STATEMENT REPORT
under the procedure for acquisition
of the scientific degree "Doctor of Science"
by candidate Assoc. Prof. Dr. Tsvetelin S. Zaevski,
of the Thesis entitled: "AN APPROACH FOR PRICING AMERICAN-STYLE DERIVATIVES"

Scientific Field: **4. Natural Sciences, Mathematics and Informatics**
Professional Field: **4.5. Mathematics**
Scientific Specialty: **"Probability Theory and Mathenmatical Statistics",**
Section: **"Operations Research, Probabilities and Statistics" (IOVS)**
Institute of Mathematics and Informatics of Bulgarian Academy of Sciences, (IMI-BAS)

The statement report is prepared by: **Prof. Dr. Maroussia Nikiforova Bojkova**, section IOVS, member of the scientific jury according to Order № 457/ 3.12.2024 of the Director of the Institute of Mathematics and Informatics - BAS and the protocol from the pre-defense with number 15/29.11.2024. The date of the first jury meeting is 09.12.2024.

1. General characteristics of the thesis and the presented materials

The presented D. Sci. thesis is written in English. It consists of 527 pages of main text, ending with a reference to contributions over 4 pages, formatted as Chapter 16, 6 pages of content, 2 pages of introduction, and 32 pages of bibliography, including 294 sources, indexes of names and terms found in the main text. The thesis is structured into 16 chapters. Additionally, at the beginning of the thesis, there is a list of 13 publications by the candidate from the period 2020-2024, followed by a 3-page preface. The first chapter provides an introduction to the developed topic, including: objectives, relevance, original contributions, and practical findings, motivation, description of American options in contemporary scientific literature, American financial instruments, and the developed methodology. Chapters 2-13 are numbered accordingly and contain the same material as the articles listed in the appendix titled "All publications for participation in the competition." In Chapter 14, the MATLAB code used for calculating/validating the theoretical methods developed by Dr. Zayevski is presented, covering pages 359-469 (110 pages). Chapter 15 contains 3 pages of concluding remarks and open problems.

2. Short CV and personal impressions of the candidate Assoc. Prof. Dr. Zayevski completed his higher education at the Faculty of Mathematics and Informatics (FMI) at Sofia University "St. Kliment Ohridski" (SU), majoring in Applied Mathematics in 1999. He defended his Master's degree under the supervision of Prof. Dr. Racho Denchev, with a thesis titled "Models of the Financial Market. Evaluation of European Derivatives." He continued his education in a doctoral program at FMI, SU, and obtained the scientific and educational degree of "Doctor" under the supervision of Prof. Dr. Racho Denchev. In

2013, he defended a Ph.D. thesis on "Combined Ito and Lévy Processes" at the Department of "Probabilities, Operations Research, and Statistics" at FMI. He has worked as an assistant professor from 2014 to 2021 and as an associate professor from 2021 to the present at the Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences. He has been elected as a head of the "IOVS" section at IMI since July 2024. I have known Assoc. Prof. Zayevski for many years, during which he has led a Master's course entitled "Mathematical Theory of the Financial Market" within the Master's program "Probabilities, Actuarial Science, and Statistics," and I hold him in high regard for his scientific and organizational qualities. For his scientific achievements, which are largely related to the publications in the current thesis, he has been selected as the Author of a Scientific Applied Achievement of IMI-BAS for 2019, 2022, and 2023.

