AN APPROACH FOR PRICING AMERICAN-STYLE DERIVATIVES

ABSTRACT OF DISSERTATION

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1 Preface

Derivatives are one of the most important instruments against the financial risks. They are based on some underlying object which can be a single asset, portfolio of assets, financial index, commodities, bonds, debt, other derivatives, volatility (Cboe Volatility Index), etc. All they are constructing to hedge the underlying against different market scenarios giving different rights and obligations for both their writer and holder. Furthermore, the more complicated instruments can be viewed as a transmission between different financial factors – assets, returns, risk, market uncertainty, etc – as well as a regulator between them. The evaluation of these constructions is one of the main tasks in financial mathematics. It is important to mention that the derivative's price is closely related to the value of the underlying object due to the strong dependence between them. The large class of the derivatives includes many different kinds – futures, forwards, options, swaps, some debt instruments, etc.

The options are some of the most tradable derivatives in the modern financial markets. In essence, they are contracts between two participants – writer and holder – that give some rights and obligations to both of them. Particularly, the options depend on an underlying asset and give their holder the right without obligation to sell or buy it at a pre-agreed price, named strike price, until or at some maturity date. The amount that the holder has to pay for this right forms the option's price and the related premium. There are two main types of options – European and American – this distinction is not geographically, but determined by their different characteristics. The main one is the moment they expire. The European contracts can be exercised only at the maturity date. Alternatively, an American option gives its holder the right to choose the expiration date. In such a way, he can capture immediately the payoff when the underlying asset reaches the desirable level. This additional right is very important for the investors and it explains the larger part that the American options have amongst all traded derivatives at the global markets. In addition, many exotic modifications exist – Asian, Bermuda, barriers, capped, straddle, strangle, cancellable, etc.

In modern financial theory, the asset prices are driven by stochastic processes – diffusions, Lévy processes, etc. We work under the log-normal assumptions staying in the base of the Black and Scholes (1973) model. Nonetheless, it does not capture some important practical phenomena observed at all financial markets – heavy tails, sudden jumps, volatility clustering, long-range dependence, leverage effects, etc. – we chose it because it is very intuitive and allows to be generalized. Something more, many of the derived results are true under quite complicated assumptions. Also, the Black-Scholes model is so fundamental, that every new approach has to be applied firstly to it for checking its scientific significance.

There are two main methods for pricing European-style derivatives. A main theorem in mathematical finance states that all discounted by the money-account price processes are martingales under the so-called risk-neutral measure. In a contrast to the natural probability measure that describes the market uncertainty (the so-called real-world measure), the risk-neutral one is a mathematical construction based on some non-arbitrage arguments. It allows the fair derivative price to be obtained as the mathematical expectation of its discounted payoffs. Alternatively, if the derivative price is considered as a function of the time and the current price of the underlying asset, then this function solves a partial differential equation with boundary conditions determined by the payoffs. The form of the equation depends on the stochastic process that describes the underlying asset via its infinitesimal generator. For example, if the price process is log-normal, then we have a parabolic heat-style equation. Alternatively, if it is Lévy one, then we reach a pseudo(integro)-differential equation. In fact, this duality is an application of the well-known relation stated by Kolmogorov – the solution of a differential equation composed by the infinitesimal generator of some Feller-Markov process can be derived as the mathematical expectation of the boundary conditions taken for this process.

On the other hand, the American option pricing problem is more complicated. Obviously, the early exercise feature leads to an optimal stopping task. We need to divide the state space into two subsets. In the first one, the optimal set, the immediate exercise is the best holder's strategy. On the contrary, keeping the option alive leads to a better financial result if the spot price falls in the so-called continuation set. This way the holder exercises when the underlying asset reaches it. The boundary between these sets is known as the optimal or early exercise boundary. Usually, the optimal stopping problems are viewed as free boundary differential tasks through the related variational inequalities. Thus the American options are evaluated by a two-dimensional dynamical system of integral equations which has to be solved numerically. Alternatively, using some first-hitting properties of the Brownian motion, we approximate the optimal boundary maximizing the holder's financial result. As a consequence, we can derive the option price with a high-enough accuracy in a real time. In addition, once the optimal boundary is known, the free boundary differential task turns into a boundary value problem in a known region for which many numerical methods are applicable. The derivatives without maturity restrictions, known as perpetuals, are specific. Their optimal boundaries are time-independent since the holder and writer are not threatened by a forced exercise at maturity. This feature allows solving the pricing problem in a closed form. Although the perpetual derivatives are rarely traded, mainly on some unofficial markets, they provide very important information. The asymptotic characteristics they describe, destine the whole behavior of the optimal boundaries as well as the prices under the finite maturities.

2 Introduction

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2.1 Aim of the dissertation

The main purpose of this work is to investigate the American-stye financial instruments and to construct a fast and accurate method for their analysis and evaluation. Our research is stated in the framework of the famous Black and Scholes (1973) model assuming that the underlying asset is driven by a log-normal stochastic process. The main feature that distinguishes the American from the other derivatives is the early exercise right that the holder may use at an arbitrary moment until maturity. However, namely this characteristic makes the analysys of such instruments difficult due to the decision-making task they lead.

We first aim to examine some classical instruments such as the usual American options as well as their capped versions working with the suggested novel approach. Based on the derived results, we shall systematize mathematically these derivatives w.r.t. to the kind of the stopping problem they lead – one-sided (hitting) task or two-sided (exit) task. We will examine in detail the properties of the so-separated groups. Thus we can fulfill the next purpose, namely to suggest and explore a rigorous mathematical method for defining and investigating new financial derivatives that meet the various needs of the investors. They include strangle strategies, quadratic strangles, powered instruments, cancellable options, and other instruments with generalized payoffs. We shall provide several such examples. Last but not least, we strive to implement the theoretical result via the platform for mathematical computing MATLAB.

2.2 Actuality of the topic

The early exercise right that characterizes the American-style derivatives implies their significance for the financial industry. This fact is supported by the largest part of the traded assets they have in the modern markets. They are important not only as traded sources but also as a regulator of market uncertainty as well as an anti-risk factor. On the other hand, the financial crisis of 2008 led to the appearance of different novel instruments against the arising new risks. Furthermore, the investors need both fast and accurate methods for evaluation. This is even more true in light of the increasing high-frequency trading in the recent decade. All these destine the outstanding importance of contemporary scientific investigations in this field – a fact supported by the increasing literature devoted to the topic.

2.3 Original theoretical and practical findings

The presented dissertation provides several theoretical generalizations as well as solutions of the problems arising from the financial practice.

Two main questions stand for the holder of an American-style financial instrument – is it optimal to exercise immediately and, if not, what is the fair price of the derivative. The main existing methods for evaluating such instruments are based on a dynamic approximation of the pair price-optimal boundary. They are relatively high time consuming. We solve this problem by building a fast and efficient method for approximating the optimal boundary at a rare time grid. As a consequence, we price the option with high precision (fourth sign after the decimal point) in real-time. Thus, we can decide almost immediately whether the moment is suitable for exercising and what the option price is if keeping is preferable. This is of outstanding importance in light of the increasing part of high-frequency trading in modern financial markets.

On the other hand, if we need a denser grid, we construct several methods for solving the pricing problem, which are based on Monte Carlo simulations as well as on various finite difference schemes.

Our methodology is based on several steps. We first solve the pricing task without maturity constraints. Then we approximate the optimal boundaries under a finite maturity horizon maximizing the financial result of the option's holder. As a consequence, we deal with the pricing problem. Later, we discuss this scheme in detail.

The so-establish method is applied to several classical instruments including the options and their capped versions. In addition, we generalize the theoretic results by defining and investigating several new classes of financial derivatives.

Another theoretical contribution of the dissertation is an approach for distinguishing the kinds of the financial derivatives w.r.t. their optimal regions. We differentiate three major types. For the call-style instruments, the optimal points are above some boundary, whereas for the put ones, they are below. In addition, we provide a condition that leads to two optimal sets – above and below the related boundaries. This leads to optimal strategies that are an exit from a strip.

To develop all the above-mentioned constructions, we need several results related to the Laplace transform of the first hitting time of a Brownian motion, or more precisely of its density. In probabilistic terms, this is the moment generating function of the stopping time. We prove the necessary statements in a separate chapter.

The practical findings of the dissertation are in several directions. First, we apply the theoretical results to some existing American-style instruments – classical options, capped options, strangle strategies, cancellable options, etc. Second, we build a method for investigating derivatives with generalized payoffs distinguishing put- and call-style instruments as well as hybrid two-sided ones. Particularly, we suggest several new American-style derivatives such as strangles with arbitrary strikes and different weights for the call and put legs, quadratic strangles, power futures contracts, cancellable options with a convertible penalty, etc. These instruments can address many requirements of the investors. The arbitrary strikes as well as the different weights in the strangles allow the construction of many novel strategies combining the put and call features. The proposed quadratic strangles allow stronger hedging of the risky positions that are far-from-the-money. The opposite is true for the near-the-money positions. This is achieved by a payoff changed from $|x-K|$ to $(x-K)^2$. These conclusions hold for the power futures contracts too. Something more, the investigated generalized payoffs can be used for the specific needs of an investor. On the other hand, the cancellable options give their writer a possibility for early canceling. The price of this right is a penalty amount above the usual option payoff. We enlarge the flexibility of these instruments by considering three component penalties – a fixed amount, some share of the underlying asset, and a proportion of the payoff.

The so-established theoretical methods are practically validated via software – we use the platform for mathematical computing MATLAB. The closed-form formulas for the perpetual instruments are implemented. On the other hand, we approximate the optimal boundaries for the finite maturities and thus provide a relatively fast pricing method. In addition, we construct different numerical approaches based on Monte Carlo simulations or different finite difference schemes based on significantly denser grids. These tools are prepared for all studied options – classical ones, capped options, powered instruments, strangles and quadratic strangles, cancellable options, etc. Something more, the proposed methods can be applied to different new instruments.

2.4 Motivation and classical methods

In recent years and even more so after the financial crisis of 2008, there has been an increased interest in the international financial markets to the financial derivatives since they are one of the major instruments against financial risk. They exhibit a very large variety – most popular are options, futures, bonds, swaps, etc. Conventionally speaking, we can recognize two types – European and American. The European derivatives have a previously fixed date at which the transaction is executed. Alternatively, the American-style instruments give its holder the right to choose when to exercise until maturity. This right makes the American derivatives preferable for the investors and determines the largest segment of the traded assets they have in the modern financial markets. Many kinds of such instruments are available, a fact which leads to a growing scientific literature devoted to the topic. Since the holder has the right to choose the exercise moment, it is natural to assume that he will follow a strategy that maximizes his profit. Thus we reach to a problem for optimal stopping – see Lamberton and Lapeyre (1996) or Wong (1996). These problems are solutions of the so-called free boundary or obstacle problems – see Bensoussan (1984), Jaillet et al. (1990), Kim (1990), Jacka (1991), Peskir and Shiryaev (2006), Pascucci (2008), or Magirou et al. (2020). For these tasks, we know the differential equation and we have to find its solution as well as the region in which it holds. The theory for this kind of equations can be found in Bather (1970), Moerbeke (1973), Friedman (1975), Friedman (2010), and Shiryaev (2009). Deriving a closed-form solution is hard and often impossible except when there are no maturity constraints. For this many authors propose different numerical solutions. Cox et al. (1979) suggest a very useful one based on the binomial trees. Other interesting numerical methods are proposed by Johnson (1983), Geske and Johnson (1984), Barone-Adesi and Whaley (1987), Bjerksund and Stensland (1993), Ho et al. (1994), Ju (1998), and Longstaff and Schwartz (2001). We mention also the works of Brennan and Schwartz (1977), Myneni (1992), Karatzas (1988), Rogers (2002), and Huang et al. (1996). They use a recursive method based on the classical works of Kim (1990), Jacka (1991), and Carr et al. (1992). Some comparisons are presented in Zhao (2018). On the other hand, the optimal boundary is time-independent for the perpetual options, which allows deriving the closed-form formulas – see Shiryaev et al. (1995) or Shiryaev et al. (1994), in addition to the mentioned above works. Some formulas for Lévy driven models can be found in Gerber and Shiu (1994), Pham (1997), Gerber and Shiu (1996), Mordecki (1999, 2002), Boyarchenko and Levendorskii (2002), Levendorskiĭ (2004), Alili and Kyprianou (2005), and Ivanov (2007). An overview of the credit derivatives is provided in Popchev and Radeva (2008).

2.5 Methodology

The financial instruments considered in this dissertation are studied through the methodology summarized in the following steps:

1. We obtain the shape of the optimal regions and the related early exercise boundaries.

2. We solve the pricing problem in a closed form when there are no maturity constraints.

3. We derive the pricing function of a derivative that expires when the log-price of the underlying asset reaches a piecewise linear boundary or exits from a strip formed by such functions.

4. We approximate the optimal boundary in a rare time grid. Thus we recognize whether the immediate exercise is optimal or not.

5. We derive the option price based on this approximation. This is a fast and relatively accurate procedure – the error is in the fourth sign after the decimal point.

6. We approximate the whole boundary if we need a denser grid.

7. We solve the related Black-Scholes style equation numerically by a finite difference scheme or by a Monte Carlo simulation.

3 Structure of the dissertation

The dissertation consists of sixteen sections based on thirteen published papers:

1. Zaevski (2020d) Laplace transforms for the first hitting time of a Brownian motion. *Comptes rendus de l'Académie bulgare des Sciences*, 73(7):934–941, 2020a. ISSN 2367-6248 (print), 2603-4832 (online).
   doi: 10.7546/CRABS. 2020.07.05.
   URL http://www.proceedings.bas.bg/index_old.html.

