Review

for the defense procedure of the dissertation entitled:

"Symmetry and metric geometry in Banach spaces"

for obtaining the
educational and scientific degree "Doctor"

by

candidate:	**Svetozar Zlatkov Stankov**
Field of higher education:	**4. Natural Sciences, Mathematics and Informatics**
PhD program:	**Mathematical Analysis**
Professional field:	**4.5. Mathematics**

This review has been prepared by: **Assoc. Prof. Dr. Milen Nikolaev Ivanov** in his capacity as a member of the scientific jury.

1. **General characteristics of the dissertation and the presented materials**

   The presented dissertation has a volume of 70 pages and consists of 4 chapters, an introduction, a conclusion, and a bibliography of 54 titles.

   The results presenthed in the thesys are published in the articles:

   [1] Stephen J. Dilworth, Denka Kutzarova, Bünyamin Sari, Svetozar Stankov, Duals of Tirilman spaces have unique subsymmetric basic sequences, Bulletin of the London Mathematical Society, 56 (2024), issue 1, 150-158, JIF=0.8(2023), Q2 Mathematics (179/490)(2023);

   [2] S. J. Dilworth, Denka Kutzarova, Svetozar Stankov, Metric embeddings of Laakso graphs into Banach spaces, Banach Journal of Mathematical Analysis, 16 (2022), no. 4, Paper No. 60, 14 pp., JIF=1.1(2023), Q1 Mathematics (98/490)(2023);

   [3] S. Stankov, On the Symmetrized Dual of Schlumprecht Space, C. R. Acad. Bulg. Sci., vol. 78(2025), no. 1, 13–18, JIF=0.3(2023), Q4 Multidisciplinary Sciences (114/134)(2023).

   According to the Regulations in force, the article [2] gives 75 points, [1] 60 points and [3] 36 points, thus 171 points in total, which is much more than the required 30.

   The article [2] is quoted in a WoS indexed journal. It is excellent that S. Stankov has his own article [3].

2. **Personal data and my impression of the candidate**

From the attached CV, S. Stankov had been excellent in math competitions in school. After that, he brilliantly graduated from the Faculty of Mathematics and Informatics. I was a reviewer of his master's thesis with Prof. Troyanski, which was impressive.

My personal impressions of him are that he has a sharp intellect, immediately spotting the logical gaps.

3. **Analysis of the scientific and applied achievements of the candidate contained in the presented dissertation and publications included in the procedure**

S. Stankov's work contributes original results to at least three major fields in the theory of Banach spaces, all with an active history of at least 50 years and many turning points. For this reason, progress in them is very difficult. I like to joke that Banach spaces are like New York from the song: who can make it there can make it everywhere in analysis; confirmed by a number of mathematicians who built their reputation in Banach spaces and then shone where not.

These three fields are:

First, the classical Banach spaces, in this case L_1.

Second, Banach space geometry. Roughly speaking: Hilbert space is the most harmonious Banach space, the most convex and the smoothest. When other spaces have properties somehow similar to Hilbert's, to what extent are they similar to Hilbert space? For example, if every monotone basis grows in norm with sufficient speed, or a binary tree grows by maximum norm or in some average, etc.

Third, the study of non-classical spaces that block the standard classification of Banach spaces. Starting as interesting counterexamples, at the end of the 20th century this area became central, and the spaces built acquired their own theory, bit like the transcendental functions in analysis. These examples are remarkable in that at first glance nothing can be calculated for them because of the complex combinatorial definition of the norm of the sequences.