3. Content analysis of the scientific and applied achievements of the candidate, contained in the presented DSci thesis and the publications to it, included in the procedure

Topic of the thesis. The research presented in the thesis falls within the field of financial mathematics and the stochastic processes associated with it, such as Brownian motion, diffusion processes, the Black-Scholes model, and more. According to MSC 2020, the articles on which the thesis is based are referenced in the following areas:

Stochastic Processes (60: Stochastic Processes), including martingales with continuous parameters. Game Theory, Economics, Finance, and Other Social and Behavioral Sciences (91: Game Theory, Economics, Finance, and Other Social and Behavioral Sciences), including two-player games, stochastic games, stochastic differential games, stochastic models in economics, derivative securities (option pricing, hedging, etc.); Partial Differential Equations (PDE) (35: Partial Differential Equations), including boundary problems for PDEs with free boundaries, PDEs related to game theory, economics, finance, and other social and behavioral sciences. As noted in the thesis and in the introduction to the abstract, the motivation for the research is the application of ideas and methods from stochastic models and games to solve classical problems related to the valuation of American-type options.

Actuality of the developed problems. I believe that the thesis addresses important and timely issues that arose after the 2008 financial crisis. Well-known techniques are employed, and new diverse techniques are developed, overcoming significant theoretical and technical challenges.

Scientific Contributions. This thesis develops a new approach for the valuation of American-type financial instruments, issued on an underlying asset modeled by a log-normal diffusion process, defined by the equation:

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{1}$$

where r is the risk-free interest rate, σ is the volatility, and $B_t, t \geq 0$ is the Brownian motion. The key feature of these instruments is the right to early exercise, which the holder can use at any time before maturity. Thus, the Black-Scholes equation (1) becomes a Partial Differential Equation (PDE) with a free boundary. The traditional approach to studying these problems is based on a system of integral equations, whose numerical solution requires relatively high computational time. As an alternative, the approach proposed in the thesis is based on certain properties of the first-passage time of Brownian motion.

If ζ is the first time the Brownian motion B_t reaches a linear function, T is the exercise date, and θ, σ and k are constants, then the following expectations $Ee^{-\theta\zeta}\mathbb{I}_{\{\zeta<T\}}$ and $Ee^{\sigma B_T}\mathbb{I}_{\{\zeta\geq T\}}$ are of interest for the approach developed in the thesis, where $\mathbb{I}$ is the indicator function. Using these results for the Markov moments of Brownian motion, the candidate approximates the optimal boundaries by maximizing the financial benefit of the derivative holder. This transforms the free boundary problem into a PDE in a known domain, for which many numerical methods exist. The thesis proposes a relatively fast Monte Carlo algorithm to compute the expectation: $Y_t = E^{t,S_t}[e^{-r(\tau-t)}G(\tau, S_\tau)]$, which gives the derivative price, where $G(t,x)$ is the payoff function and τ is the moment of the first exit from the domain D, associated with the maturity of the option.

As an alternative to this approach, the thesis suggests that the derivative price be written as a function of time and the current price of the underlying asset, i.e., $Y_t = V(t, S_t)$.

The Kolmogorov equation shows that the function $V(t,x)$ satisfies the following PDE:

$$\begin{aligned} V(t,x) + (\mathcal{A}^{\mathbb{Q}})(t,x) - rV(t,x) &= 0, \quad (t,x) \in D, \\ V(t,x) &= G(t,x), \quad (t,x) \in \partial D, \end{aligned}$$

where $\mathcal{A}^{\mathbb{Q}}$ is the infinetisamal diffusion S_t, satisfying (1).

If the asset pays continuous dividends at a rate λ, then the dynamics (1) become:

$$dS_t = (r - \lambda)S_t dt + \sigma S_t dB_t.$$

It turns out that the adapted Crank-Nicolson method is relatively faster and more accurate compared to other methods. If financial contracts have no maturity date, the optimal boundaries are fixed over time due to the Markov property of the stochastic processes governing the underlying assets. This allows for the derivation of closed-form formulas for the boundaries and for the fair price of the derivative using the method of maximizing the holder's financial outcome. This approach is applied to traditional American options in Chapter 4. Their modification, called capped options, is studied in Chapter 5. The key feature of these options is the level above which a call (or below which a put) option cannot be exercised.