2. Zaevski (2021b) Laplace transforms of the Brownian motion's first exit from a strip. *Comptes rendus de l'Académie bulgare des Sciences*, 74(5):669–676, 2021a. ISSN 2367-6248 (print),2603-4832 (online).
   doi: 10.7546/CRABS.2021.05.04
   URL http://www.proceedings.bas.bg/index_old.html.

3. Zaevski (2024a) Some limits for the Laplace transform of the Brownian motion's first hit to a linear function. *Serdica Mathematical Journal*, 50(2):183–202, 2024a. ISSN 1310-6600 (print), 2815-5297 (online).
   doi:10.55630/serdica.2024.50.183-202.
   URL https://serdica.math.bas.bg/index.php/serdica/article/view/87.

4. Zaevski (2021a) A new approach for pricing discounted American options. *Communications in Nonlinear Science and Numerical Simulation*, 97:105752, 2021b. ISSN 1007-5704 (print), 1878-7274 (online).
   doi: https://doi.org/10.1016/j.cnsns.2021.105752.
   URL https://www.sciencedirect.com/science/article/pii/S1007570421000630.

5. Zaevski (2022a) Pricing discounted American capped options. *Chaos, Solitons & Fractals*, 156:111833, 2022a. ISSN 1007-5704 (print), 1878-7274 (online).
doi: https://doi.org/10.1016/j.chaos.2022.111833.
URL https:// www.sciencedirect.com/science/article/pii/S0960077922000443.

6. Zaevski (2024c) On some generalized American style derivatives. *Computational and Applied Mathematics*, 43(3):115, 2024b. ISSN 2238-3603 (print), 1807-0302 (online).
doi: https://doi.org/10.1007/s40314-024-02625-6.
URL https://link.springer.com/article/10.1007/s40314-024-02625-6.

7. Zaevski (2023b) American strangle options with arbitrary strikes. *Journal of Futures Markets*, 43(7):880–903, 2023a. ISSN 0270-7314 (print), 1096-9934 (online).
doi: https://doi.org/10.1002/fut.22419.
URL https://onlinelibrary.wiley.com/doi/abs/10.1002/fut.22419.

8. Zaevski (2024b) Quadratic American strangle options in light of two-sided optimal stopping problems. *Mathematics*, 12(10):1449, 2024c. ISSN 2227-7390.
doi: 10.3390/math12101449.
URL https://www.mdpi.com/2227-7390/12/10/1449.

9. Zaevski (2020b) Discounted perpetual game call options. *Chaos, Solitons & Fractals*, 131:109503, 2020b. ISSN 0960-0779 (print), 1873-2887 (online).
doi: https://doi.org/10.1016/j.chaos.2019.109503.
URL http://www.sciencedirect.com/science/article/pii/S0960077919304552.

10. Zaevski (2020c) Discounted perpetual game put options. *Chaos, Solitons & Fractals*, 137:109858, 2020c. ISSN 0960-0779 (print), 1873-2887 (online).
doi: https://doi.org/10.1016/j.chaos.2020.109858.
URL http://www.sciencedirect.com/science/article/pii/S0960077920302587.

11. Zaevski (2020a) Perpetual game options with a multiplied penalty. *Communications in Nonlinear Science and Numerical Simulation*, 85:105248, 2020d. ISSN 1007-5704 (print), 1878-7274 (online).

doi: https://doi.org/10.1016/j.cnsns.2020.105248.
URL http://www.sciencedirect.com/science/article/pii/S1007570420300812.

12. Zaevski (2023a) Perpetual cancellable American options with convertible features. *Modern Stochastics: Theory and Applications*, 10(4):367–395, 2023b. ISSN 2351-6046 (print), 2351-6054 (online).
doi: 10.15559/23-VMSTA230.
URL https://www.vmsta.org/journal/VMSTA/article/273/read

13. Zaevski (2022b) Pricing cancellable American put options on the finite time horizon. *Journal of Futures Markets*, 42(7):1284–1303, 2022b. ISSN 0270-7314 (print), 1096-9934 (online).
doi: https://doi.org/10.1002/fut.22331.
URL https://onlinelibrary.wiley.com/doi/abs/10.1002/fut.22331.

All of the papers are independent. The article with numbers 1, 9, 10, and 11 are used in a previous procedure for the position of associated professor. However, the results obtained in these papers are essentially used in the later publications. Some MATLAB codes that implement the derived results are presented in a separate chapter. The reference lists contains 294 sources.

4 Main results

4.2 First hitting time properties

In Chapter 2 we obtain some results about the Brownian motion first hits to a (piecewise) linear boundaries. The necessity of these results is motivated by the log-normal process we use to model the underlying asset

$$dS_t = rS_t dt + \sigma S_t dB_t$$

If the boundary is denoted by $b(t) = b_1 t + b_2 > K$, the first hit to it by τ, the maturity date by T, and the strike by K, then the present value of the call payoff can be written as

$$\begin{aligned}\mathbb{E}\left[e^{-\theta t}\left(S_t - K\right)^+\right] &= S_0 \mathbb{E}\left[e^{-\left(\theta - r + \frac{\sigma^2}{2}\right)\tau + \sigma b(\tau)} \Lambda_T\right] - K\mathbb{E}\left[e^{-\theta\tau}\Lambda_T\right] \\ &+ S_0 e^{-\left(\theta - r + \frac{\sigma^2}{2}\right)T} \mathbb{E}\left[e^{\sigma B_T} I_{\tau \geq T, S_T > K}\right] - K\mathbb{P}\left(S_T \in (K, b(T)), \tau \geq T\right),\end{aligned}$$

where Λ_t is the indicator τ to happen before t. Thus we need the Laplace transform of the stopping time if the first hit is before the maturity, and the Laplace transform of the Brownian motion if the hit is after the maturity. These results are proven in Chapter 2.2 as Theorems 2.1 and 2.2:

Theorem 2.1 (Theorem 3.1 from Zaevski (2020d))**.** *Let $\theta > 0$. The Laplace transform of τ before T is given by*

$$L(T,\theta;b_1,b_2) = \mathbb{E}\left[e^{-\theta\tau}\Lambda_T\right] = e^{b_2\left(\sqrt{b_1^2+2\theta}-b_1\right)} g\left(T;\sqrt{b_1^2+2\theta},b_2\right), \tag{1}$$

where the function $g(\cdot)$ is

$$g(T;b_1,b_2) \equiv \mathbb{P}(\tau < T) = 1 - N\left(\frac{b_1 T + b_2}{\sqrt{T}}\right) + \exp(-2b_1 b_2) N\left(\frac{b_1 T - b_2}{\sqrt{T}}\right).$$

Theorem 2.2 (Theorem 3.2 from Zaevski (2020d))**.** *If $z < b(T)$, then*

$$\begin{aligned} &V(\theta,z,T;b_1,b_2) \equiv \mathbb{E}\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right] = \\ &= \exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} N\left(\frac{b(T)-T\theta}{\sqrt{T}}\right) - N\left(\frac{z-T\theta}{\sqrt{T}}\right) \\ +e^{2b_2(\theta-b_1)}\left(N\left(\frac{z-T\theta-2b_2}{\sqrt{T}}\right) - N\left(\frac{b(T)-T\theta-2b_2}{\sqrt{T}}\right)\right) \end{array}\right]. \end{aligned} \tag{2}$$

If the function $b(t)$ is piecewise linear, then these theorems turn into

Theorem 2.3 (Theorem 4.1 from Zaevski (2020d))**.** *Let $\theta > 0$. The Laplace transform of the first hitting time in the m-th interval is given by*

$$\begin{aligned} &\mathbb{E}\left[e^{-\theta\tau} I_{\tau\in(t_{m-1},t_m]}\right] \\ &= \int_{-\infty}^{\beta_1,\ldots,\beta_{m-1}} \left(\begin{array}{l} \prod_{i=1}^{m-1}\left(1-\exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{-\theta t_{m-1}} L\left(t_m-t_{m-1},\theta;b_{1,m},\beta_{m-1}-x_{m-1}\right) \end{array}\right) dx_1 \ldots dx_{m-1}, \end{aligned}$$

where the function $L(\cdot)$ is given by equation (1).

Theorem 2.4 (Theorem 4.2 from Zaevski (2020d)). *If $z < b(T)$, then the Laplace transform when first hitting is after the terminal moment is*

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T > z, \tau > T}\right] = \int_{-\infty}^{\beta_1,\ldots,\beta_{n-1}} \left(\begin{array}{l} \prod_{i=1}^{n-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i - x_i)}{t_i - t_{i-1}}\right)\right) \\ \prod_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ e^{\theta x_{n-1}} V\left(\theta, z - x_{n-1}, t_n - t_{n-1}; b_{1,n-1}, \beta_{n-1} - x_{n-1}\right) \end{array} \right) dx_1 \ldots dx_{n-1},$$

where the function $V(\cdot)$ is given by equation (2).

If we have a financial derivative with bounded from both sides continuation region, then the stopping time turns to a first exit from a strip. Such derivatives are the straddle strategies, strangles, and cancellable options. The desired results are obtained and presented in theorems 2.6, 2.7, and 2.8 from Chapter 2.4. Also, we prove the results when the boundaries are piecewise linear and one of them vanishes after some moment – Theorem 2.9.

We investigate also in this chapter some limits related to the studied Laplace transforms. Their necessity arises when we examine the so-called perpetual derivatives, i.e. without maturity constraints – $T = \infty$. The results are formulated in two theorems – 2.10 and 2.11:

Theorem 2.10 (Theorem 3.1 from Zaevski (2024a)). *Let θ be a positive number and ζ be the first hit of a Brownian motion to the linear function $b(t) = b_1 t + b_2$. The following statements hold.*

1. *If $\{b_2 = 0\}$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.*
2. *If $\left\{b_2 \neq 0, k < -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.*
3. *If $\left\{b_2 \neq 0, b_1 = \theta, k = -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.*
4. *If $\left\{b_2 \neq 0, b_1 = \theta, k > -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.*
5. *If $\left\{b_2 > 0,\ b_1 < \theta,\ k = -\frac{\theta^2}{2}\right\}$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.*

6. *If* $\left\{b_2 > 0,\ b_1 > \theta,\ k = -\frac{\theta^2}{2}\right\}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 1 - e^{2b_2(\theta - b_1)}$.

7. *If* $\left\{b_2 > 0,\ b_1 > \theta,\ k > -\frac{\theta^2}{2}\right\}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.

8. *If* $\left\{b_2 > 0,\ b_1 < \theta,\ -\frac{\theta^2}{2} < k \le \frac{b_1^2}{2} - \theta b_1\right\}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.

9. *If* $\left\{b_2 > 0,\ b_1 < \theta,\ \frac{b_1^2}{2} - \theta b_1 < k\right\}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.

10. *If* $\left\{b_2 < 0,\ b_1 > \theta,\ k = -\frac{\theta^2}{2}\right\}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.

11. *If* $\left\{b_2 < 0,\ b_1 < \theta,\ k = -\frac{\theta^2}{2}\right\}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 1 - e^{2b_2(\theta - b_1)}$.

12. *If* $\left\{b_2 < 0,\ b_1 < \theta,\ k > -\frac{\theta^2}{2}\right\}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.

13. *If* $\left\{b_2 < 0,\ b_1 > \theta,\ -\frac{\theta^2}{2} < k \le \frac{b_1^2}{2} - \theta b_1\right\}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.

14. *If* $\left\{b_2 < 0,\ b_1 > \theta,\ \frac{b_1^2}{2} - \theta b_1 < k\right\}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.

Theorem 2.11 is about a similar problem when the Brownian motion is restricted by another linear function.

4.3 Preliminaries.

As we mentioned above, our investigation is stated in the framework of the famous Black and Scholes (1973) model. The underlying asset is driven by the log-normal diffusion

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{3}$$

where r is the risk-free rate of return and σ is the volatility. Note that the dynamics (3) is under the risk-neutral measure. Suppose that a European-style derivative expires when the asset leaves a region D whose boundary is

∂D. If τ is the first exit and the payoff function is $G(t,x)$, then the risk-neutral pricing principle states that the derivative price can be obtained as the process

$$Y_t = \mathbb{E}^{t,S_t}\left[e^{-r(\tau-t)}G(\tau,S_\tau)\right]. \tag{4}$$

Alternatively, we can derive the price as a function of time and the current asset price – $Y_t = V(t,S_t)$. The Kolmogorov backward equation states that the function $V(\cdot,\cdot)$ solves the boundary value problem (BVP)

$$\begin{aligned} &V_t(t,x) + \left(\mathcal{A}^{\mathbb{Q}}V\right)(t,x) - rV(t,x) = 0, \quad (t,x)\in D \\ &V(t,x) = G(t,x), \quad (t,x)\in \partial D, \end{aligned} \tag{5}$$

where $\mathcal{A}^{\mathbb{Q}}$ is the infinitesimal generator of diffusion (3) w.r.t. the risk-neutral measure $\mathbb{Q}$. Equation (5) is the famous Black-Scholes equation. If the asset pays continuously dividends at rate λ, then risk-neutral dynamics (3) turns into

$$dS_t = (r-\lambda)S_t dt + \sigma S_t dB_t.$$

Of course, Black-Scholes equation (5) needs a little modification. Alternatively, we introduce dividends using an approach suggested by Shiryaev et al. (1995). The risk-neutral dynamics (3) keeps its form but the payoff is changed to $N(t,x) = e^{\lambda t}g(x)$. This method is possible due to the following theorem:

Theorem 3.1 (See Proposition 2.3 from Zaevski (2020c)). *A (r,λ,δ)-model is equivalent to a $(r-\delta,\lambda+\delta,0)$-one in the sense that the derivative has the same price under both models at the initial moment.*

Thus the call and put payoffs are given by

$$\begin{aligned} N(t,x) &= e^{-\lambda t}(x-K)^+ \quad \text{call} \\ N(t,x) &= e^{-\lambda t}(K-x)^+ \quad \text{put}. \end{aligned}$$

Let us discuss the American pricing task. The holder's right to exercise prematurely leads to an optimal stopping problem. Thus Black-Scholes equation (5) turns into a free boundary problem – we need to derive its solution as well as the region D where it holds. The classical method for solving these problems uses a two-dimensional integro-differential system for the free boundary and the price function.

