Dilworth, Denka Kutzarova, Bünyamin Sarı, Svetozar Stankov, Duals of Tirilman spaces have unique subsymmetric basic sequences, submitted.

[11] S. J. Dilworth, Denka Kutzarova, Svetozar Stankov. Metric embeddings of Laakso graphs into Banach spaces, to appear in Banach Journal of Mathematical Analysis.

[12] Per Enflo, Banach spaces which can be given an equivalent uniformly convex norm, Israel J. Math. 13 (1972), 281–288.

[13] T. Figiel and W. B. Johnson, A uniformly convex Banach space which contains no ℓ_p, Compositio Math. 29 (1974), 179–190.

[14] D. J. H. Garling, Symmetric bases of locally convex spaces, Studia Math. 30 (1968), 163–181.

[15] W. T. Gowers, A new dichotomy for Banach spaces. Geom. Funct. Anal. 6 (1996), no. 6, 1083вЂ“1093.

[16] W. T. Gowers, An infinite Ramsey theorem and some Banach-space dichotomies. Ann. of Math. (2) 156 (2002), no. 3, 797-833.

[17] W. T. Gowers, A solution to Banach's hyperplane problem, Bull. London Math. Soc. 26 (1994), no 6, 523-530

[18] B. Bollobas, The work of William Timothy Gowers, Proc. Internat. Congress Math. Berlin 1998, 109-118

[19] W. T. Gowers, B. Maurey (1993) The unconditional basic sequence problem, J. Amer. Math. Soc. 6, pages 851-874

[20] A. Gupta, I. Newman, Y. Rabinovich and A. Sinclair, Cuts, trees and ℓ_1-embeddings of graphs, Combinatorica 24 (2004) 233–269; Conference version in: *40th Annual IEEE Symposium on Foundations of Computer Science*, 1999, pp. 399–408.

[21] R. C. James, Super-reflexive Banach spaces, Can. J. Math. 24 (1972), 896–904.

[22] R. C. James and J.J. Schaffer, Super-reflexivity and the girth of spheres, Israel J. Math. 11 (1972), 398–404.

[23] Johnson, William B.; Lindenstrauss, Joram Basic concepts in the geometry of Banach spaces. Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001.

[24] Johnson, William B.; Lindenstrauss, Joram Basic concepts in the geometry of Banach spaces. Handbook of the geometry of Banach spaces, Vol. II, North-Holland, Amsterdam, 2003.

[25] W .B. Johnson and G. Schechtman, Diamond graphs and super-reflexivity, J. Topol. Anal. 1 (2009), 177–189.

[26] M. Junge, D. Kutzarova and E. Odell, On asymptotically symmetric Banach spaces, Studia Math. 173 (2006), no. 3, 203–231.

[27] D. Kutzarova, A. Manoussakis, and A. Pelczar-Barwacz, Isomorphisms and strictly singular operators in mixed Tsirelson spaces, J. Math. Anal. Appl. 388 (2012), no. 2, 1040–1060.

[28] D. Kutzarova, P. K. Lin (2000) Remarks about Schlumprecht space, Proceedings of the American Mathematical Society 128, no. 7, pages 2059-2068

[29] T. J. Laakso, Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincare inequality, Geom. Funct. Anal. 10 (2000), 111–123.

[30] Urs Lang and Conrad Plaut, Bilipschitz Embeddings of Metric Spaces into Space Forms, Geom. Dedicata 87 (2001), 285–307.

[31] J. R. Lee and P. Raghavendra, Coarse differentiation and multi-flows in planar graphs, Discrete Comput. Geom. 43 (2010), 346–362.

[32] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. I, Springer-Verlag, Berlin-New York, 1977.

[33] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. II. Function spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 97. Springer-Verlag, Berlin-New York, 1979.

[34] A. Manoussakis, On the structure of a certain class of mixed Tsirelson spaces, Positivity 5 (2001), no. 3, 193–238.

[35] B. Maurey, V. D. Milman and N. Tomczak-Jaegermann, Asymptotic infinite-dimensional theory of Banach spaces, Oper. Theory: Adv. Appl. 77 (1995), 149–175.

[36] Assaf Naor, An introduction to the Ribe program, Jpn. J. Math. 7 (2012), no. 2, 167–233.

[37] M. I. Ostrovskii, Metric Embeddings: Bilipschitz and Coarse Embeddings into Banach Spaces, de Gruyter Studies in Mathematics, 49. Walter de Gruyter & Co., Berlin, 2013.

[38] Mikhail I. Ostrovskii and Beata Randrianantoanina, A new approach to low-distortion embeddings of finite metric spaces into non-superreflexive Banach spaces, J. Funct. Anal. 273 (2017), 598–651.

[39] Odell, Edward; Schlumprecht, Thomas The distortion problem. Acta Math. 173 (1994), no. 2, 259 – 281.

[40] G. Pisier, Martingales in Banach spaces, Cambridge Stud. Adv. Math., vol. 155, Cambridge University Press, 2016.

[41] M. Ribe, On uniformly homeomorphic normed spaces, Ark. Mat. 14 (1976), 237 – 244.

[42] B. Sari, Envelope functions and asymptotic structures in Banach spaces, Studia Math. 164 (2004), no. 3, 283–306.

[43] J. J. Schaffer and K. S. Sundaresan, Reflexivity and the girth of spheres, Math. Ann. 184 (1970), 163–168.

[44] Th. Schlumprecht, An arbitrarily distortable Banach space, Israel J. Math. 76 (1991), no. 1–2, 81–95.

[45] Svetozar Stankov, On the symmetrized dual of Schlumprecht's space, submitted.

[46] Andrew Swift, A coding of bundle graphs and their embeddings into Banach spaces, Mathematika 64 (2018), 847–874.

[47] B. S. Tsirelson, It is impossible to imbed ℓ_p or c_0 into an arbitrary Banach space, Funkcional. Anal. i Priložen. 8 (1974), no. 2, 57–60 (Russian).

[48] L. Tzafriri, On the type and cotype of Banach spaces, Israel J. Math. 32 (1979), no. 1 32–38.

[49] F. Baudier, G. Lancien, Th. Schlumprecht, The coarse geometry of Tsirelson's space and applications, J. American Mathematical Society 31 (2018), 699-717

[50] J. Hoffmann-Jorgensen, Sums of independent Banach space valued random variables, Studia Math. 52 (1974), 159-186

[51] P. G. Casazza, W. B. Johnson, L. Tzafriri, On Tsirelson's space. Israel J. Math. 47 (1984), no. 2-3, 81-98.

[52] G. Androulakis, T. Schlumprecht, The Banach space S is complementably minimal and subsequentially prime. Studia Math. 156 (2003), no. 3, 227-242.

[53] P.G. Casazza, N.J. Kalton, D. Kutzarova and M. Mastylo, Complex interpolation and complementably minimal spaces, Interaction between functional analysis, harmonic analysis and probability (Columbia, MO, 1994), Lecture Notes in Pure and Appl. Math., 175, Dekker, New York, 1996, 135-143.

[54] S. Argyros, I. Deliyanni, D. Kutzarova and A. Manoussakis, Modified mixed Tsirelson spaces, J. Funct. Anal. 159 (1998), 43-109.