REVIEW

by Prof. D.Sc. Mancho Hristov Manev
Faculty of Mathematics and Informatics
Plovdiv University "Paisii Hilendarski"

on a dissertation for awarding
the educational and scientific degree "Doctor"

Area of Higher Education: 4. Natural Sciences,
Mathematics and Informatics
Professional Field: 4.5 Mathematics
Doctoral Program: Geometry and Topology

Author: Victoria Gerasimova Bencheva-Petrova

Title: **Differential Geometry of Time-like Surfaces in Four-dimensional Minkowski Space**

Scientific Supervisors: Prof. Dr. Velichka Milousheva (IMI-BAS),
Assoc. Prof. Dr. Milen Hristov (VTU)

By Order No. 191/02.07.2024 of the Director of the Institute of Mathematics and Informatics at the BAS (abbr., IMI), issued on the basis of a decision of the Scientific Council of IMI (Protocol No. 9/28.06.2024) and in accordance with Article 30 of the Regulations for the Implementation of Development of the Academic Staff in the Republic of Bulgaria Act (abbr., DASRB Act), I have been designated as a member of the Scientific Jury in the procedure for the defense of the mentioned dissertation thesis.

According to the decision of this jury (Protocol No. 1/03.07.2024), I am assigned to write a review of the dissertation and the documents for the procedure.

In my role as a member of the scientific jury, I have received electronic access to all documents submitted for participation in the procedure.

1. Data and personal impressions about the doctoral candidate

Doctoral candidate Viktoria Bencheva-Petrova was born in 1991 and graduated with a Bachelor's degree in "Mathematics and Informatics" at the Veliko Tarnovo University "St. Cyril and St. Methodius" (abbr., VTU) in 2014. Two years later, she graduated as a Master in "Technologies for Teaching Mathematics" again at VTU. On 07.05.2017, she started part-time doctoral studies at IMI under the doctoral program "Geometry and Topology". Viktoria Bencheva completed her doctoral studies with the right of defense on 07.05.2022. She was a part-time Assistant Professor in Advanced Mathematics at VTU in the first semester of 2016/17 and immediately after that she was an Assistant Professor there for 4 years, leading seminars in several mathematical disciplines. In addition, for the past more than 10 years, she has also worked as a web site developer in a company.

I have positive, albeit brief, impressions of Viktoria Bencheva's scientific activity from her talks at the International Colloquium on Differential Geometry and its Related Fields in Veliko Tarnovo in 2014 and 2016. Then she presented her results on another topic of differential geometry together with her scientific supervisor Assoc. Prof. Milen Hristov.

2. Characteristics of the dissertation and the materials of the procedure

I got acquainted with the documents that were submitted by the doctoral candidate Viktoria Bencheva in the procedure for the defense of her dissertation thesis. I believe that they are in accordance with the requirements of the regulatory framework, namely the DASRB Act, the Regulations for the Implementation of the DASRB Act, and the Regulations for the Terms and Conditions for Acquiring Scientific Degrees and for Occupying Academic Positions in IMI of the BAS (abbr., TCASDOAP Regulations in IMI) for procedures opened before 04.05.2018.

The author's abstract (36 pages) and the author's reference for the contributions in the dissertation thesis are prepared according to the requirements and correctly reflect the results and scientific contributions in the dissertation.

I have not noticed any form of plagiarism in the dissertation and the publications submitted for evaluation by the doctoral candidate.

The topic of the dissertation is on the development of the local differential geometry of time-like surfaces in a 4-dimensional Minkowski space and application of the results to the study of such surfaces of rotational type. This dissertation continues the differential-geometric studies of 4-dimensional Euclidean and pseudo-Euclidean spaces, started in our country by Georgi Ganchev and Velichka Milousheva and continued by Krasimir Kanchev and Yana Alexieva.

The research methods are analogous to those used for surfaces in 4-dimensional Euclidean space and on the space-like surfaces in 4-dimensional Minkowski space.

The presented thesis of Victoria Bencheva is written in Bulgarian and has 112 pages. It consists of an introduction (10 pages), two chapters with a total of 7 sections (93 pages), a bibliography of 67 titles and a table of contents. The topic and content of the dissertation fully correspond to the doctoral program and the scientific specialty.