In the present investigation, we suggest a different approach based on maximization of the holder's utility. Along with its intuitiveness, this method has another major advantage – it needs a relatively low computational time. On the other hand, often the holder does not need to know what is the fair option price, but only whether it is optimal to exercise immediately or not. If we use the classical approach, we need to solve the two-dimensional system whereas our approach returns immediately the optimal value at the current moment.

In its essence, our method separates the free-boundary problem into two parts. We first approximate the optimal boundary. Thus the task turns into a terminal value problem in a known region. We can solve it in two ways – via expectation (4) or via BVP (5). We shall construct later some Monte Carlo methods for deriving expectation (4). Alternatively, we modify the explicit, implicit, and Crank-Nicolson finite difference methods to the BVP. Quite expected, the latter is most appropriate due to its stability (A-stability, not L-stability).

We shall use the following constants during the whole work:

$$\begin{aligned} p &:= \frac{x_1 - x_2}{\sigma} = 2\sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} \\ q &:= -\frac{x_2}{\sigma} = \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \frac{r}{\sigma^2} - \frac{1}{2}. \end{aligned} \tag{6}$$

They are related to the roots of a quadratic equation that characterizes the first hit of the Brownian motion to a linear boundary. We have that $p \geq q+1$. The equality holds only in the undiscounted case $\lambda = 0$.

4.4 A new approach for pricing discounted American options.

In this chapter, we investigate the American options through our approach in the presence of additional discounting. For some fundamental results in the area we refer to works Kim (1990), Jacka (1991), Jacka (1991), and Carr et al. (1992).

All main properties of the optimal boundaries and the related sets are proven. As usual, the holder of an American call exercises when the underlying asset is above the optimal boundary – the opposite for a put. The results for the perpetual options are summarized in Theorems 4.1 and 4.2:

Theorem 4.1. *The price of a discounted American perpetual call option written on the asset with an initial price below the exercise boundary* $S_0 = x < c$ *and strike price* K *is*

$$V(x) = \left(\frac{x}{p-q}\right)^{p-q} \left(\frac{p-q-1}{K}\right)^{p-q-1},$$

where p *and* q *are defined by formulas* (6)*. Note that* $p > q+1$ *when* $\lambda > 0$.

If the initial asset value is above the exercise boundary, then the option price is $V(x) = x - K$*. The exercise boundary is*

$$c = \frac{p-q}{p-q-1} K.$$

The optimal stopping moment is the first hit to the interval $[c, \infty)$.

Theorem 4.2. *If the initial asset price is above the early exercise boundary,* $S_0 = x > c$*, then the price of a discounted American put is*

$$V(x) = \left(\frac{K}{q+1}\right)^{q+1} \left(\frac{q}{x}\right)^{q},$$

where q *is defined in formulas* (6)*. If the initial asset value is below the exercise boundary, then the option price is*

$$V(x) = K - x.$$

The exercise boundary is

$$c = \frac{q}{q+1} K.$$

The optimal stopping moment is the first hit to the interval $[0, c]$.

We summarize now the approach for approximating the optimal boundary for an American put by the exponent of piecewise linear functions. Let the time interval $[0, T]$ be divided into n sub-intervals $0 \equiv t_0 < t_1 < ... < t_n \equiv T$. Suppose that the holder's strategy ζ is to exercise when the asset reaches the level $\exp(a_i t + b_i)$ if this happens in the interval $[t_{i-1}, t_i)$, $i = 1, 2, ..., n$. We also state continuity $\exp(a_i t_i + b_i) = \exp(a_{i+1} t_i + b_{i+1}) \equiv C_i$. Assume that the underlying asset starts from the value x, i.e. $S_0 = x$. Therefore the exercise happens when the Brownian motion touches the level

$$\frac{1}{\sigma}\left(\left(a_i - r + \frac{\sigma^2}{2}\right) t + b_i - \log(x)\right) = A_i t_i + B_i$$

for

$$\begin{aligned} A_i &= \frac{1}{\sigma}\left(a_i - r + \frac{\sigma^2}{2}\right) \\ B_i &= \frac{b_i - \log(x)}{\sigma}. \end{aligned}$$

Let us define a derivative which pays amount of $\exp(-\lambda(\zeta \wedge T))(K - S_{\zeta\wedge T})^+$ at the moment $\zeta \wedge T$. We denote its price by

$$\begin{aligned} &V(x; \{t_0, ..., t_n\}; \{C_0, ..., C_n\}) = \mathbb{E}^x\left[e^{-(r+\lambda)(\tau\wedge T)}(K - S_{\tau\wedge T})^+\right] \\ &= \mathbb{E}^x\left[e^{-(r+\lambda)\tau}(K - S_\tau)\Lambda_T\right] + E^x\left[e^{-(r+\lambda)T}(K - S_T)^+\Phi_T\right] \\ &= K\mathbb{E}\left[e^{-\alpha_1\tau}\Lambda_T\right] - x\sum_{m=1}^{n} e^{\sigma B_m} E\left[e^{-\alpha_{2,m}\tau} I_{t_{m-1}<\tau\le t_m}\right] \\ &+ Ke^{-\alpha_1 T}\mathbb{Q}(B_T < k, \Phi_T = 1) - xe^{-\alpha_3 T}\mathbb{E}\left[e^{\sigma B_T} I_{B_T<k, \Phi_T=1}\right] \end{aligned} \tag{7}$$

for

$$\begin{aligned} \alpha_1 &= r + \lambda \\ \alpha_{2,m} &= (r + \lambda) - \left(r - \frac{\sigma^2}{2}\right) - A_m\sigma = \frac{\sigma^2}{2} - A_m\sigma + \lambda \\ \alpha_3 &= \lambda + \frac{\sigma^2}{2} \\ k &= \frac{1}{\sigma}\log\left(\frac{K}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) T. \end{aligned} \tag{8}$$

Our algorithm is based on equations (7) and (8).

1. The value of the exercise boundary at the maturity, C_n, is given by

$$c\,(T) = \min\left(\frac{r+\lambda}{\lambda}, 1\right) K.$$

2. Suppose that we have found values C_m, C_{m+1}, ..., C_n for some $m \leq n$. Let us fix some $x \leq K$ and denote by $C\,(x)$

$$C\,(x) = \arg\max\left\{C : V\left(x; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\}; \{C, C_m, ..., C_n\}\right)\right\}.$$

We can find C_{m-1} through one of the formulas

$$C_{m-1} = \max\left\{x : C\,(x) = x\right\}$$
$$C_{m-1} = \max\left\{x : V\left(x; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\}; \{C\,(x), C_m, ..., C_n\}\right) = K - x\right\}.$$

Functions $V\left(x; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\}; \{C, C_m, ..., C_n\}\right)$ are calculated via equations (7) and (8).

Let us discuss this algorithm. If the true optimal boundary is indeed an exponent of a piecewise linear function, then the constant $C\,(x)$ will not depend on the particular value of the initial asset price x. So we could fix one x and derive the value of C. However, this assumption is unreasonable and thus we need a different approach. We look for the largest x for which $C\,(x) = x$ – this is the largest initial asset value that makes the immediate exercise optimal.

It turns out that this algorithm approximates the optimal boundary very precisely using only three steps. The result for the option price is also near to the real one. The computational time is negligible. If we need extremely high precision, then we may run several times the algorithm with different time-divisions and this way approximate the boundary at a dense grid. Based on it, we create a Monte Carlo method for deriving expectation (4) having in mind some results of Wang and Pötzelberger (1997):

1. We generate $m-1$ normal random numbers with zero expectation and standard deviation one. They form the vector u.

2. Let D be the $(m-1)\times(m-1)$ diagonal matrix composed by values $\sqrt{T/N}$ and M be a $(m-1)\times(m-1)$ lower triangle matrix with values ones. We define the vector x as $x = MDu$.

3. We calculate the values of the function

$$w_j = e^{-\alpha t_{m-1}} L\left(t_m - t_{m-1}, \alpha; a_m, b_m - x_{m-1}\right).$$

4. We derive the values of the function

$$v_j = v\left(x_1, ..., x_{m-1}\right) = \prod_{i=1}^{m-1} I_{x_i<\overline{c}_i}\left(1 - \exp\left(-\frac{2\left(\overline{c}_{i-1} - x_{i-1}\right)\left(\overline{c}_i - x_i\right)}{\frac{T}{N}}\right)\right).$$

5. We calculate the term $p_j = w_j v_j$.

6. We calculate the truncated Laplace transform repeating the above procedure H times and averaging $\left(\sum_{i=1}^{H} p_i\right)\Big/ H$.

The last two terms in equation (7) are obtained in the same way, by taking $m = n$ and changing the term $e^{-\alpha t_{m-1}} L(\cdot)$ in w (step 3) by $e^{-\alpha x_{m-1}} U(\cdot)$ and $e^{-\alpha x_{m-1}} V(\cdot)$, respectively.

Some symmetries obtain the results for the call options.

4.5 Pricing discounted American capped options

For some important results we refer to Broadie and Detemple (1995) and Detemple and Tian (2002). The capped option restricts the spot value at which the holder can exercise by a fixed level – say L. Thus the payoffs turn into

$$N(t,x) = e^{-\lambda t}\left(S_t \wedge L - K\right)^+$$
$$N(t,x) = e^{-\lambda t}\left(K - S_t \vee L\right)^+,$$

Accordingly with Broadie and Detemple (1995) we establish the shape of the optimal boundaries:

Theorem 5.1 (Theorem 3.1 of Zaevski (2022a))**.** *If $c^A(t)$ is the boundary for the uncapped option, then the exercise boundary of the discounted American capped put option is $c(t) = c^A(t) \vee L$.*

Theorem 5.3 (Theorem 4.1 of Zaevski (2022a))**.** *The exercise boundary of an American call capped option is $c(t) = c^A(t) \wedge L$.*

It is important to note that we prove these theorems by a technique based on the infinitesimal generators – it allows us to generalize the results for different stochastic processes.

We establish the following theorem for the put price under the finite maturity horizon.

Theorem 5.2 (Theorem 3.2 of Zaevski (2022a))**.** *Let the constants D_1 and D_2 be defined as*

$$
\begin{aligned}
D_1 &= \min\left(\frac{r+\lambda}{\lambda}, 1\right) K \\
D_2 &= \frac{q}{q+1} K.
\end{aligned}
$$

In the case $L \in (D_2, D_1)$, we denote by τ^* this time to maturity for which $c^A(\tau^*) = L$. The price of an American capped put option can be derived through one of the following statements.

1. *Suppose that $L \in [D_1, K)$. If $S_0 \leq L$, then the price is $V = K - L$. Otherwise, if $S_0 > L$, then the price is*

$$
\begin{aligned}
V &= (K-L)\, e^{b_2\left(\sqrt{b_1^2+2(r+\lambda)}-b_1\right)} g\left(T, \sqrt{b_1^2+2\,(r+\lambda)}, b_2\right) \\
&\quad + K e^{-(r+\lambda)T} W\left(0, d\left(T,K\right), T; b_1, b_2\right) - S_0 e^{-\left(\lambda+\frac{\sigma^2}{2}\right)T} W\left(-\sigma, d\left(T,K\right), T; b_1, b_2\right) \\
&\quad g\left(T; b_1, b_2\right) = 1 - N\left(\frac{b_1 T + b_2}{\sqrt{T}}\right) + \exp\left(-2 b_1 b_2\right) N\left(\frac{b_1 T - b_2}{\sqrt{T}}\right) \\
&\quad d\left(t, x\right) = \frac{\ln S_0 - \ln x}{\sigma} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) t \\
&\quad W\left(\theta, z, T; b_1, b_2\right) \\
&\quad = \exp\left(\frac{T\theta^2}{2}\right) \left[\begin{array}{l} N\left(\frac{b(T)-T\theta}{\sqrt{T}}\right) - N\left(\frac{z-T\theta}{\sqrt{T}}\right) \\ +e^{2b_2(\theta-b_1)}\left(N\left(\frac{z-T\theta-2b_2}{\sqrt{T}}\right) - N\left(\frac{b(T)-T\theta-2b_2}{\sqrt{T}}\right)\right) \end{array}\right] \\
b_1 &= \frac{r}{\sigma} - \frac{\sigma}{2} \\
b_2 &= \frac{\ln S_0 - \ln L}{\sigma}.
\end{aligned}
$$

2. *Suppose that $L \in (D_2, D_1)$ and $T > \tau^*$. If $S_0 \leq L$, then $V = K - L$. If $S_0 > L$, then the price of the American capped option is given by*

$$
\begin{aligned}
V &= (K-L)\, e^{b_2\left(\sqrt{b_1^2+2(r+\lambda)}-b_1\right)} g\left(T-\tau^*, \sqrt{b_1^2+2\,(r+\lambda)}, b_2\right) \\
&\quad + e^{-(r+\lambda)(T-\tau^*)} \int_{-\infty}^{d(T-\tau^*,L)} A\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau^*)-\sigma y}, \tau^*\right) f\left(T-\tau^*, y\right) dy,
\end{aligned}
\tag{9}
$$

where A is the price of the related ordinary American option.

