REPORT

according to the procedure for the defence of a PhD Thesis entitled
„Symmetry and metric geometry in Banach spaces "

for the acquisition of PhD degree of the Bulgarian Academy of Sciences
by the PhD student: **Svetozar Zlatkov Stankov**
in the field of higher education: **4. Natural Sciences, Mathematics and Informatics**
professional field: **4.5. Mathematics (Mathematical Analysis)**

The referee report is prepared by Stanimir Troyanski, DSc, associated member of the Institute of Mathematics and Informatics, Bulgarian Academy of Science.

The PhD Thesis is written in very good English and is a text of 70 pages, A4 format, 54 titles are cited. The Thesis consists of six parts - an introduction, four parts, divided into paragraphs, in which the results of the dissertation are presented, and a conclusion.

The topic of the Thesis is in the field of Functional Analysis and in particular in the most important section of the Geometry of Banach spaces: the structure of infinite-dimensional Banach spaces.

In the 1970s, B. Tsirelson negatively solved the fundamental structural question of Banach Space Theory: Does every infinite-dimensional Banach space contain a subspace isomorphic to c_0 or l_p, $1 \leq p < \infty$. Tsirelson's idea is simple: one inductively constructs a series of norms in the space c_{00}, whose finite-dimensional subspaces have increasingly "worse" geometric characteristics. Finally, by taking the closed hull , with respect to the limit norm, one obtains a reflexive Banach space that is not superreflexive. The implementation of this idea is technically extremely difficult. The construction of Tsirelson is complex, as we have already noted, a sequence of norms is defined recursively. A complex combinatorial analysis is used. This is the beginning of the development of the machinery of "combinatorial" Banach spaces. T. Figiel and W. Johnson construct a superreflexive space that does not contain c_0 or l_p, $1 \leq p < \infty$. Latter, it turns out that the original Tsirelson's space does not contain symmetric basic sequences. Tsirelson space had an enormous impact on the theory of Banach spaces. Many constructions, similar in spirit, were defined to solve other important problems. Among those, we mention Dichotomy of T. Gowers(1996): Every infinite-dimensional Banach space X contains an infinite-dimensional subspace Y which either has an unconditional basis or is hereditarily indecomposable (the latter property means that no subspace of Y is isomorphic to a topological direct sum of infinite-dimensional spaces). In 2011 S. Argyros and R. Haydon solved the famous scalar-plus-compact problem by

providing an example of an infinite dimensional separable Banach space X so that every bounded linear operator from X to X has the form $\lambda Id + K$, where K compact operator. In fact, X is an isomorphic l_1 -predual hereditarily indecomposable space.Their construction depended on the method of J. Bourgain and F. Delbaen to produce "exotic" $\mathcal{L}^\infty$-spaces. For the study of the structure of Banach spaces, in addition to the names mentioned, we should also note the merits of T. Odell, Th. Schlumprecht, B. Maurey, P. Casazza, and others.

This is the area where most of the dissertation's results are. S. Stankov studies a particular space of Tsirelson type and shows that it contains a subspace isomorphic to l_1. This statement shows how carefully one must carry out the construction of these spaces.
The result of Theorem 7 of the dissertation is impressive. P. Casazza and T. Shura introduce a class of Banach spaces $\mathcal{T}$, which originated with L. Tzafriri.

In the thesis is proved that the canonical basis $(e_j)_{j=1}{}^\infty$ of the unit vectors of any particular $X \in \mathcal{T}$ is not symmetric, but every subsymmetric basis series in X is equivalent to $(e_j)_{j=1}{}^\infty$. The proof is complicated and is based on several facts that are of independent interest.

R. James find an effective characterisation of super -reflexive spaces in therms of finite dyadic trees. Using this characterisation P. Enflo showed that every super-reflexive is uniformly convex and uniformly smooth renormable. G.Pisier

using probabilistic analog of dyadic tree, the Walsh - Palay martingales showed that every super-reflexive Banach space can be renormed in such a way that its moduli of convexity and smoothness admits power type estimates. W. Johnson and G. Schechtman, characterized Banach spaces which are not super-reflexive in terms the diamond graphs and the Laakso graphs embed with uniformly bounded distortion. Using James' characterisation of non super- reflexivity (called *J*-convexity) S.Stankov maps Laakso graph $\mathcal{L}_n$ in non super-reflexive Banach space X. The most interesting case is when X is $L_1[0,1]$.(see Theorem 49/52 p.47/49)

For each $n \geq 1$, there exists a mapping $f_n : \mathcal{L}_n \to L_1[0, 1]$ such that, for all $a, b \in \mathcal{L}_n$,

$$\frac{3}{4}\, d(a, b) \leq \|f_n(a) - f_n(b)\|_1 \leq d(a, b).$$

If $f : \mathcal{L}_2 \to L_1[0, 1]$ satisfy

$$d(a,b) \leq \|f(a) - f(b)\|_1 \leq c\, d(a,b) \text{ for all } a, b \in \mathcal{L}_2. \text{ Then, } c \geq 9/8.$$

The dissertation is based on three articles. One was published in the prestigious general mathematical journal Bull. London Math. Soc., the second in the specialized journal Banach J. Math. Anal., and the third in C. R. Acad. Bulg. Sci. This gives me reason to believe that the results of the dissertation have been subjected to serious international expertise. Parts of the dissertation have been reported at various scientific forums. The autoreferat of the Thesis correctly reflects the content of

the dissertation. In conclusion, I would like to note that the PhD student has mastered a significant amount of mathematical methods and has skillfully applied it in solving specific problems. In my opinion, the results of the dissertation, both in quantity and quality, more than meet the standard requirements for a doctoral dissertation.

I am fully convinced that Svetozar Zlatkov Stankov deserves to be awarded the educational and scientific degree "DOCTOR" and I strongly recommend that the Scientific Jury award it to him.

Murcia, May 2025.