The Introduction thoroughly informs the reader about the state of the local differential geometry of special surfaces in 4-dimensional spaces equipped with different metrics according to the signature, citing publications related to this topic. This overview of the problems related to the topic and the known results for them shows a good knowledge of the problem set on the doctoral candidate's perspective.

The purpose of the dissertation is clearly formulated, namely: Study of time-like surfaces in the 4-dimensional Minkowski space $\mathbb{R}^4_1$ and development of their local theory by analogy with the local theory of surfaces in the 4-dimensional Euclidean space $\mathbb{R}^4$ and the theory of space-like surfaces in $\mathbb{R}^4_1$.

The doctoral candidate copes with the given task by using a geometrically determined moving frame field at each point of the surface, with respect to which geometric functions and a system of partial differential equations (abbr. PDEs) are obtained that determine the surface up to motion in $\mathbb{R}^4_1$.

Another task that is also set is to introduce special parameters on other classes

of time-like surfaces in $\mathbb{R}^4_1$ besides the minimal ones, to allow the minimization of the number of functions and the number of equations mentioned above. The problem is solved for the class of time-like surfaces with a parallel normalized mean curvature vector field by introducing special isotropic parameters, which are called canonical. In this case, 3 functions and 3 equations are obtained.

After that, the Introduction summarizes the research done and the results obtained in each section and subsection of the dissertation.

At the end of the Introduction, in addition to acknowledgements to the supervisors and the family, a list of the 3 published papers with results included in the dissertation is given. It is noteworthy that all three papers appeared in the last 2 years in journals with an impact factor: *Turkish Journal of Mathematics*, *Filomat* and *Proceedings of the Bulgarian Academy of Sciences*. Considering the latest data for 2023, there are 2 papers in 2nd quartile journals and one – in the 4th quartile on Web of Science. The authors of these articles are the doctoral candidate and Prof. Milousheva, as her supervisor. I suppose that the participation of the two co-authors is equal.

In Chapter 1 of the dissertation, fundamental theorems for time-like surfaces in $\mathbb{R}^4_1$ are formulated and proved, i.e. theorems for existence and uniqueness (up to motion) of the surface of the studied type. The chapter begins with §1.1, where the necessary background information and concepts for two-dimensional surfaces M^2 in $\mathbb{R}^4_1$ are given. In §1.2, time-like surfaces M^2 without inflection points are considered, i.e. the first normal space of M^2 in $\mathbb{R}^4_1$ is 2-dimensional.

A second fundamental form II of M^2 is defined and the formulas for their coefficients under arbitrary change of the surface parameters are found. The invariance of II under changes of the bases of the normal and tangent spaces of M^2, respectively, is studied. Through II, an analogue of the Weingarten map γ is considered and its invariants $k = \det\gamma$ and $\varkappa = -\frac{1}{2}\mathrm{tr}\,\gamma$ are studied. Asymptotic and principal lines are defined, then the condition $\varkappa^2 - k > 0$ is found under which there exists a local parametrization with respect to the principal lines. In the case $\varkappa^2 - k < 0$, the

use of isotropic parameters is proposed.

In §1.3, it is proved that M^2 is a minimal time-like surface in $\mathbb{R}^4_1$ if and only if M^2 is umbilical.

In §1.4, time-like surfaces in $\mathbb{R}^4_1$ without umbilical points are considered. In the case of the existence of a local parameterization with respect to the principal lines, a geometric frame field of M^2 defined by the principal directions and the mean curvature vector field H is used to prove a fundamental theorem (Theorem 1.4.6.) for this class of surfaces in terms of 8 geometrically determined functions satisfying a system of 8 PDEs. The geometric meaning of some of these functions is shown. In the alternative case, M^2 is studied with respect to a pair of isotropic parameters, and analogous geometric interpretations of the used characteristic functions related to the Gaussian curvature K and H are obtained.