3. *If $L \leq D_2$ or $L \in (D_2, D_1) \cap T \leq \tau^*$, then the option is an ordinary American put and its price can be found using the approach provided in Chapter 4.*

This theorem provides a closed-form formula in the first case, whereas the option is a usual American in the third one. Let us discuss the second case. The numerical approach for pricing American options used in Chapter

4 (no matter finite difference or Monte Carlo) allows deriving the prices of the usual put for different initial values of the underlying asset at once. Thus the integral in formula (9) can be easily evaluated making pricing the capped option very fast.

The results for the call options are obtained by similar manner.

4.6 On some generalized American style derivatives.

In this chapter, we provide a method to distinguish when a payoff leads to a pricing task similar to those for the classical American options in the sense that the state space can be divided into two connected parts – the continuation and the optimal regions. We say that the derivative is call style if the optimal region is above. If it is below, we name the derivative put style. Let the payoff be

$$N(t,x) = e^{-\lambda t} G(x)$$

We define the following related to the infinitesimal generator differential operator over the C^2 functions

$$(\mathcal{B}g)(x) = (\mathcal{A}g)(x) - (r+\lambda)\, g(x) \,. \tag{10}$$

We prove that the following conditions lead to such instruments:

1. Call-style: if $(\mathcal{B}G)(x) < 0$ for some x, then $(\mathcal{B}G)(y) < 0$ for all $y \geq x$.
2. Put-style: if $(\mathcal{B}G)(x) < 0$ for some x, then $(\mathcal{B}G)(y) < 0$ for all $y \leq x$.

Let us discuss the perpetual calls. Let the function $g_c(c)$ be defined as

$$g_c(c) = \frac{G(c)}{c^{p-q}},$$

If the asset starts at x, then the financial result of the strategy of the first hit to c is

$$V^c(x;c) = g_c(c)\, x^{p-q} + \lim_{T\to\infty} \mathbb{E}^x \left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c} \right]. \tag{11}$$

We prove that it has no more one local maximum in the interval $[x, \infty)$. If we denote the global one by $c(x)$, then we prove the following result:

1. If there exists an initial point x, for which x is strictly less than $c(x)$, $x < c(x)$, then $c(x)$ is the optimal boundary.

2. If $x = c(x)$ for all x, then the optimal boundary is zero, i.e. all points are optimal.

3. If $x < c(x)$ for every x, then the optimal boundary is infinite, i.e. early exercising is never optimal.

Thus we obtain the optimal boundary c and the price via formula (11). The put value is given by

$$V^p(x;c) = \frac{g_p(c)}{x^q} + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G(S_T) I_{T<\zeta^c}\right],$$

where the function $g_p(\cdot)$ is defined as

$$g_p(c) = G(c)\, c^q.$$

In this case we maximize in the interval $(0, x]$. The result for the optimal boundary is:

1. If $c(x) < x$ for some x, then the optimal boundary is namely $c(x)$.

2. If $c(x) = x$ for all x's, then all points are optimal.

3. If $c(x) < x$ for all x, then early exercising is never optimal.

Finite maturity instruments are evaluated by the approach for the usual American options presented in Chapter 3.

Next we discuss the power payoffs of the form $G(x) = Mx^n + K$, where $n > 0$ and $M \in \{-1, 1\}$. It turns out that the following constant is very important:

$$L = (n-1)\left(r + \frac{\sigma^2}{2} n\right) - \lambda.$$

Let us denote by A_1 and A_2 the endpoints of the optimal boundary of a put-style instrument, and by B_1 and B_2 the call ones. Note that a derivative may combine both features – this happens when early exercising is always or never optimal. We establish the following result:

Theorem 6.3 (Theorem 3 of Zaevski (2024c)). *The following statements hold.*

1. *If* $\{L = 0,\ K \geq 0\}$, $\{L < 0,\ M = 1,\ K \geq 0\}$, *or* $\{L > 0,\ M = -1,\ K \geq 0\}$, *then the derivative is combined put-call and the immediate exercise is optimal everywhere. The call boundaries are* $B_1 = B_2 = 0$, *and the put ones are* $A_1 = A_2 = \infty$. *The perpetual price is given by* $V^c(x) = Mx^n + K$.

2. *If* $\{L = 0,\ K < 0\}$, $\{L < 0,\ M = -1,\ K \leq 0\}$, *or* $\{L > 0,\ M = 1,\ K \leq 0\}$, *then the derivative is again put-call style but the immediate exercise is never optimal. The call boundaries are* $B_1 = B_2 = \infty$ *and the put ones are* $A_1 = A_2 = 0$. *The perpetual price is* $V(x) = Mx^n$ *in the first case, it is* $V(x) = 0$ *in the second one, and* $V(x) = \infty$ *in the third case.*

3. *If* $\{L < 0,\ M = 1,\ K < 0\}$ *or* $\{L > 0,\ M = -1,\ K < 0\}$, *then the derivative is call-style. The values of the optimal boundaries are given by formulas (12) and (13):*

$$\begin{aligned} B_1 &= \left(K\frac{r+\lambda}{L}\right)^{\frac{1}{n}} \\ B_2 &= \left(-K\frac{p-q}{p-q-n}\right)^{\frac{1}{n}} \end{aligned} \tag{12}$$

$$\begin{aligned} B_1 &= \left(-K\frac{r+\lambda}{L}\right)^{\frac{1}{n}} \\ B_2 &= \left(-K\frac{p-q}{n-p+q}\right)^{\frac{1}{n}}, \end{aligned} \tag{13}$$

respectively. The perpetual price is $V^c(x) = Mx^n + K$ *if* $x \geq B_2$ *and by* $V^c(x) = \frac{B_2^n + K}{B_2^{p-q}}x^{p-q}$ *or* $V^c(x) = \frac{-B_2^n + K}{B_2^{p-q}}x^{p-q}$, *otherwise (first and second case, respectively).*

4. *If* $\{L < 0,\ M = -1,\ K > 0\}$ *or* $\{L > 0,\ M = 1,\ K > 0\}$, *then we have a put-derivative. The boundaries are given by formulas*

$$A_1 = \left(-K\frac{r+\lambda}{L}\right)^{\frac{1}{n}}$$
$$A_2 = \left(\frac{Kq}{q+n}\right)^{\frac{1}{n}},$$

for the first case; the perpetual price is $V^p(x) = Mx^n + K$ *if* $x \leq A_2$ *and* $V^p(x) = \frac{\left(-A_2^n+K\right)A_2^q}{x^q}$ *otherwise. The initial optimal boundary* A_1 *for the second case is* $A_1 = \left(K\frac{r+\lambda}{L}\right)^{\frac{1}{n}}$ *whereas the perpetual one is zero,* $A_2 = 0$*. The perpetual price in this case is* $V^p(x) = \infty$*.*

It may seem that the twice-differentiability is quite restrictive since many traded instruments do not satisfy it – for example the options in the strike. In fact, this limitation is not so strong because all real financial derivatives admit regularization. For example, the option payoffs can be approximated by the following functions:

$$G_\epsilon(x) = \begin{cases} 0, & \text{if } x < K \\ \frac{(x-K)^2}{2\epsilon}, & \text{if } K \leq x < K+\epsilon \\ x - K - \frac{\epsilon}{2}, & \text{if } K+\epsilon \leq x. \end{cases} \quad call$$

$$G_\epsilon(x) = \begin{cases} K + \frac{\epsilon}{2} - x, & \text{if } x < K-\epsilon \\ \frac{(K-x)^2}{2\epsilon}, & \text{if } K-\epsilon \leq x < K \\ 0, & \text{if } K \leq x. \end{cases} \quad put$$

4.7 American strangles with arbitrary strikes

The so-called American strangles are examined in this chapter. For some fundamental results we refer to Beibel and Lerche (1997) and Shiryaev (1999), Jeon and Oh (2019), and Qiu (2020). Their main characteristic is the combined put and call feature. The holder has the right to exercise prematurely choosing the option's style – put or call. We abandon the traditional assumption that the put strike is below the call one considering arbitrary values. We also assume that the put and call weights are different. We prove that these instruments lead to the first exit from a strip problem. Namely this strip is the continuation region. If the investor prefers a put characteristic, he receives $C_1 > 0$ shares of a put option with strike K_1. Analogously, if the holder chooses a call feature, he/she receives the payoff of $C_2 > 0$ call options with strike K_2. Hence, the payoff of a strangle can be written as

$$N(t,x) = e^{-\lambda t} \max\left\{C_1 (K_1 - x)^+, C_2 (x - K_2)^+\right\}.$$

Let D_0 be the value that makes the put and call payoffs equal

$$D_0 := \frac{C_1 K_1 + C_2 K_2}{C_1 + C_2}.$$

We name it the put-call barrier. We have two early exercise boundaries – between the continuation region and the put or call optimal regions. We shall denote these boundaries by $A(t)$ and $B(t)$, respectively. We prove that the put boundary $A(\tau)$ is non-increasing w.r.t. the time to maturity, whereas the call one $B(\tau)$ is non-decreasing. We prove that the initial boundary values are

$$\begin{aligned} D_1 \equiv A(0) &= \min\left\{K_1, \frac{C_1 K_1 + C_2 K_2}{C_1 + C_2}, \frac{r+\lambda}{\lambda} K_1\right\} \\ D_2 \equiv B(0) &= \max\left\{K_2, \frac{C_1 K_1 + C_2 K_2}{C_1 + C_2}, \frac{r+\lambda}{\lambda} K_2\right\}. \end{aligned}$$

It turns out that the immediate exercise is never optimal as a call if the additional discount factor λ is zero. Suppose now that $\lambda > 0$. We prove that the following equation has a unique root – it determines the perpetual optimal boundaries:

$$\begin{aligned} &a^{p+1} C_1 C_2 K_2 \alpha - a^p C_1 C_2 K_1 \beta - a^{p-q} C_2^2 K_2 (\beta - \alpha) \\ &- a^{q+1} C_1^2 K_1 (\beta - \alpha) - a C_1 C_2 K_2 \beta + C_1 C_2 K_1 \alpha = 0, \end{aligned} \tag{14}$$

where the constants α and β are

$$\begin{aligned} \alpha &= \frac{q}{q+1} \\ \beta &= \frac{p-q}{p-q-1}. \end{aligned}$$

Theorem 7.1 (Theorem 4.1 of Zaevski (2023b)). *Suppose that $\lambda > 0$ and let $\overline{a}$ be the unique solution of equation (14). Then the optimal boundaries can be derived as $\overline{A} = A_1(\overline{a})$ and $\overline{B} = \frac{\overline{A}}{\overline{a}}$, where*

$$A_1(a) = a\frac{(p-q)\left(a^q C_1 K_1 + C_2 K_2\right) + q\left(C_2 K_2 + \frac{C_1 K_1}{a^{p-q}}\right)}{(p-q-1)\left(a^{q+1} C_1 + C_2\right) + (q+1)\left(C_2 + \frac{C_1}{a^{p-q-1}}\right)}.$$

These boundaries lead to option price

$$f(A, B, x) = C_1(K_1 - A)\left(\frac{A}{x}\right)^q \frac{B^p - x^p}{B^p - A^p} + C_2(B - K_2)\left(\frac{B}{x}\right)^q \frac{x^p - A^p}{B^p - A^p}.$$

Suppose that $\lambda = 0$. As we already mentioned, the early exercise as a call is never optimal. We obtain the following results

Theorem 7.2 (Theorem 4.2 of Zaevski (2023b)). *If $\lambda = 0$, then the early exercising of a perpetual strangle is never optimal as a call. On the contrary, the option holder exercises as a put when the underlying asset reaches the level $\overline{A}$, where $\overline{A}$ is given by*

$$\overline{A} = \frac{2rC_1K_1}{(C_1 + C_2)(2r + \sigma^2)},$$

If the starting point is below this value, $x \leq \overline{A}$, then the option price is $C_1(K_1 - x)$. Otherwise, if $x > \overline{A}$, then the price is $Y\left(\overline{A}\right)$, where the function $Y(\cdot)$ is

$$Y(A) = C_2 x + \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}\left[-A(C_1 + C_2) + C_1 K_1\right].$$

Regardless that exercising as a call is never optimal, the put boundary and the option price depend on the call feature by the number of shares C_2, but not on the strike K_2.

Suppose now that the maturity is finite, $T < \infty$. For the functions $A(t)$ and $B(t)$, $A(t) < B(t)$, we define a European style derivative, that expires as a put if the asset falls below $A(t)$ and as a call if it rises above $B(t)$. We shall name these instruments $(A(t), B(t))$-European options. The corresponding stopping times shall be denoted by ζ^A and ζ^B, and the lower between them by ζ, $\zeta = \zeta^A \wedge \zeta^B$.

