REVIEW

by Stefan Petrov Ivanov, Dr. Sci.,

Professor at the Sofia University “St. Kliment Ohridski” - FMI and IMI-BAS

of a dissertation for awarding the educational and scientific degree “doctor”

in the Area of Higher Education 4. Natural Sciences, Mathematics and Informatics,

Professional Field 4.5 Mathematics, Doctoral Program “Geometry and Topology”

Author: Viktoria Gerasimova Bencheva-Petrova

Title: DIFFERENTIAL GEOMETRY OF TIME-LIKE SURFACES IN FOUR-DIMENSIONAL MINKOWSKI SPACE

Scientific supervisor: Prof. Dr. Velichka Milousheva - IMI-BAS and Assoc. Prof. Dr. Milen Hristov - VTU “St. Cyril and Methodius”

General description of the presented materials.

By order No. 191 of 02.07.2024. of the Director of IMI-BAS, I have been appointed as a member of the Scientific Jury in the procedure for the defense of a dissertation thesis on the topic DIFFERENTIAL GEOMETRY OF TIME-LIKE SURFACES IN FOUR-DIMENSIONAL MINKOWSKI SPACE for the acquisition of the educational and scientific degree “doctor” in the Area of Higher Education 4. Natural Sciences, Mathematics and Informatics, Professional Field 4.5 Mathematics, Doctoral Program “Geometry and Topology”.

The author of the dissertation is Viktoria Gerasimova Bencheva-Petrova - a part-time doctoral student trained at the Analysis Geometry and Topology (AGT) Section of IMI-BAS with scientific supervisors Prof. Dr. Velichka Milousheva from IMI-BAS and Assoc. Prof. Dr. Milen Hristov from VTU “St. Cyril and Methodius”.

The set of materials submitted by Victoria Bencheva is in accordance with Article 36 (1) of the Regulations for the Development of the Academic Staff of IMI-BAS and includes the following documents:

- a request to the Director of IMI-BAN for starting the procedure for the defense of a dissertation thesis;
- curriculum vitae in European format;
- protocol from the discussion of the dissertation thesis by the extended scientific unit of the AGT Section;

– dissertation thesis;
– abstract;
– a list of scientific publications on the subject of the dissertation;
– copies of scientific publications;
– report on the contributions in the dissertation;

The doctoral student has attached to her dissertation 3 publications in peer-reviewed journals with an Impact Factor.

I have no remarks or comments on the submitted documents.

Brief biographical data about the doctoral student

The candidate was born on September 17, 1991 in Bulgaria. She graduated from Mathematics High School "Vasil Drumev", Veliko Tarnovo, in 2010. She received her higher education from VTU "St. St. Cyril and Methodius" as a bachelor in Mathematics and Informatics in 2014 and as a Master of Technology for Mathematics Education in 2016. Then, during the period from 2017 until now, she has been a part-time doctoral student at the AGT Section of IMI-BAS. During the period 2017-2021 she was an assistant at VTU "St. St. Cyril and Methodius". She led exercises in Mathematical Analysis, Linear Algebra, Analytical Geometry, Statistics, etc. From October 2013 up to now, she is a Web site programmer, development and design of web sites and online stores at "GS Vision" EOOD, Veliko Tarnovo.

Relevance of the topic and appropriateness of the set goals and objectives

The presented dissertation is devoted to the local geometry of smooth surfaces in pseudo-Euclidean space, specifically surfaces in flat Lorentz space, Minkowski space, on which the induced metric has a Lorentz signature, time-like surfaces. A fundamental issue in local surface theory is finding a Bonnet-type theorem, namely determining a minimal number of functions and corresponding PDEs that define the surface up to a motion, a pseudo-Euclidean isometry. The present dissertation is devoted to this fundamental question for two-dimensional time-like surfaces in 4-dimensional Minkowski space. For a two-dimensional time-like surface in a 4-dimensional Minkowski space, a geometrically defined moving frame is introduced at each point of the surface, with respect to which a set of geometric functions and a system of partial differential equations are obtained which define the surface up to a motion.

The theory of surfaces in a Minkowski space is of extreme interest in mathematics and mathematical physics as it finds application in the theory of space-time and the theory of black

holes, a special class of such surfaces, called marginally trapped, was introduced and studied by the famous physicist Roger Penrose, surfaces with a parallel mean curvature vector field were studied by famous mathematicians and physicists such as the Fields Medalist S.-T. Yau and others.

Based on this, I believe that the topic is relevant enough on a global scale.

Characteristics, evaluation and contributions of the thesis

The dissertation submitted for review contains an introduction, an exposition in two chapters of 5 and 2 sections, respectively, and a list of references. The total volume of the text is 113 pages in Bulgarian, and the list of references contains a total of 67 papers. The dissertation thesis is based on 3 papers published in international scientific journals with an Impact Factor such that the total Impact Factor = 2.1.

In the introduction, an overview of the most significant results in the subject of the dissertation is given, the problems and goals considered in the dissertation are described, as well as the results obtained.