In the next part of the section, fundamental theorems for time-like surfaces of general type are proved, i.e. surfaces with non-parallel normalized H. Isotropic parameters and the corresponding geometric pseudo-orthonormal frame field are used. Three types of surfaces of general type are considered, for each of which a fundamental theorem is proved: Theorems 1.4.15–1.4.17.

In §1.5, time-like surfaces in $\mathbb{R}^4_1$ with parallel normalized H are considered. Special isotropic parameters, called canonical, are locally introduced for them. The introduction of canonical parameters is shown separately for the cases determined by the sign of the quantity $K - H^2$. At the end of the section, the corresponding fundamental theorems formulated in terms of canonical parameters are proved: Theorems 1.5.6–1.5.7.

Chapter 2 is devoted to the study of time-like rotational surfaces M^2 in $\mathbb{R}^4_1$ and begins with §2.1, where generalized M^2 of two types are studied – with time-like and space-like meridian curves, respectively. On these surfaces, the Weingarten map γ is shown to be diagonalizable, and a parameterization with respect to the principal directions of M^2 can be used, applying the theory of §1.4.1. As a result, for the two types of M^2, the 8 functions of the corresponding fundamental theorem are

found. Furthermore, it is proved that if M^2 has no minimal points, it is a nontrivial Chen-surface (Theorem 2.1.1). The remaining part of §2.1 describes analytically these two types of surfaces with different conditions for K, H, $\varkappa$, and k.

In §2.2, the so-called meridian time-like surfaces M^2 in $\mathbb{R}^4_1$ which are 1-parameter systems of meridians of a rotational hypersurface in $\mathbb{R}^4_1$ with a time-like axis are studied. In this case, γ is not diagonalizable and M^2 are studied with respect to isotropic parameters, applying the theory of §1.4.2 and §1.4.3. Various classes of such surfaces lying on a rotational hypersurface with time-like axis are considered. Classification results are obtained in the cases of constant Gaussian curvature, constant mean curvature, and constant Weingarten map determinant.

3. Approbation of the results of the dissertation

According to Art. 6, item 3 of the TCASDOAP Regulations in IMI, in addition to a dissertation, publications reflecting essential parts of the work must be submitted, which must be at least 3 in peer-reviewed publications, at least one of which is a journal. From the presented journal articles with an impact factor, it can be seen that this requirement is precisely met quantitatively and exceeded qualitatively.

There are no citation data for the publications. In the curriculum vitae, two reports on the results of the dissertation were reported, one was presented at the General Seminar of the Analysis, Geometry and Topology Section, IMI, 2021, and the other at the International Conference "Mathematics Days in Sofia", 2023.

4. Critical notes and recommendations

It is noteworthy that the presentation and layout of the dissertation is at a very good level. However, I will make some notes in order to further improve the research presentation, which can be used in the future to teach other doctoral students.

– The adjectives "Euclidean", "Riemannian", "Lorentzian", "Gaussian" and similar must be written in lowercase letters in Bulgarian. The rule is formulated in item 42.7. from *The Official Spelling Dictionary of the Bulgarian Language* (2012, p. 47).

– It is good to formulate the conclusions as statements in §1.2.

– Dissertations should be styled like textbooks. It is not good to have sentences without a predicate, even almost without words, e.g. that after (1.26).

– Page 37, line 2 of the proof is interrupted.

– Clarified terms should be used, e.g. "meridian time-like surfaces" or "time-like meridian surfaces".

5. Conclusion

Victoria Bencheva's dissertation contains topical scientific results that represent an original contribution to differential geometry.

The submitted documents fully meet all the requirements of the DASRB Act, the Regulations for the Implementation of the DASRB Act and the TCASDOAP Regulations in IMI.

Due to the above, I give my **positive assessment** for the scientific research presented by the dissertation, the author's abstract, the achieved results and contributions presented above, and I propose to the honorable Scientific Jury **to award Viktoria Gerasimova Bencheva-Petrova** the educational and scientific degree **„Doctor"** in Area of Higher Education: 4. Natural Sciences, Mathematics and Informatics; Professional Field: 4.5. Mathematics; Doctoral Program: "Geometry and Topology".

20 August 2024
Plovdiv, Bulgaria

Signature:

(Prof. D.Sc. Mancho Manev)