Let $0 \equiv t_0 < t_1 < t_2 < ... < t_n \equiv T$ be an increasing time sequence and $a(t)$ and $b(t)$ be two continuous piecewise linear functions w.r.t. it

$$
\begin{aligned}
a(t) &= \sum_{i=1}^{n} a_i(t)\, I_{t\in[t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (a_{1,i}t + a_{2,i})\, I_{t\in[t_{i-1},t_i]} \\
b(t) &= \sum_{i=1}^{n} b_i(t)\, I_{t\in[t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (b_{1,i}t + b_{2,i})\, I_{t\in[t_{i-1},t_i]},
\end{aligned}
$$

$a_i(t_i) = a_{i+1}(t_i)$ and $b_i(t_i) = b_{i+1}(t_i)$, $i = 1, 2, ..., n-1$. We impose the condition $a(t) < b(t)$ and additionally $a(0) < 0 < b(t)$. As we have mentioned above, we shall approximate the optimal boundaries as exponents of such functions – $A(t) = \exp(a(t))$ for the put boundary, and $B(t) = \exp(b(t))$ for the call one. The values of these functions at the grid nodes shall be denoted by $\alpha_i = a(t_i)$, $\beta_i = b(t_i)$, $A_i = A(t_i)$ and $B_i = B(t_i)$, $i = 0, 1, ..., n$. Let us introduce the functions

$$
\begin{aligned}
c(t) &= \sum_{i=1}^{n} c_i(t)\, I_{t\in(t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (c_{1,i}t + c_{2,i})\, I_{t\in(t_{i-1},t_i]} \\
d(t) &= \sum_{i=1}^{n} d_i(t)\, I_{t\in(t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (d_{1,i}t + d_{2,i})\, I_{t\in(t_{i-1},t_i]}
\end{aligned}
$$

for

$$
\begin{aligned}
c_{1,i} &= \frac{a_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \;\; i = 1, ..., n \\
c_{2,i} &= \frac{a_{2,i} - \ln(x)}{\sigma}, \;\; i = 1, ..., n \\
d_{1,i} &= \frac{b_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \;\; i = 1, ..., n \\
d_{2,i} &= \frac{b_{2,i} - \ln(x)}{\sigma}, \;\; i = 1, ..., n.
\end{aligned}
$$

The stopping times ζ^A and ζ^B can be viewed as the first hitting moments of the Brownian motion to the functions $c(t)$ and $d(t)$, respectively. Using the notations above, we present the price of the $(A(t), B(t))$-European option as

$$
\begin{aligned}
&G\left(x,T;A\left(t\right),B\left(t\right)\right)=\mathbb{E}^{x}\left[C_{1}e^{-(r+\lambda)\zeta^{A}}\left(K_{1}-S_{\zeta^{A}}\right)^{+}I_{\zeta^{A}=\zeta,\zeta<T}\right]\\
&+\mathbb{E}^{x}\left[C_{2}e^{-(r+\lambda)\zeta^{B}}\left(S_{\zeta^{B}}-K_{2}\right)^{+}I_{\zeta^{B}=\zeta,\zeta<T}\right]\\
&+\mathbb{E}^{x}\left[C_{1}e^{-(r+\lambda)T}(K_{1}-S_{T})^{+}I_{T\le\zeta,S_{T}\in(D_{1},D_{0})}\right]+\mathbb{E}^{x}\left[C_{2}e^{-(r+\lambda)T}(S_{T}-K_{2})^{+}I_{T\le\zeta,S_{T}\in(D_{0},D_{2})}\right]\\
&=C_{1}K_{1}\sum_{i=1}^{n}\mathbb{E}\left[e^{-(r+\lambda)\zeta^{A}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{A}}\right]-C_{1}x\sum_{i=1}^{n}e^{\sigma c_{2,i}}\mathbb{E}\left[e^{-\psi_{1,i}\zeta^{A}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{A}}\right]\\
&+C_{2}x\sum_{i=1}^{n}e^{\sigma d_{2,i}}\mathbb{E}\left[e^{-\psi_{2,i}\zeta^{B}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{B}}\right]-C_{2}K_{2}\sum_{i=1}^{n}\mathbb{E}\left[e^{-(r+\lambda)\zeta^{B}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{B}}\right]\\
&+C_{1}K_{1}e^{-(r+\lambda)T}\mathbb{Q}\left(v_{1}<B_{T}<l,T\le\zeta\right)-C_{1}xe^{-\psi_{3}T}\mathbb{E}\left[e^{\sigma B_{T}}I_{v_{1}<B_{T}<l,T\le\zeta}\right]\\
&+C_{2}xe^{-\psi_{3}T}\mathbb{E}\left[e^{\sigma B_{T}}I_{l<B_{T}<v_{2},T\le\zeta}\right]-C_{2}K_{2}e^{-(r+\lambda)T}\mathbb{Q}\left(l<B_{T}<v_{2},T\le\zeta\right),
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
\psi_{1,i}&=(r+\lambda)-\left(r-\frac{\sigma^{2}}{2}\right)-\sigma c_{1,i}=\lambda+\frac{\sigma^{2}}{2}-\sigma c_{1,i}\\
\psi_{2,i}&=(r+\lambda)-\left(r-\frac{\sigma^{2}}{2}\right)-\sigma d_{1,i}=\lambda+\frac{\sigma^{2}}{2}-\sigma d_{1,i}\\
\psi_{3}&=\lambda+\frac{\sigma^{2}}{2}\\
l&=\frac{1}{\sigma}\ln\left(\frac{D_{0}}{x}\right)-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T\\
v_{1}&=\frac{1}{\sigma}\ln\left(\frac{D_{1}}{x}\right)-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T\\
v_{2}&=\frac{1}{\sigma}\ln\left(\frac{D_{2}}{x}\right)-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T.
\end{aligned}
$$

We derive the expectations in formula (15) using the results of Chapter 2. We use the following algorithm to approximate the optimal boundaries:

1. The boundaries at the maturity are $A_n = D_1$ and $B_n = D_2$.

2. Suppose that we have derived all values $A_m, A_{m+1}, ..., A_n$ and $B_m, B_{m+1}, ..., B_n$ for some $m < n$.

3. We derive the put boundary in the following way. For the constants $A < x$, let $B(x, A)$ be the value that maximizes

$$G(x; 0, t_m - t_{m-1}, ...t_n - t_{m-1}; A, A_m, ..., A_n; B, B_m, ..., B_n) \tag{16}$$

amongst all $B > x$. Let us view equation (16) as a function of A, and $A(x)$ be the argument that maximizes

$$G(x; 0, t_m - t_{m-1}, ...t_n - t_{m-1}; A, A_m, ..., A_n; B(x, A), B_m, ..., B_n).$$

Our approximation for the put boundary value A_{m-1} is the largest x for which $x = A(x)$. In fact, this is the largest initial value of the underlying asset for which the immediate exercise as a put is optimal.

4. We obtain the call boundary analogously. Let for a fixed $x < B$, $A(x, B)$ be the value that maximizes function (16) w.r.t. the variable A. Also, let $B(x)$ maximizes

$$G(x; 0, t_m - t_{m-1}, ...t_n - t_{m-1}; A(x, B), A_m, ..., A_n; B, B_m, ..., B_n)$$

amongst all $B > x$.

5. We approximate the call boundary B_{m-1} as the smallest x for which $x = B(x)$. Our approximation is the smallest x for which this difference is zero.

Note again that if the optimal boundaries are indeed exponents of piecewise linear functions, then the values of $A(x, B)$, $B(x)$, $B(x, A)$, and $A(x)$ should not vary w.r.t. to x. Of course, this assumption is unreasonable – this motivates the algorithm above.

Once we approximate the exercise boundaries, we can view the option pricing task as the boundary value problem for which we apply the Crank-Nicolson finite difference approach.

4.8 Quadratic American strangles in the light of two-sided optimal stopping problems.

The aim of this chapter is to examine some American-style financial instruments that lead to two-sided optimal hitting problems. We pay particular attention to derivatives that are similar to strangle strategies but have a quadratic payoff function. We consider these derivatives in light of much more general payoff structures under certain conditions which guarantee that the optimal strategy is an exit from a strip. We impose a necessary condition for operator (10) applied to the payoff $G(x)$ that leads to a such task:

There exist constants $C \leq D$*, such that* $(\mathcal{B}G)(x) < 0$ *for* $x < C$ *and* $x > D$*, and* $(\mathcal{B}G)(x) \geq 0$ *for* $x \in [C, D]$*.*

Thus there are two optimal boundaries – $c(t) < d(t)$. The continuation region is between them. The optimal set consists of two parts – one below the continuation region and another above it. We prove that the values of the optimal boundaries when time to maturity is zero are namely $c(0) = C$ and $d(0) = D$. Next we obtain the perpetual values $c(\infty) = A$ and $d(\infty) = B$ as the solution of the following system:

$$\left(\frac{A}{B}\right)^{q} = \frac{G'(B)B - (p-q)G(B)}{G'(A)A - (p-q)G(A)}$$
$$\left(\frac{A}{B}\right)^{p-q} = \frac{G'(A)A + qG(A)}{G'(B)B + qG(B)}.$$

Next we define the so-called the quadratic strangles via the payoffs:

$$G(x) = (x-K)^2.$$

First we derive the price of the related European derivative

$$P(S_0) = S_0^2 e^{\left(r+\sigma^2-\lambda\right)T} - 2KS_0 e^{-\lambda T} + K^2 e^{-(r+\lambda)T}.$$

After that we turn to the American-style instrument. The value of the operator $\mathcal{B}$ applied to the payoff is

$$(\mathcal{B}G)(x) = x^2\left(r+\sigma^2-\lambda\right) + 2\lambda Kx - (r+\lambda)K^2.$$

We have to examine separately the cases $\lambda > r+\sigma^2$ and $\lambda \leq r+\sigma^2$. Let us consider the first one. The above imposed condition shows that we have indeed a two-sided exit problem. The values of C and D are

$$\{C,D\}=K\frac{\lambda\mp\sqrt{r^2+\sigma^2(r+\lambda)}}{\lambda-r-\sigma^2}.$$

We obtain the perpetual boundary values through the following theorem:

Theorem 8.1 (Theorem 1 of Zaevski (2024b))**.** *Let $\overline{a}$ be the solution of*

$$\frac{(p-q-1)\left(1-a^{q+1}\right)-\sqrt{(p-q-1)^2\left(1-a^{q+1}\right)^2-(p-q)(p-q-2)\left(1-a^q\right)\left(1-a^{q+2}\right)}}{(p-q)\left(1-a^q\right)}$$
$$=a\frac{(q+1)\left(1-a^{p-q-1}\right)+\sqrt{(q+1)^2\left(1-a^{p-q-1}\right)^2-q(q+2)\left(1-a^{p-q}\right)\left(1-a^{p-q-2}\right)}}{q\left(1-a^{p-q}\right)}$$

in the interval $(0,1)$ and $\overline{x}$ be defined as

$$\overline{x}=\frac{(p-q-1)\left(1-\overline{a}^{q+1}\right)-\sqrt{(p-q-1)^2\left(1-\overline{a}^{q+1}\right)^2-(p-q)(p-q-2)\left(1-\overline{a}^q\right)\left(1-\overline{a}^{q+2}\right)}}{(p-q)\left(1-\overline{a}^q\right)}.$$

The optimal boundaries of a perpetual quadratic strangle are $\overline{A}=\frac{\overline{a}}{\overline{x}}K$ and $\overline{B}=\frac{K}{\overline{x}}$. The derivative price is

$$f\left(\overline{A},\overline{B};x\right)=\left(x-\overline{A}\right)^2\left(\frac{\overline{A}}{x}\right)^q\frac{\overline{B}^p-x^p}{\overline{B}^p-\overline{A}^p}+\left(x-\overline{B}\right)^2\left(\frac{\overline{B}}{x}\right)^q\frac{x^p-\overline{A}^p}{\overline{B}^p-\overline{A}^p}.$$

The finite maturity quadratic strangles are investigated via the approach of Chapter 7. The pricing function of the financial instrument related to the first exit of the strip with piecewise linear boundaries is

$$
\begin{aligned}
&F\left(x,T;c\left(t\right),d\left(t\right)\right)=\mathbb{E}^{x}\left[e^{-(r+\lambda)\zeta^{A}}\left(S_{\zeta^{A}}-K\right)^{2}I_{\zeta^{A}=\zeta,\zeta<T}\right]\\
&+\mathbb{E}^{x}\left[e^{-(r+\lambda)\zeta^{B}}\left(S_{\zeta^{B}}-K\right)^{2}I_{\zeta^{B}=\zeta,\zeta<T}\right]+\mathbb{E}^{x}\left[e^{-(r+\lambda)T}(S_{T}-K)^{2}I_{T\le\zeta}\right]\\
&=K^{2}\sum_{i=1}^{n}\left(\mathbb{E}\left[e^{-(r+\lambda)\zeta^{A}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{A}}\right]+\mathbb{E}\left[e^{-(r+\lambda)\zeta^{B}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{B}}\right]\right)\\
&-2Kx\sum_{i=1}^{n}\left(e^{\sigma a_{2,i}}\mathbb{E}\left[e^{-\psi_{1,i}\zeta^{A}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{A}}\right]+e^{\sigma b_{2,i}}\mathbb{E}\left[e^{-\psi_{2,i}\zeta^{B}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{B}}\right]\right)\\
&+x^{2}\sum_{i=1}^{n}\left(e^{2\sigma a_{2,i}}\mathbb{E}\left[e^{-\eta_{1,i}\zeta^{A}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{A}}\right]+e^{2\sigma b_{2,i}}\mathbb{E}\left[e^{-\eta_{2,i}\zeta^{B}}I_{\zeta\in(t_{i-1},t_{i}],\zeta=\zeta^{B}}\right]\right)\\
&+K^{2}e^{-(r+\lambda)T}\mathbb{Q}\left(v_{1}<B_{T}<v_{2},T\le\zeta\right)-2Kxe^{-\psi_{3}T}\mathbb{E}\left[e^{\sigma B_{T}}I_{v_{1}<B_{T}<v_{2},T\le\zeta}\right]\\
&+x^{2}e^{-\psi_{4}T}\mathbb{E}\left[e^{2\sigma B_{T}}I_{v_{1}<B_{T}<v_{2},T\le\zeta}\right],
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
\psi_{1,i}&=(r+\lambda)-\left(r-\frac{\sigma^{2}}{2}\right)-\sigma a_{1,i}=\lambda+\frac{\sigma^{2}}{2}-\sigma a_{1,i}\\
\psi_{2,i}&=(r+\lambda)-\left(r-\frac{\sigma^{2}}{2}\right)-\sigma b_{1,i}=\lambda+\frac{\sigma^{2}}{2}-\sigma b_{1,i}\\
\eta_{1,i}&=(r+\lambda)-2\left(\left(r-\frac{\sigma^{2}}{2}\right)-\sigma a_{1,i}\right)=\lambda+\sigma^{2}-r-2\sigma a_{1,i}\\
\eta_{2,i}&=(r+\lambda)-2\left(\left(r-\frac{\sigma^{2}}{2}\right)-\sigma b_{1,i}\right)=\lambda+\sigma^{2}-r-2\sigma b_{1,i}\\
\psi_{3}&=\lambda+\frac{\sigma^{2}}{2}\\
\psi_{4}&=\lambda+\sigma^{2}-r\\
v_{1}&=\frac{1}{\sigma}\ln\left(\frac{C}{x}\right)-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T\\
v_{2}&=\frac{1}{\sigma}\ln\left(\frac{D}{x}\right)-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T.
\end{aligned}
$$

The expectations in statement (17) can be derived through the results of Chapter 2. Note that for some values of the parameters we need to use the analytic continuation of the erf-function.