Chapter 6 examines some financial instruments with a generalized payoff function, which has to be twice differentiable. Some sufficient conditions are derived that transform the valuation of such derivatives into one-sided first-passage problems. The method is based on infinitesimal generators. Roughly speaking, the condition is satisfied if the differential operator, applied to the payoff function, splits the state space into two connected subsets — one containing positive values, while the other consists of negative values. If the first-passage occurs from below, the derivative is associated with put options. If it is from above, we have a call-like contract. The method is applied to these derivatives, with special attention given to power payoff functions. Although differentiable payoff structures are considered, the presented method can also be applied to traditional options, whose payoff functions are nondifferentiable at the strike — $(x-K)^+$ or $(K-x)^+$. This is achieved by approximating them with twice-differentiable functions.

Chapter 7 is dedicated to so-called "strangle" strategies. These are a mix of call and put options — their payoff function is $\max\{C_1(x-K_1), C_2(K_2-x)\}$, where $K_1 \leq K_2$. The approach can be applied even without this last restriction. Furthermore, various put and call weights are considered through the number of shares C_1 and C_2. It turns out that the state space can be divided into three connected subsets. If the asset price is in the lowest subset, it is optimal for the holder to exercise as a put. The upper subset contains the points where exercising as a call is optimal. The middle subset makes it preferable to retain the option. This results in a two-sided exit problem from a strip. Formulas are derived for the case without maturity restrictions. It is important to note that if the underlying asset does not pay dividends (equivalent to a model with no additional discounting), early exercise as a call is never optimal — a phenomenon that applies to many other American call derivatives, including conventional options. Interestingly, it is worth noting that, despite this, the call specification has an effect — it appears through the number of call shares, but not through the call strike.

Chapter 8 investigates payoff structures that lead to a similar two-sided optimal stopping problem, for which sufficient conditions are derived. It turns out that this is the case if the infinitesimal generator, applied to the payoff function, divides the space $\mathbb{R}^+$ into three intervals — the generator is positive in the middle and negative in the others. To illustrate the model, so-called quadratic strangles are introduced with a payoff function $(x-K)^2$. These instruments can be useful for investors who prefer to hedge positions more strongly far from the strike and less so near it. It turns out that these derivatives can lead to one-sided as well as two-sided problems, depending on the position of the discount rate λ relative to the constant $r+\sigma^2$.

The remaining part of the thesis is dedicated to so-called "cancelable" American options, also known as Israeli options. In addition to the holder's right to exercise early, cancelable options grant the issuer the right to terminate the contract by paying a penalty over the usual payoff function. These instruments lead to two-sided exit problems from a strip. Unlike strangles, which maximize two-dimensional functionals, cancelable options are related to finding a saddle point in the space of random Markov moments. The call and put variants are examined using this approach in Chapters 9 and 10 (based on the publications with the same numbers, which have already been used in the procedure for the associate professor position).

In Chapter 11, perpetual options are studied, where the penalty is a proportion of the usual payoff function. The results for the optimal sets are similar. The main difference is that all points below the strike are optimal for the call-cancelable options. On the other hand, assuming that the holder will exercise later if it ensures the same financial result, these points can be considered part of the retention set.

Finally, in Chapter 12, a new class of cancelable options is defined, introducing some convertible features. The penalty that the writer owes for its early termination right consists of three parts — a proportion of the usual payoff function, the number of shares of the underlying asset, and a fixed sum. Results for the optimal boundaries and corresponding sets are obtained. It seems that this generalization is non-trivial and, in a certain sense, closes the set of cancelable options. This research is left for future work.

Cancelable (put) options with a finite time horizon are discussed in Chapter 13. The approach used in the thesis is adapted to these instruments. The main difference is that the holder maximizes their profit while the issuer minimizes their financial outcome.

Chapter 14 presents selected MATLAB code for evaluating the financial instruments discussed. These implement the constructed algorithms based on the theoretical results obtained.

In conclusion, I believe that the results achieved in the thesis represent an indisputable and significant contribution to the development of financial market theory and the application of stochastic modeling for the analysis of financial derivatives.