In the first chapter, the local theory of two-dimensional time-like surfaces in the four-dimensional Minkowski space is developed analogously to the theory of surfaces in the four-dimensional Euclidean space. For the different classes of surfaces, parametrization with respect to different, specially chosen parameters is used, and fundamental theorems – existence and uniqueness theorems – are proved. An essential difference with the theory of two-dimensional surfaces in the Euclidean 4-dimensional space, where a parametrization with respect to principal lines always exists, for time-like surfaces in Minkowski space there is also a case in which there are no principal lines, and for this class of surfaces a new approach based on the isotropic directions of the surface is developed, which is an essential contribution of the doctoral student.

In Section 1.2, a Weingarten-type linear map W and two functions k=det(W) and h=-1/2tr(W) which are invariant under changes of the parametrization are introduced, and it is shown that the function h is exactly the curvature of the normal connection. In the cases when $h^2-k>0$ and $h^2-k=0$, the theory of time-like two-dimensional surfaces in the 4-dimensional Minkowski space is developed analogously to the well-known theory of space-like surfaces in the Minkowski space developed by Ganchev-Milousheva, and I will point out Proposition 1.3.1, Theorems 1.4.1 - 1.4.5, whose proofs are a preparation for the proof of the fundamental theorem on the existence of two-dimensional time-like surfaces admitting parametrization with respect to principal lines, Theorem 1.4.6. This main result of the dissertation asserts that the 8 geometrically introduced functions

satisfying a system of 8 PDEs define a time-like surface up to a motion, the proof requiring specific non-trivial computations, which the PhD student has demonstrated.

The main contributions of this dissertation are contained in Sections 1.4.2 and 1.4.3, where for a two-dimensional time-like surface without umbilical points in a 4-dimensional Minkowski space a parametrization with respect to isotropic parameters is used, a geometric moving frame field defined by the isotropic directions of the surface is introduced, 6 geometric functions are defined and fundamental existence theorems of Bonnet type are proved: Theorem 1.4.15, where 6 geometric functions satisfying a system of 4 PDEs define a time-like surface of the first type up to a motion, Theorem 1.4.16 where 5 geometric functions satisfying a system of 4 PDEs define a time-like surface of the second type up to a motion, Theorem 1.4.17 where 3 geometric functions satisfying a system of 3 PDEs define a time-like surface of the third type up to a motion.

In Section 1.5, two-dimensional time-like surfaces in a 4-dimensional Minkowski space with parallel normalized mean curvature vector field H and Gaussian curvature K were studied and characterized, and the fundamental theorem for them, Theorem 1.5.6 uniquely determines such a surface, for which $K-H^2 > 0$ or $K-H^2 < 0$, with 3 functions satisfying a system of 3 PDEs, and Theorem 1.5.7 uniquely defines such a surface for which $K-H^2 = 0$ with 3 functions satisfying a system of 2 PDEs.

In the second chapter of the thesis, the local geometry of two-dimensional time-like general rotational and meridian surfaces in the 4-dimensional Minkowski space is investigated by applying the theory of two-dimensional time-like surfaces already developed in Chapter 1, thereby providing examples on which the developed local theory is verified and applied.

Two types of such surfaces are defined and it is shown that time-like generalized rotational surfaces of the first and second type without minimal points in 4-dimensional Minkowski space are non-trivial Chen surfaces. Some basic classes of time-like generalized rotation surfaces given by conditions on the Gaussian curvature, the normal curvature or the mean curvature vector field are also studied. It is worth noting Theorem 2.1.5, where the minimal time-like generalized rotation surfaces of the first and second type are explicitly described, Theorem 2.1.7 which explicitly defines the time-like general rotation surfaces of the first and second type with a parallel normalized mean curvature vector field, etc.

Assessment of the publications on which the dissertation is based

The results of the dissertation are included in 3 papers joint with the supervisor V. Milousheva which are published in international mathematical journals with Impact Factor, namely Filomat, Turkish Journal of Mathematics and C. R. Acad. Bulg. Sci., as the total IF = 2.1.

The author of the dissertation has not provided any citations of the dissertation.

Author's abstract

The abstract synthesizes the main research in the thesis by formulating both the main and auxiliary results. In my opinion, the author's abstract is done as required and reflects the main results achieved in the dissertation.

I have no critical remarks or recommendations.

CONCLUSION

The dissertation contains scientific, scientific-applied and applied results, which represent an original contribution to science and meet all the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria (ZRASRB), the Regulations for the Implementation of ZRASRB and the relevant Regulations of IMI-BAS. The presented materials and dissertation results fully correspond to the specific requirements of IMI-BAS, adopted in connection with the BAS Regulations for the application of the ZRASRB.

The dissertation shows that the doctoral student Viktoria Bencheva-Petrova possesses in-depth theoretical knowledge and professional skills in the Professional Field 4.5 Mathematics, demonstrating qualities and skills for doing independent scientific research.

Due to the above, I confidently give my **positive assessment** of the conducted research, presented by the above-reviewed dissertation thesis, abstract, achieved results and contributions, and **I propose to the honorable Scientific Jury to award the educational and scientific degree "doctor"** to Victoria Gerasimova Bencheva-Petrova in the Area of Higher Education: 4. Natural Sciences Mathematics and Informatics, Professional Field 4.5 Mathematics, Doctoral Program "Geometry and Topology".

23.08.2024 г. Reviewer:.....

Prof. Dr. Sci. Stefan Ivanov