Finally, we consider the case $\lambda \leq r + \sigma^2$. It turns out that we have a one-sided put-style optimal stopping problem – tasks studied in Chapter 6. We prove that the boundary value at maturity is

$$\begin{aligned} C &= \tfrac{r+\lambda}{2\lambda} K \qquad if\ \lambda = r + \sigma^2 \\ C &= \tfrac{\sqrt{r^2+\sigma^2(r+\lambda)}-\lambda}{r+\sigma^2-\lambda} K \qquad if\ \lambda < r + \sigma^2.. \end{aligned}$$

The perpetual one as well as the price are derived in the following theorem:

Theorem 8.2 (Theorem 2 of Zaevski (2024b))**.** *If $\lambda < r + \sigma^2$, then the early exercise is never optimal for a perpetual quadratic strangle. Its price is infinitely large.*

If $\lambda = r + \sigma^2$, then all points below $\overline{A}$, given by

$$\overline{A} = \frac{q}{2(q+1)} K,$$

are optimal. The price is $(x - K)^2$ when the initial asset value $S_0 = x$ is below $\overline{A}$ and it is given by equation

$$F\left(x; \overline{A}\right) = \frac{K^2 \overline{A}^q - 2K \overline{A}^{q+1}}{x^q} + x^2,$$

otherwise.

We derive the desired results for the finite maturity quadratic strangles by the approach of Chapter 6.

4.9-12. Perpetual cancellable options

We discuss now jointly Chapters 8-12. The major characteristic of the cancellable American options, also known as game or Israeli, is the existing writer's right to cancel the contract prematurely paying some penalty amount. These options are introduced by Kifer (2000). Some later important works are Kifer (2000), Kyprianou (2004), Kühn and Kyprianou (2007), Suzuki and Sawaki (2007), Emmerling (2012), and Yam et al. (2014). Alternatively, we apply our approach for maximizing the holder's and writer's utility to these instruments. We present mainly Chapter 12 because it generalized the results of Chapters 9-11. However, the proofs in Chapter 12 are based substantially on Chapters 9-11. Hence, the clarity of presentation necessitates the presence of these chapters.

The main purpose of this chapter is to introduce and examine a new subclass of such options. It encloses the set of cancellable options in some sense. Let the function $N_1(t,x)$ present the amount that the writer owes if the holder exercises the option at the moment t at the spot price $S_t = x$. Analogously, the function $N_2(t,x)$ defines the amount that the writer has to pay if he cancels the contract. Suppose that the penalty consists of three parts – the constant $\eta_1 \geq 1$ leads to a proportion of the usual option payment, $\eta_2 \geq 0$ is the number of shares of the underlying asset, and $\eta_3 \geq 0$ is a fixed amount. Thus the functions $N_1(t,x)$ and $N_2(t,x)$ are

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(x-K)^+ \\ N_2(t,x) &= e^{-\lambda t}\big(\eta_1 (x-K)^+ + \eta_2 x + \eta_3\big) \end{aligned}$$

or

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(K-x)^+ \\ N_2(t,x) &= e^{-\lambda t}\big(\eta_1 (K-x)^+ + \eta_2 x + \eta_3\big). \end{aligned}$$

for the call- or put-style options, respectively. In fact, this is a problem for the field of stochastic games – see Dynkin (1969). Hence, the main difference between strangles and cancellable options is that the first ones lead to a problem of finding the maximum of a two-dimensional functional over space of the stopping times, whereas the second ones need a saddle point. Our approach allows us to investigate both classes of financial instruments by similar techniques.

We can divide the state space into three parts – holder's optimal (Υ^b), writer's optimal (Υ^s), and continuation (Υ). We shall denote the price function by $V(t,x)$.

We need to restrict the writer's optimal set in some marginal cases to keep the generality of the presentation. In fact, we impose that the writer would not cancel the option immediately even if this is optimal, when some future strategy provides the same result. This assumption is not so restrictive from a financial point of view.

Let the option be out-of-the-money, $\lambda = 0$, and $\eta_3 = 0$. Suppose that $V(t,x) = N_2(t,x)$ and there exists a stopping time $\zeta > t$ a.s. such that $N_2(t,x) = M(t,x;\zeta,B(\zeta;x))$. Then $(t,x) \notin \Upsilon^s$.

Let us consider call options. We prove the following statements that characterize the optimal boundaries:

1. If $x < K$, then $x \in \overline{\Upsilon}$.
2. If $\eta_3 \geq \eta_1 K$, then $\Upsilon^s = \emptyset$.
3. Suppose that a larger than the strike constant x is optimal for the writer, $x \in \Upsilon^s$. Let y be another constant such that $K < y < x$. Then $y \in \Upsilon^s$.
4. If $\eta_1 = 1$, $\eta_2 = 0$, and $\eta_3 = 0$, then $\overline{\Upsilon} = (0, K)$.
5. If $x \in \Upsilon^b$ and $y > x$, then $y \in \Upsilon^b$.
6. If $\lambda = 0$, then the holder's exercise region is empty.
7. If $r < 0$, then $\Upsilon^s \equiv \emptyset$ or $\Upsilon^s \equiv \{K\}$.

These statements indicate that the holder's exercise region has the form $\Upsilon^b = [B, \infty)$ for some constant $B > K$, whereas the writer's one is the interval $\Upsilon^s = [K, A]$ (A is a constant less than B, $K < A < B$), the singleton $\Upsilon^s = \{K\}$, or the empty set $\Upsilon^s \equiv \emptyset$.

Let x be the initial value of the underlying asset. For a fixed value of B, we denote $a = \frac{A}{B}$, $k = \frac{K}{B}$, $\xi = \frac{\eta_3}{B}$, and $y = \frac{x}{B}$. We prove that the equation

$$\begin{aligned}&a^{p+1}(\eta_1+\eta_2)(p-q-1) - a^p(\eta_1 k - \xi)(p-q)\\ &-a^{p-q}p(1-k) + a(q+1)(\eta_1+\eta_2) - q(\eta_1 k - \xi) = 0\end{aligned}$$

has a unique root which we denote by $a(B)$. Analogously, let $b = \frac{B}{A}$, $k = \frac{K}{A}$, $\xi = \frac{\eta_3}{A}$, and $y = \frac{x}{A}$ for a fixed value of A. We prove that the equation

$$\begin{aligned}&-b^{p+1}(p-q-1) + b^p k(p-q)\\ &+b^{p-q}p(\eta_1+\eta_2-\eta_1 k+\xi) - b(q+1) + qk = 0\end{aligned}$$

has just one root in the interval $(1, \infty)$ except in the marginal case $\eta_2 = \eta_3 = 0$ examined in Chapter 11. We denote this root by $b(A)$. Let us denote by $\overline{\eta}$ the price of the at-the-money pure American option:

$$\overline{\eta} = \frac{K}{\gamma}\left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1} \tag{20}$$

for $\gamma = p - q$. We prove the following theorem:

Theorem 12.1 (Theorem 3.8 of Zaevski (2023a)). *Let $\lambda > 0$, $\eta_2 + \eta_3 > 0$, and the true boundaries be denoted by A^* and B^*.*

1. *If $\eta_3 \geq \eta_1 K$, then $\Upsilon^s = \emptyset$ and the option is ordinary American – for more details about these options, see Chapter 4.*

2. *If $\eta_3 < \eta_1 K$ in addition to $\eta_2 + \eta_3 > 0$, then $\overline{A}$ is defined as the solution of the equation $b\left(y\right) a\left(yb\left(y\right)\right) = 1$. The following statements hold:*

   (a) *If $\overline{A} \geq K$, then $A^* = \overline{A}$ and $B^* = \overline{A} b\left(\overline{A}\right)$. The exercise regions for the writer and holder are $\Upsilon^s = [K, A^*]$ and $\Upsilon^b = [B^*, \infty)$, respectively. The option price $V\left(x\right)$ is given by:*

      i. *formula*

      $$V\left(x\right) = \left(\eta_2 K + \eta_3\right) \mathbb{E}^x \left[e^{-(r+\lambda)\tau} I_{\tau<\infty}\right] = \left(\eta_2 K + \eta_3\right) \left(\frac{x}{K}\right)^{\gamma}$$

      when $x \leq K$;

      ii. *$V\left(x\right) = \left(\eta_1 + \eta_2\right) x - \eta_1 K + \eta_3$ when $K < x < A^*$;*

      iii. *$V\left(x\right) = x - K$ when $x > B^*$;*

      iv. *formula*

      $$V\left(x\right) =$$
      $$\left(\left(\eta_1 + \eta_2\right) A^* - \eta_1 K + \eta_3\right) \left(\frac{A^*}{x}\right)^q \frac{B^{*p} - x^p}{B^{*p} - A^{*p}} + \left(B^* - K\right) \left(\frac{B^*}{x}\right)^q \frac{x^p - A^{*p}}{B^{*p} - A^{*p}},$$

      when $A^* \leq x \leq B^*$.

   (b) *If $\overline{A} < K$ and $\eta_2 K + \eta_3 \leq \overline{\eta}$, $\overline{\eta}$ is given by equation (20), then $A^* = K$ and $B^* = Kb\left(K\right)$. The exercise regions are $\Upsilon^s = \{K\}$ and $\Upsilon^b = \left[\overline{B}, \infty\right)$. The option price $V\left(x\right)$ is determined as in the previous case.*

   (c) *If $\overline{A} < K$ and $\eta_2 K + \eta_3 > \overline{\eta}$, then the option is again ordinary American.*

If the additional discount factor is zero, then the it is never optimal for the holder to exercise. We prove the following results:

Theorem 12.2 (Theorem 3.9 of Zaevski (2023a)). *Let $\lambda = 0$.*

1. *Suppose that $M < K$ – the constant M is defined as:*

$$M = \frac{2r\left(\eta_1 K - \eta_3\right)}{\left(\eta_1 + \eta_2 - 1\right)\left(2r + \sigma^2\right)}.$$

   (a) *If $\eta_2 K + \eta_3 > K$, then early exercising is never optimal for both participants and the option price is $V\left(x\right) = x$.*

   (b) *If $\eta_2 K + \eta_3 \leq K$, then the writer's exercise region is the strike. The price is given by equation*

      i. $V\left(x\right) = x + \left(\frac{K}{x}\right)^{\frac{2r}{\sigma^2}}\left(K\left(\eta_2 - 1\right) + \eta_3\right)$ *when $x \geq K$.*

      ii. $V\left(x\right) = \frac{\left(\eta_2 K + \eta_3\right)x}{K}$ *when $x < K$.*

2. *If $M \geq K$, then the writer's exercise region is $\Upsilon^s = [K, M]$. The price is*

   (a) $V\left(x\right) = \frac{\left(\eta_2 K + \eta_3\right)x}{K}$ *when $x < K$;*

   (b) $V\left(x\right) = x\left(1 - \frac{\sigma^2\left(\eta_1 + \eta_2 - 1\right)}{2r}\left(\frac{2r\left(\eta_1 K - \eta_3\right)}{x\left(\eta_1 + \eta_2 - 1\right)\left(2r + \sigma^2\right)}\right)^{\frac{2r}{\sigma^2} + 1}\right)$ *when $M < x$;*

   (c) $V\left(x\right) = \left(\eta_1 + \eta_2\right)x - \eta_1 K + \eta_3$ *when $K \leq x \leq M$.*

Theorem 11.1 (Theorem 4.1 of Zaevski (2020a)). *Let $\eta_2 = \eta_3 = 0$ and $\lambda > 0$.*

1. *If $r > 0$, then the optimal boundaries are $A^* = \max\left(K, \overline{A}\right)$ and $B^* = A^* b\left(A^*\right)$. The constant $\overline{A}$ is the solution of equation $b\left(A\right) a\left(A b\left(A\right)\right) = 1$. Thus the call price V is*

   (a) *If $x \leq A^*$, then $V = \eta_1 (x - K)^+$.*

   (b) *If $A^* < x < B^*$, then*

$$V = \eta_1\left(A^* - K\right)\left(\frac{A^*}{x}\right)^q \frac{B^{*p} - x^p}{B^{*p} - A^{*p}} + \left(B^* - K\right)\left(\frac{B^*}{x}\right)^q \frac{x^p - A^{*p}}{B^{*p} - A^{*p}}.$$

   (c) *If $B^* \leq x$, then $V = x - K$.*

2. *If $r \leq 0$, then $\Upsilon^s = (0, K]$ and $\Upsilon^b = (K, \infty)$. If $x \leq K$, then $Y = 0$, and $Y = x - K$, otherwise.*

Let us discuss the case $\eta_2 = \eta_3 = 0$. All points below the strike are considered as writer optimal since the writer owes nothing. On the other hand, these points can be viewed also as belonging to the continuation region since the first hitting to the strike strategy gives the same result – we can apply the above-imposed condition.