Actuality of the developed problems. I consider that the thesis tackles and solves important and timely problems that arose after the 2008 financial crisis, related to American-style options. Well-known techniques are used, and new, diverse techniques are developed, overcoming significant principled and technical challenges.

Applications of the results. Considering the period in which the articles included in the dissertation were published, one can conclude that the obtained results are well known to the scientific community. This is evidenced by the citations presented. As an additional merit of the candidate, I would highlight that many unsolved problems are formulated throughout the text, some of which are serious challenges for mathematicians working in the field.

It is clear to me that the candidate has an excellent understanding of the results in their area of research.

In conclusion, I would note that, in my opinion, the thesis can serve as the foundation for a monograph, which would undoubtedly be very useful both for specialists in the field and for popularizing the subject.

4. Approbation of the results According to the information provided in the abstract, the thesis is based on 9 articles published between 2020 and 2024, of which 4 have been used in the procedure for the position of Associate Professor. These four articles are included for clarity in the exposition, where the previously obtained results are applied. I will comment on the results of the remaining 9 articles that were not used in earlier procedures. All of them have been published in reputable journals: 2 in the Journal of Futures Markets, and one each in Communications in Nonlinear Science and Numerical Simulation, Computational and Applied Mathematics, Chaos, Solitons and Fractals, Comptes rendus de l'Académie bulgare des Sciences, Serdica Mathematical Journal, Mathematics, and Modern Stochastics: Theory and Applications. Out of these publications, 4 are in the Q1 quartile, 1 in Q2, 1 in Q3, 2 in Q4, and one is referenced in Zentralblatt and MathSciNet. All publications are independent.

a) The scientific works meet the minimum national requirements (according to Art.

2b, para. 2 and 3 of the ZRASRB) and, accordingly, the additional ones of the IMI - BAS for obtaining the scientific degree "Doctor of Sciences" in the scientific field and professional direction of the procedure. With minimum requirements of 100 points for groups of indicators G and D, the candidate has presented data for 495 and 232 points, respectively.

b) The results presented by the candidate in the dissertation and the scientific works (9 articles) do not repeat those from previous procedures for obtaining an academic title and position.

c) No plagiarism has been proven by the legally established procedure in the presented thesis and scientific works for this procedure.

5. Qualities of the abstract The abstract and the report on the contributions accurately reflect the content and main results of the thesis. I consider that the abstract and the report on the contributions meet all the requirements for their preparation.

6. Critical notes and recommendations I have no particular critical notes, except for some unedited spelling errors in the Bulgarian language texts resulting from translation. I have one recommendation for Assoc. Prof. Zaevski: to put more effort into attracting fresh talents and followers in this modern and fascinating field of mathematics.

7. Conclusion Having become acquainted with the DSci thesis presented in the procedure and the accompanying scientific works, and based on the analysis of their significance and the scientific and applied contributions contained in them, **I confirm** that the presented thesis and the scientific publications related to it, as well as the quality and originality of the results and achievements they contain, meet the requirements of the ZRASRB, the Rules for its application, and the relevant Rules of BAS and IMI - BAS for obtaining the scientific degree "Doctor of Sciences" in the scientific field 4. Natural Sciences, Mathematics and Informatics, and professional direction 4.5. Mathematics. In particular, the candidate satisfies the minimum national requirements in the professional direction and the additional requirements of IMI - BAS, and no plagiarism has been detected in the scientific works presented for the competition.

Based on the above, **I strongly recommend** to the scientific jury to award Assoc. Prof. Dr. Tsvetelin Stefanov Zaevski the scientific degree "Doctor of Sciences" in the scientific field 4. Natural Sciences, Mathematics and Informatics, professional direction 4.5. Mathematics (scientific specialty: "Probability Theory and Mathematical Statistics").

Date: 21.02.2025 Sign:

/Prof. Dr. Sci. Maroussia Bojkova/