Report

by Acad. Julian Revalski

on the PhD thesis of Svetozar Zlatkov Stankov entitled

Symmetry and Metric Geometry in Banach Spaces

for the award of the educational and scientific degree "Doctor" in the professional field 4.5 Mathematics

Svetozar Stankov was enrolled to an individual doctoral program in January 2023 with a duration of 3 years. He was withdrawn from the program with the right to defend on 23.12.2024, and the pre-defense took place on 17.01.2025.

Stankov's dissertation comprises 71 pages, consisting of an introduction, four chapters, a conclusion, and references consisting of 54 titles. The dissertation is written in English, and the abstract is also presented in Bulgarian.

The PhD thesis is dedicated to a classical and significant area in Functional Analysis—metric geometry of Banach spaces. Research in this area stems from classical results concerning whether every infinite-dimensional Banach space contains an infinite-dimensional Banach subspace which is isomorphic to one of the spaces c_0 or l_p, $1 \leq p < +\infty$. Despite the negative answer to this question provided by Tsirelson, interest in this topic continues, with numerous important results concerning the geometric structure of Banach spaces achieved by world renowned experts in the theory of Banach spaces. Related to these properties is the question of the existence of symmetric basic sequences in a given Banach space. Recall that a sequence is basic if it is a Schauder basis for its closed span, and symmetric if it is equivalent to all its permutations. The dissertation explores the existence of such sequences and their generalization—subsymmetric basic sequences—in certain Banach spaces. Subsymmetric basic sequences are unconditional and equivalent to all of their subsequences.

Chapter One contains the introduction, outlining the main questions explored in the dissertation and summarizing the main results achieved.

Chapter Two addresses the existence of subsymmetric basic sequences in the dual space of the so-called Tirilman space. These investigations are motivated by a question, posed by several authors, concerning the existence of a space with a unique (up to equivalence) subsymmetric basic sequence that is not symmetric. The Tirilman space was introduced by Casazza and Shura as a modification of a well-known space of Tzafriri and is named after his Romanian surname. This space, denoted $Ti(p, \gamma)$, where $1 < p < +\infty$, $0 < \gamma < 1$, is defined as the completion of c_{00} equipped with a suitably defined norm depending on p and γ. The main theorem in this chapter, Theorem 29, states that for sufficiently small γ, every subsymmetric basic sequence in the dual space $Ti^*(p, \gamma)$ is equivalent to the subsymmetric canonical basis, which

is not symmetric. In other words, the dual space of Tirilman has a unique (up to equivalence) subsymmetric basic sequence, which is not symmetric. As an auxiliary result (Lemma 27), it is shown that $Ti^*(p,\gamma)$ does not contain an isomorphic copy of l_q, where q is the conjugate of p $(1/p + 1/q = 1)$. Consequences related to quotient spaces of the Tirilman space are also derived.

Chapter Three is dedicated to the geometric structure of the symmetrization of the dual space of the Schlumprecht space. The Schlumprecht space S is defined as the completion of c_{00} equipped with a suitably defined norm. It is known that the canonical basis of S is subsymmetric but not symmetric. The symmetrized version $S(S^*)$ of the dual space S^* is the space S^* equipped with a symmetrized norm. The main result in this chapter, Theorem 33, states that the space $S(S^*)$ contains a subspace isomorphic to l_1. As noted by the author, this result does not hold for the symmetrization of the dual of the classical Tsirelson space.

Chapter Four deals with the finite representability of c_0 in a given Banach space. A known result by Junge, Kutzarova, and Odell states that c_0 is finitely representable in the Tirilman space $Ti(2,1/2)$ disjointly with respect to the canonical basis. The author builds on ideas from this proof and the so-called "yardstick" construction to prove a similar result for a much broader class of parameters. Namely, for every $1 < p < +\infty$ and $0 < \gamma < 3^{-1/q}$, where $1/p + 1/q = 1$, the space c_0 is finitely representable in $Ti(p,\gamma)$ disjointly with respect to the canonical basis. The proof also provides an alternative argument that the canonical basis in the corresponding Tirilman space is not symmetric.

Chapter Five studies the geometry of Banach spaces through one of the classical approaches: searching for spaces that admit bilipschitz embedding into the given space. Motivated by results by Bourgain, Johnson, and Schechtman, which characterize non-superreflexive spaces through embeddings of binary trees, the so-called diamond graphs, and Laakso graphs, with uniformly bounded distortion, the author considers similar questions about embeddings of diamond and Laakso graphs. In particular, in Theorem 47, for every Banach space X that is not superreflexive, the author constructs an embedding of a Laakso graph of arbitrary level $n \geq 1$ into X with distortion $2 + \varepsilon$, $\varepsilon > 0$. A similar result for embeddings of diamond graphs was proved by Pisier. For the space $L_1[0,1]$, Theorem 49 proves even more: for every Laakso graph of level $n \geq 1$, there exists an embedding into $L_1[0,1]$ with distortion $4/3$. The corresponding result for diamond graphs was obtained by Lee and Raghavendra. For the specific cases $n = 2$ for Laakso and diamond graphs, Theorems 52 and 54, respectively, show embeddings into $L_1[0,1]$ with distortions bounded below by $9/8$ and $5/4$ respectively. The author has also developed a computer program to obtain these latter two estimates.

The doctoral student's research lies in a modern and difficult area of mathematics—Geometry of Banach spaces. He has studied a wide range of existing results on the dissertation topic and mastered a complex and highly non-trivial set of tools

to achieve his results. The proofs of many of the statements in the dissertation are quite difficult and extensive. The author's results are original contributions to Geometry of Banach spaces and, for the most part, constitute further developments of results obtained by highly prominent specialists in the field.

The doctoral student has three publications based on the dissertation, all in journals with impact factor, co-authored, fully meeting the requirements of the national low for development of the academic staff and the regulations of the Bulgarian Academy of Sciences and the Institute of Mathematics and Informatics. In my opinion, the doctoral student's contribution to the joint publications is equivalent.

The abstract accurately reflects the content of the dissertation. Some notes to be made include discrepancies in the numbering of key results compared to the dissertation itself, as well as that the Bulgarian translation is not good enough in terms of both style and grammar.

Conclusion: Based on the above, I consider Svetozar Stankov's PhD thesis to be an original contribution to the theory of Banach spaces, which fully satisfies the requirements of the national low for development of the academic staff and the regulations of BAS and IMI for the award of the educational and scientific degree Doctor. Therefore, I strongly recommend that the esteemed jury propose to the Scientific Council of IMI-BAS to confer upon Svetozar Zlatkov Stankov the educational and scientific degree "Doctor" in Mathematics.

Sofia, 21.05.2025

Acad. Julian Revalski