Let us turn to the put-style options. The following statements are proven

1. If $x > K$, then $x \in \overline{\Upsilon}$.
2. If $\eta_2 \geq \eta_1$, then $\Upsilon^s \equiv \emptyset$.
3. If $\eta_1 = 1$, $\eta_2 = 0$, and $\eta_3 = 0$, then $\overline{\Upsilon} = (K, \infty)$.
4. If $x \in \Upsilon^b$ and $y < x$, then $y \in \Upsilon^b$.
5. If $x < K$, $x \in \Upsilon^s$, and $x < y < K$, then $y \in \Upsilon^s$.
6. If $r > 0$, then $\Upsilon^s \equiv \emptyset$ or $\Upsilon^s \equiv \{K\}$.

These statements indicate that the holder's exercise region has the form $\Upsilon^b = (0, A]$ for some constant A, whereas the writer's set has one of the following three forms – $\Upsilon^s = [B, K]$, $\Upsilon^s = \{K\}$, or $\Upsilon^s = \emptyset$. We define now the functions $a\,(B)$ and $b\,(A)$ as the roots of

$$-a^{p+1}\,(p-q-1) + a^p k\,(p-q) - a^{p-q} p\,(\eta_1 k - \eta_1 + \eta_2 + \xi) - a\,(q+1) + qk = 0$$

$$b^{p+1}\,(p-q-1)\,(\eta_1 - \eta_2) - b^p\,(\eta_1 k + \xi)\,(p-q) +$$
$$+ b^{p-q} p\,(k-1) + b\,(q+1)\,(\eta_1 - \eta_2) - q\,(\eta_1 k + \xi) = 0$$

The at-the-money put price is

$$\overline{\eta} = \frac{K}{q+1}\left(\frac{q}{q+1}\right)^q.$$

We prove the following theorem for the optimal boundaries and the option price:

Theorem 12.3 (Theorem 4.6 of Zaevski (2023a)). *Suppose that $\eta_2 + \eta_3 > 0$. Let $\overline{B}$ be the solution of the equation $a(y) b(ya(y)) = 1$. We recognize the following cases:*

1. *If $\eta_2 \geq \eta_1$, then the option is ordinary American.*

2. *Suppose now that $\eta_2 < \eta_1$ and $\eta_2 + \eta_3 > 0$.*

   (a) *If $\overline{B} \leq K$, then $B^* = \overline{B}$ and $A^* = B^* a(B^*)$ – thus the exercise regions for the writer and holder are $\Upsilon^s = [B^*, K]$ and $\Upsilon^b = [0, A^*)$, respectively. The option price $V(x)$ is given by:*

      i. $V(x) = (\eta_2 K + \eta_3) \left(\frac{K}{x}\right)^q$ *when $x \geq K$;*

      ii. *by formula*

      $$V(x) = (K - A^*) \left(\frac{A^*}{x}\right)^q \frac{B^{*p} - x^p}{B^{*p} - A^{*p}} + (\eta_1 K - (\eta_1 - \eta_2) B^* + \eta_3) \left(\frac{B^*}{x}\right)^q \frac{x^p - A^{*p}}{B^{*p} - A^{*p}}.$$

      when $\overline{A} \leq x \leq \overline{B}$;

      iii. *by $V(x) = -(\eta_1 - \eta_2) x + \eta_1 K + \eta_3$ when $\overline{B} < x < K$;*

      iv. *by $V(x) = K - x$ when $x < \overline{A}$.*

   (b) *If $K < \overline{B}$ and $\eta_2 K + \eta_3 \leq \overline{\eta}$, then $B^* = K$ and $A^* = Ka(K)$. The exercise regions are $\Upsilon^s = \{K\}$ and $\Upsilon^b = \left(0, \overline{A}\right]$. The option price is determined as in the previous case.*

   (c) *If $K < \overline{B}$ and $\eta_2 K + \eta_3 > \overline{\eta}$, then the option is ordinary American.*

Theorem 11.3 (Theorem 6.1 of Zaevski (2020a)). *Suppose that $\eta_2 = \eta_3 = 0$.*

1. *If $r \geq 0$, then the exercise regions are $\Upsilon^s = [K, \infty)$ and $\Upsilon^b = (0, K)$. If $x < K$, then the option price is $V = K - x$ and it is zero, otherwise.*

2. *If $r < 0$, then the exercise regions are $\Upsilon^s = [B^*, \infty)$ and $\Upsilon^b = (0, A^*)$, where $B^* = \min\left(\overline{B}, K\right)$ and $A^* = B^* a(B)$. The value of $\overline{B}$ is obtained as the solution of equation $1 = a(B) b(Ba(B))$. Thus the option price is*

   (a) $V = K - x$, *if $x < A^*$;*

(b) $V = (A^* - K)\left(\frac{A^*}{x}\right)^q \frac{B^{*p} - x^p}{B^{*p} - A^{*p}} + \eta\,(B^* - K)\left(\frac{B^*}{x}\right)^q \frac{x^p - A^{*p}}{B^{*p} - A^{*p}}$*, if* $x \in [A^*, B^*)$*;*

(c) $V = \eta(K - x)^+$*, if* $B^* \leq x$*.*

If $\eta_2 = \eta_3 = 0$*, then all points above the strike are considered as writer optimal, but they can be viewed also as a part of the continuation region since the first hit to the strike provides the same financial result.*

4.13. Pricing cancellable American put options on the finite time horizon

We consider now finite maturity cancellable put options with a constant penalty, i.e. $\eta_1 = 1$, $\eta_2 = 0$, and $\eta_3 = \eta > 0$. A special role in our examination has a subclass of the American derivatives. Their owner has the right to exercise at every moment before the maturity receiving the usual option payment. In addition, if the underlying asset hits the strike when the remaining time to maturity is larger than some previously defined value τ, the derivative expires paying some amount η. We shall call this derivative a (τ, η)-American option. Obviously, if the time to maturity is less than τ, then the (τ, η)-American option coincides with the ordinary American one.

We have two critical values for the time to maturity. The first one, τ_1, makes the price of the usual American at-the-money option equal to the penalty:

$$V^{am}(K; \tau_1) = \eta. \tag{21}$$

The characterization of the second one, τ_2, is more complicated – we do this numerically. We prove the following theorem:

Theorem 13.1. (Theorem 3.1 of Zaevski (2022b))**.** *The holder's exercise boundary is a decreasing function starting from the point*

$$\min\left(\frac{r+\lambda}{\lambda}, 1\right) K.$$

The form of the writer's exercise boundary is more complicated. Let B *be its perpetual value if it exists. The following statements characterize the boundary curve:*

1. *If B does not exist, equivalently to $\eta \geq \overline{\eta}$ for $\overline{\eta}$ given in equation (21), then $\tau_1 = \tau_2 = \infty$ and $\Upsilon^s \equiv \emptyset$.*

2. *If $B = K$, then $\tau_1 < \infty$, but $\tau_2 = \infty$. Also, $\Upsilon^s_\tau \equiv \emptyset$ for $\tau \leq \tau_1$ and $\Upsilon^s_\tau \equiv \{K\}$ otherwise. Note that this is the case when $r \geq 0$.*

3. *If $B < K$, then $\tau_1 < \tau_2 < \infty$. Thus the writer's exercise boundary does not exist for τ less than τ_1, it coincides with the strike for $\tau \in (\tau_1, \tau_2)$, and it is a decreasing tending to B function for $\tau \geq \tau_2$.*

In such a way the option is ordinary American when $\tau \in (0, \tau_1]$, it is (τ_1, η)-American for $\tau \in (\tau_1, \tau_2)$, and a real cancellable option for $\tau \in [\tau_2, \infty)$.

We need to obtain first which case of Theorem 13.1. holds. If the penalty is larger than the critical value $\overline{\eta}$, given in equation (21), then the option is ordinary American. Suppose now that $\eta < \overline{\eta}$. We can calculate the writer's exercise boundary of the perpetual cancellable put, B, using the results of Chapter 10. It can be equal to or less than the strike.

The next step is to divide the time to maturity interval into $n \geq 2$-sub-intervals, $0 \equiv t_0 < t_1 < ... < t_n \equiv T \equiv \tau$. We can think that $\tau_1 < \tau$, because in the opposite case, the option is non-cancellable. We impose two requirements – τ_1 to be a grid node and the division to be relatively uniform. To do this, we use the following procedure. First, we divide the interval into two parts – $(0, \tau_1)$ and (τ_1, τ). After, that we divide uniformly both intervals into m_1 and m_2 parts, respectively, such that $m_1 + m_2 = n$ and

$$m_1 = \min\left(\max\left(1, Round\left(\frac{\tau_1}{\tau} n\right)\right), n - 1\right). \qquad (22)$$

We have used above the notation $Round(x)$ for the nearest to x integer. Formulation (22) guarantees that $m_1 \geq 1$ and $m_2 \geq 1$, i.e. there is at least one sub-interval before τ_1 as well as after. Also, note that $t_{m_1} = \tau_1$.

We have to modify our approach for approximating optimal boundaries, having in mind their features. We first define the following European-style derivatives for some functions $0 < a(t) < b(t)$. They expire at the maturity date or when the underlying asset exits the strip $(a(t), b(t))$. The derivatives pay the amount of $N_1(t, a(t))$ or $N_2(t, b(t))$ if the exit happens from the lower or upper boundary, respectively. We shall name these derivatives $(a(t), b(t))$-European options. The price of these derivatives can be obtained as

$$
\begin{aligned}
&G(x,T;a(t),b(t)) = \mathbb{E}^x\left[e^{-(r+\lambda)(\zeta_1\wedge T)}(K-S_{\zeta_1\wedge T})^+ I_{(\zeta_1\wedge T)\le\zeta_2}\right]\\
&+\mathbb{E}^x\left[e^{-(r+\lambda)\zeta_2}\left((K-S_{\zeta_2})^+ +\eta\right) I_{\zeta_2<(\zeta_1\wedge T)}\right]\\
&= K\sum_{i=1}^{n}\mathbb{E}\left[e^{-(r+\lambda)\zeta_1} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_1}\right] - x\sum_{i=1}^{n} e^{\sigma c_{2,i}}\mathbb{E}\left[e^{-\psi_{1,i}\zeta_1} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_1}\right]\\
&+(K+\eta)\sum_{i=1}^{m_1}\mathbb{E}\left[e^{-(r+\lambda)\zeta_2} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_2}\right] - x\sum_{i=1}^{m_1} e^{\sigma d_{2,i}}\mathbb{E}\left[e^{-\psi_{2,i}\zeta_2} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_2}\right]\\
&+Ke^{-(r+\lambda)T}\mathbb{Q}\left(B_T<k, T\le\zeta\right) - xe^{-\psi_3 T}\mathbb{E}\left[e^{\sigma B_T} I_{B_T<k,T\le\zeta}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_{1,i} &= (r+\lambda) - \left(r-\frac{\sigma^2}{2}\right) - \sigma c_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma c_{1,i}\\
\psi_{2,i} &= (r+\lambda) - \left(r-\frac{\sigma^2}{2}\right) - \sigma d_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma d_{1,i}\\
\psi_3 &= \lambda + \frac{\sigma^2}{2}\\
k &= \frac{1}{\sigma}\ln\left(\frac{K}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T.
\end{aligned}
$$

As we mentioned above, the cancellable options lead to the task of finding a saddle point. Thus we may use an algorithm similar to those presented for the ordinary American options or for the strangles. The main difference is that we have to minimize the result for the writer's boundary – the lower one.

Once we approximate the optimal boundaries, we may find the option price very precise by the Crank-Nicolson finite difference scheme. First, suppose that the initial asset price x is between the optimal boundaries. We propose in addition a Monte Carlo method based on some results of Wang and Pötzelberger (1997) and Pötzelberger and Wang (2001).

1. We generate $n-1$ random numbers using the standard normal distribution. These numbers form the vector $\overline{u}$.

2. Let $m \le n$ and the vector u consists of the first $m-1$ elements of $\overline{u}$. Let D be a $(m-1)\times(m-1)$ diagonal matrix with elements $\sqrt{\Delta t_i/n}$.

Note that the length of the intervals, Δt_i, differs before and after the moment $T-\tau_1$. We calculate the vector x as $x = MDu$, where M is a $(m-1)\times(m-1)$ lower triangle matrix with values one.

3. If $t_{m-1} < T-\tau_1$ we derive the values v as

$$v = v\left(x_1, ..., x_{m-1}\right) = \prod_{i=1}^{m-1} I_{c_i<x_i<d_i}\left(1-\sum_{j=1}^{\infty} q_{ij}\left(x_{i-1}, x_i\right)\right).$$

Otherwise, if $m_1 \leq m-1$, then v is obtained as

$$v = v\left(x_1, ..., x_{m-1}\right) = \prod_{i=1}^{p-1} I_{c_i<x_i<d_i}\left(1-\sum_{j=1}^{\infty} q_{ij}\left(x_{i-1}, x_i\right)\right)$$
$$\times \prod_{i=p}^{m-1} I_{c_i<x_i}\left(1-\exp\left(-\frac{2\left(c_{i-1}-x_{i-1}\right)\left(c_i-x_i\right)}{\Delta t_i}\right)\right).$$

4. We derive the values w as $w = e^{-\xi t_{m-1}} L_{1,2}\left(\cdot\right)$ using the results of Chapter 2.4.

5. We calculate the product $P = vw$.

6. We repeat the procedure above H times and after averaging, we derive the necessary expectations as $\frac{1}{H}\sum\limits_{i=1}^{H} P_i$.