In the second chapter of the dissertation it is proved that a class of spaces has a unique subsymmetric basic sequence up to equivalence. What does this mean? It is clear that the standard bases of l_p and c_0 are symmetric in the sense that the order of indices plays no role, or put more scientifically, the canonical norm is invariant under any permutation of indices. In the extensive and deep study of the structure of Banach (sub)spaces, it is natural to consider basic sequences, i.e., those that are a basis for their closed linear span. Such basic sequences are symmetric if they are equivalent to their rearrangements, in the sense that the resulting Banach spaces are isomorphic - not the most workable definition, but the clearest at first glance. A remarkable property of l_p and c_0 is that every symmetric basic sequence is equivalent to the standard basis, i.e., in some sense, these spaces have only one

type of symmetric basic sequences. In the very recent work [8], an example is given of a space with a unique subsymmetric basic sequence in this sense. Subsymmetric means being unconditional and equivalent to its subsequences. From the current perspective, it is clear that this should be a substantially weaker condition, but how to give an example is not clear. And the examples are not at all easy to calculate, based on Tsirelson's space: the first example of a Banach space that does not contain l_p and c_0, because it does not contain any symmetric basic sequence. The present work develops the theory initiated in [8].

Any "non-symmetric" sequence space can be symmetrized by taking the supremum over the permutations of the indices. It is clear that the new space will be substantially different. Schlumprecht's space is a famous "Tsirelson-like" space in which the basis of unit vectors (sequences with exactly one 1 and the rest zeros) is subsymmetric but not symmetric. Its symmetrization contains a copy of l_1. The third chapter proves that this also holds for the symmetrization of its dual. This is a significant result because Schlumprecht's space can now be considered almost classical.

The fourth chapter is devoted to the key construction allowing a finite representation of c_0 as blocks of the basis of unit vectors of the spaces from the second chapter.

The fifth chapter investigates another topic. The closest the Hilbert space large class of Banach spaces is that of super-reflexive spaces, i.e., those that have an equivalent uniformly convex norm. They can be characterized by analytic-geometric "moduli", which don't have much to do with the norm, and this is one of the most exciting spheres in the theory of Banach spaces. The first result is that of Bourgain, who proves that binary trees cannot be embedded with finite distortion into a super-reflexive space. A binary tree is a graph where all vertices except the leaves are the midpoint of the line segment connecting two vertices from the next level. Distortion is a measure of how well metric relationships are preserved during embedding. Subsequently, similar results have been proven for two other types of graphs that can be defined iteratively, like binary trees: diamond and Laakso graphs. In the present work, the estimate of the minimal distortion with which Laakso graphs can be embedded in an arbitrary non-super-reflexive space is significantly improved.

4. **Approbation of the results**

The results have been published in 3 articles in good journals. One of them has already been cited. They have been presented at conferences.

5. **Quality of the abstract**

The abstract well presents the main results of the dissertation without going into details of the proofs, of course, but arousing curiosity to understand the latter.

The work is well placed in the context of the abundant history of the studied areas.

There are sufficient definitions of concepts that are quite specific and technical, and without definitions, the reader would be forced to search for them.

The volume is adequate.

The author's own contributions are highlighted clearly and accurately.

The Bulgarian translation of the abstract has a non-optimal distortion.

6. **Critical remarks and recommendations**

There are some mathematically insignigicant typos, like eg:

"...никое подпространство може да бъде представено..."(Definition 2, p. 3 of the Bulgarian abstract)

"The theorem in Chapter ?? leaves the question of symmetric minimal space open."(p.7 of the abstract)

Some definitions are repeated, e.g., the definition of a basic sequence on pages 11 and 25.

7. **Conclusion**

After reviewing the dissertation and accompanying scientific papers presented in the procedure, and based on the analysis of their significance and the scientific and applied contributions contained therein, **I confirm** that the presented dissertation and scientific publications accompanying it, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria, the Regulations for its implementation, and the relevant Regulations of the Bulgarian Academy of Sciences for the acquisition by the candidate of the educational and scientific degree "Doctor" in the scientific field 4. Natural Sciences, Mathematics and Informatics and professional field 4.5. Mathematics. In particular, the candidate satisfies the minimum national requirements in the professional field, and no plagiarism has been established in the scientific papers presented in the competition.

Based on the above, **I recommend** to the scientific jury to award Svetozar Zlatkov Stankov the educational and scientific degree "Doctor" in the scientific field 4. Natural Sciences, Mathematics and Informatics, professional field 4.5. Mathematics (Mathematical Analysis).

05/20/2025

Reviewer:

Assoc. Prof. Dr. M. Ivanov