If the initial asset price is above the strike we derive the following semi-closed form formula.

$$\begin{aligned}
V\left(\tau, S_0\right) &= \mathbb{E}\left[e^{-(r+\lambda)\zeta}\eta I_{\zeta\leq T-\tau_1}\right] \\
&\quad + e^{-r(T-\tau_1)}\int_d^{\infty} e^{-\lambda(T-\tau_1)} V_{am}\left(\tau_1, S_0 e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau_1)+\sigma y}\right) d\mathbb{Q}\left(B_{T-\tau_1} < y, \zeta > T-\tau_1\right) \\
&= \eta e^{-a_2\left(\sqrt{a_1^2+2(r+\lambda)}+a_1\right)} g\left(T-\tau_1, -\sqrt{a_1^2+2\left(r+\lambda\right)}, a_2\right) \\
&\quad + e^{-(r+\lambda)(T-\tau_1)}\int_d^{\infty} V_{am}\left(\tau_1, S_0 e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau_1)+\sigma y}\right) f\left(y; T-\tau_1\right) dy,
\end{aligned}$$

where

$$
\begin{aligned}
a_1 &= -\frac{r}{\sigma} + \frac{\sigma}{2} \\
a_2 &= -\frac{\ln S_0 - \ln K}{\sigma} \\
d &= a_1 \left(T - \tau_1\right) + a_2 \\
f\left(y; t\right) &= \frac{1}{\sqrt{2\pi t}} \left(1 - \exp\left(-\frac{2a_2\left(a_1 t + a_2 - y\right)}{t}\right)\right) \exp\left(-\frac{y^2}{2t}\right) \\
g\left(T; b_1, b_2\right) &= N\left(\frac{b_1 T + b_2}{\sqrt{T}}\right) + \exp\left(-2b_1 b_2\right) N\left(\frac{-b_1 T + b_2}{\sqrt{T}}\right).
\end{aligned}
$$

4.14. MATLAB codes

The derived in the previous chapters theoretical results are implemented by the use of MATLAB. We provide some of the most important codes – their total number is larger than two hundred. All of them are available and can be provided upon request. Note that they are not professionally prepared, but are rather for personal use. We discuss also some specifics of the used algorithms.

5 Concluding remarks and further works

In this dissertation, we have established a novel fast, approach for pricing American-style financial instruments. This task was solved through the following steps. We first obtain the shape of the optimal regions and the related boundaries. Next, we consider options without maturity constraints deriving the closed-form formulas for the option prices as well as for the optimal boundaries. After this, we examine the finite maturities approximating the optimal boundaries on a relatively rare grid. Thus we received a very fast algorithm with a high accuracy – errors in the fourth sign after the decimal point. For denser grids, we constructed several numerical methods based on Monte Carlo simulations and finite difference schemes. To do all this, we have proved several results for the first hitting moments of a Brownian motion. Two kinds of stopping times have been examined – the first hit to a piecewise linear boundary and the first exit from a strip formed by two such functions. Also, some limits of these Laplace transforms have been considered.

The above-mentioned methodology was first applied to the classical American options as well as to a modification named capped options. These instruments led to a one-sided hitting problem. It turned out that we can generalize significantly the set of these derivatives since we have used a method based on the infinitesimal generators. Something more, we have established a sufficient criterion for recognizing whether the payoff leads to a one-sided task – put- or call-style. As a particular example, we have suggested a new class of instruments that can be viewed as a generalization of the futures contracts.

We continued our investigation with the so-called straddles and strangles which are hybrid strategies between a put and a call option. We have made this without any restrictions for the strikes as well as for the put and call weights. These instruments led to a task for a first exit from a strip. Furthermore, we have established a criterion which guarantees that a given payoff leads to a such two-sided problem. As an example, we have defined and examined a new class of derivatives – we named them quadratic strangles. It is interesting to note that some parameter values lead to two-sided exit problems, while others lead to one-sided put-style tasks. These instruments are interesting for the investors that prefer to hedge strongly the risky positions that are deeply far-from-the-money.

Next, we have examined the cancellable options for which the writer has the right to stop the contract prematurely. These instruments led to two-sided stopping problems too – one of the boundary for the holder's exercise right and another for the writer's one. We have studied separately the options with and without maturity constraints. We have generalized these derivatives assuming that the writer's penalty consists of three parts – a proportion of the usual option payoff, some shares of the underlying asset, and a fixed amount. It turns out that this generalization is not trivial – it seems that this way we can enclose the cancellable options in a way to guarantee a put-call duality similar to one existing for the classical options. This is a task for further work.

Last but not least, we have prepared many MATLAB codes for the studied derivatives to check and validate the derived theoretical results. We have provided the main of them.

The findings of this dissertation can be extended in several directions. First, the ever-changing financial realities lead to an increasing interest in novel instruments against different risks. As we mentioned above, the proposed technique represents a significant generalization and thus it allows the building of various new derivatives. In addition, many new strategies can be considered through the so-developed approach.

On the other hand, it is well-observed in all financial markets that the Gaussian assumptions of the Black and Scholes (1973) model are not consistent with the realities. For this, many authors turn to different alternatives – Lévy processes (we refer to Rachev and Mittnik (2000), Boyarchenko and Levendorskiĭ (2002), Bianchi et al. (2008), Cont and Tankov (2004), Rachev et al. (2005), etc.), stochastic volatilities (Heston (1993), Bates (1996), Zaevski et al. (2014), etc.) and other more general dynamics (Zaevski and Kounchev (2018) and Zaevski et al. (2019)). The approach, used in this dissertation and based on the infinitesimal differential operators, can be further applied to the above-mentioned models since they are based on other Feller-Markov stochastic processes.

The derivatives studied in this dissertation are powerful instruments against different financial risks. In this light, the results can be viewed as another method for analyzing market uncertainty. This is a very important but difficult task since everyone knows what the risk is but there is not a consensus on how to measure it. Different novel techniques are proposed for solving this problem – for example, the block-chains methods, Popchev and Taneva (2018) and Popchev et al. (2021b). For other interesting results, we refer to Denchev (1996), Denchev and Gumnerov (2006), Rachev et al. (2008), Popchev and Radeva (2019), Popchev et al. (2021a), Popchev et al. (2021c), and Zaevski and Nedeltchev (2023).

Last but not least, the personal MATLAB codes prepared for the needs of this dissertation can be extended and prepared as a MATLAB package for analyzing and evaluating the American-style financial instruments.

6 Scientific Contributions

In this dissertation, we develop a novel approach for evaluating American-style instruments written on an underlying asset modeled by a log-normal diffusion process. Their main feature is the early exercise right that the holder may use at any time before maturity. Thus Black-Scholes equation (5) turns into a free-boundary differential task. The traditional approach for examining these problems is based on several integral equations. However, their numerical solving needs relatively much computation time. Alternatively, the approach we suggest is based on several first-hitting and exiting properties of the Brownian motion. Let us denote by ζ the first hit of a

Brownian motion to a piecewise linear function or the exit of such strip. Let also T be a terminal date and θ , σ, and k be constants. We interested in the terms $\mathbb{E}\left[e^{-\theta\zeta}I_{\zeta<T}\right]$ and $\mathbb{E}\left[e^{\sigma B_T}I_{\zeta\geq T}\right]$. The desired results are provided in Chapter 2. Also, we prove in this chapter several results for some important limits of the form $\lim_{T\to\infty}e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta}\right]$ for the first hit to a linear boundary. In addition, we derive the necessary results for the limit $\lim_{T\to\infty}e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta,B_T>z(T)}\right]$, where $z\left(t\right)$ is another linear function.

Using these results for the Brownian motion's stopping times, we approximate the optimal boundaries by maximizing the holder's financial utility. Thus we convert the free boundary task into a boundary value problem. Many numerical methods are available for their solving. We create a relatively fast Monte Carlo algorithm for calculated expectation (4) that gives the price. Alternatively, we adapt several finite difference schemes to the arising differential task (5). It turns out that the Crank-Nicolson method is relatively faster and more accurate. If the financial contracts are not restricted by maturity constraints, then the optimal boundaries are flat due to the Markov property of the stochastic processes that drive the underlying assets. This allows us to derive closed-form formulas for the boundaries and the fair price using the method of maximizing the holder's financial result. This approach is applied to the traditional American options in Chapter 4. Their modification named capped options is studied in Chapter 5. The main characteristic they exhibit is the cap level above which a call option cannot be exercised (below for the puts). Some closed and semi-closed form formulas for the prices are obtained.

Chapter 6 considers some American style instruments with generalized payoffs – the main restriction we impose is a twice differentiability. We obtain sufficient conditions that turn the pricing of such derivatives into one-sided hitting problems. The method is based on the infinitesimal generators. Roughly said, the condition is satisfied if this differential operator applied to the payoff divides the state space into two connected subsets – the first one contains the positive values whereas the other consists of the negative ones. If the hit is below, then the derivative is related to the put options. On the contrary, if the hit is above, then we have a call-style contract. Our method is applied to these derivatives paying special attention to the power payoffs. Although the differentiable payoffs are considered, the presented method can be applied to the traditional options too. Note that their payoffs are not differentiable at the strike – $(x-K)^{+}$ or $(K-x)^{+}$. The goal is to approximate them by twice differentiable functions.

We examine in Chapter 7 the so-called American strangles. They appears as a combination between call and put options – the payoff is $\max\{C_1(x-K_1), C_2(K_2-x)\}$. The traditional assumption is that the put strike is lower than the call one, $K_1 \leq K_2$. Our approach allows us to abandon this restriction. In addition, we consider different impacts of the call and put features through the number of shares C_1 and C_2. It turns out that the time-state space can be divided into three connected parts. If the asset price is in the lowest one, then it is optimal for the holder to exercise as a put. The upper one contains the points that make the exercise as a call optimal. The middle set makes keeping the option alive preferable. Thus a two-sided optimal exit problem arises. Closed-form formulas for the perpetual options are obtained. It is important to mention that if the underlying asset does not pay dividends (equivalent to a model without additional discounting), then early exercising as a call is never optimal – a phenomenon that holds for many other American call-style derivatives including the usual options. It is interesting to note that despite this, the call feature has its impact – it appears through the number of the call shares but not via the call strike.

In Chapter 8 we investigate which payoffs lead to a similar two-sided optimal stopping problem – we obtain a sufficient condition. It says that this is the case when the infinitesimal generator applied to the payoff divides the state space $\mathbb{R}^+$ into three intervals – the generator is positive in the middle one and negative in the rest. To illustrate our model, we introduce and examine the so-called quadratic strangles with payoffs $(x-K)^2$. These instruments can be useful for investors who prefer to hedge strongly the far-from-the-money positions and weakly the near-the-money ones. It turns out that these derivatives may lead to one-sided hitting problems as well as two-sided ones depending on the position of the discount rate λ w.r.t. the constant $r+\sigma^2$ – both cases are investigated separately.

The rest of the dissertation is devoted to the so-called cancellable American options, also known as game or Israeli options. In addition to the holder's right to exercise prematurely, the cancellable ones provide to their writer the right to execute the contract paying some amount above the usual payoff. As a rule, these instruments lead to two-sided exit problems. Different from the strangles that maximize two-dimensional problems, the game options are related to finding a saddle point in the space of the stopping times. The call and put options are explored under our approach in Chapters 9 and 10. We prove that the optimal boundaries solve a two-dimensional non-linear system that achieves a unique solution. If we have a call-style option, then the holder's exercise set consists of all points above some level whereas the writer's one may be an interval with a left end-point equal to the strike, the singleton $\{K\}$, or even the empty set. In the last case, the option turns to ordinary American. For all other points, keeping the option is better than the immediate exercise for both participants. The results for the put options are similar but in some sense inverse –the holder's region consists of all points below some boundary, whereas the writer's one may be an interval $(B, K]$, the singleton $\{K\}$, or the empty set. In Chapter 11 we investigate options whose writer's penalty is a proportion of the usual payoff. The results for the optimal regions are similar. The main difference is that all points below the strike are optimal for cancellable calls. The same is true for the points above the strike for the puts. On the other hand, under the assumption that the holder would exercise later if this provides the same financial result, these points can be viewed as part of the continuation region. An interesting result is that both optimal boundaries for a put option coincide with the strike when $r \geq 0$. The same is true for the call options when $r \leq 0$ and $\lambda > 0$. Finally, we define in Chapter 12 a new class of cancellable options introducing some convertible features. The penalty that the writer owes for his early canceling right is composed of three parts – a proportion of the usual payoff, some shares of the underlying asset, and a fixed amount. We derive the related results for the optimal boundaries and corresponding regions. It seems that this generalization is not trivial, but it closes the set of cancellable options in some sense. This investigation is left for further work. Roughly said, as large penalties as the option is close to the usual American one. The cancellable (put) options under a finite maturity horizon are studied in Chapter 13. The used in this dissertation approach is adapted to these instruments. The main difference is that the holder maximizes his profit, but the writer minimizes the financial result. It turns out that there are two critical values for the time to maturity $0 \leq \tau_1 \leq \tau_2 \leq \infty$. For small enough maturities, $\tau \leq \tau_1$, the option is ordinary American. If $\tau \in (\tau_1, \tau_2]$, then the writer's optimal boundary is the strike. Note that the case $\tau_1 = \infty$ is possible. Finally, if $\tau > \tau_2$, then the option is real cancellable – the writer's optimal region is an interval $(K, A(\tau))$ for call options and $(B(\tau), K)$ for the puts.

At last but not least, we present in Chapter 14 some selected MATLAB codes for pricing the considered financial instruments. They implement the constructed algorithms based on the derived theoretical results.

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