REPORT

by Prof. Galya Vasileva Nakova, PhD
Faculty of Mathematics and Informatics,
St. Cyril and St. Methodius University of Veliko Tarnovo
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of a dissertation for awarding the educational and scientific degree **"doctor"**, area of higher education 4. Natural Sciences, Mathematics and Informatics,
professional field 4.5. Mathematics, doctoral program "Geometry and Topology"

Author: Victoria Gerasimova Bencheva-Petrova

Title: Differential geometry of timelike surfaces in the Four-dimensional Minkowski Space

Supervisors:
Prof. Velichka Vasileva Milousheva, PhD – IMI-BAS
Assoc. Prof. Milen Jordanov Hristov, PhD – University of Veliko Tarnovo

1 Data and personal impressions about the doctoral student

Victoria Bencheva-Petrova completed her secondary education in 2010 at Vasil Drumev High School of Science and Mathematics in Veliko Tarnovo, profile "Mathematics and German". In 2014, she graduated from VTU St. St. Cyril and Methodius as a bachelor in the specialty "Pedagogy of Mathematics and Informatics Education", and in 2016 she obtained a master's degree at VTU in the specialty "Technologies of Mathematics Education". In the period 2017–2022, she was a part-time PhD student, PhD program "Geometry and Topology"at IMI-BAS. From 2016 to 2017, Viktoria Bencheva-Petrova was a part-time assistant at FMI at VTU, and from 2017 to 2021 she was an assistant at the Department of Mathematical Analysis and Applications of FMI at VTU. She has taught exercises in mathematical analysis, linear algebra, analytical geometry, statistics. From 2013 until now, she also works as a web site developer.

I have known Victoria Bencheva-Petrova since 2010. I have taught her various geometric disciplines. My personal impressions are that she has excellent mathematical training and great creative potential. In support of my opinion are her successes at the following mathematical competitions: NSOM - 1 silver and 4 bronze medals; Inter-university competition in mathematics, organized by FMI of VTU - 2 gold and 1 bronze medals; SEEMOUS - 1 bronze medal; first places in full-time and part-time rounds of SOM at VTU. I will especially note that, even as a student, she conducted scientific research in the field of differential geometry under the supervision of Assoc. Prof. Milen Hristov, who is her scientific supervisor of the graduate works for bachelor's and master's degrees. From the time of our joint work at FMI of VTU, I can say that Viktoria Bencheva-Petrova is a responsible teacher respected by the students. She is a fair and responsive colleague. She actively participated in various FMI initiatives at VTU, such as the preparation and holding of a Mathematical competition for students, preparation of students for participation in NSOM and NSOCM. She was a member of the scientific teams of 2 projects at the Bulgarian National Science Fund and 6 projects at the FMI of VTU.

2 Relevance of the topic, goals and tasks

The study of surfaces in pseudo-Euclidean spaces is a current direction in modern differential geometry, which is conditioned by their diverse applications in theoretical and mathematical physics. It is known that there exist the following three types of surfaces in a pseudo-Euclidean space depending on the restriction of the indefinite metric of the ambient space on the tangent space of the surface: spacelike, timelike, and lightlike, when the induced metric is Riemannian, Lorentzian and degenerate, respectively. Different classes of the three types of surfaces in the 4-dimensional Minkowski space $\mathbb{R}^4_1$ are intensively studied by many authors worldwide. Georgi Ganchev and Velichka Milousheva have a significant contribution to the development of the invariant theory of spacelike surfaces in $\mathbb{R}^4_1$.

Object of study in the presented dissertation are timelike surfaces in $\mathbb{R}^4_1$. The aim of the research is to develop the local theory of these surfaces. To achieve the set goal, the doctoral student has creatively applied the approach to the study of surfaces in 4-dimensional Euclidean space $\mathbb{R}^4$ and spacelike surfaces in $\mathbb{R}^4_1$. It is based on the introduction of a geometric frame field at each point of the surface, with respect to which a set of geometric functions and a system of partial differential equations are derived that define the surface up to a motion in $\mathbb{R}^4_1$.

3 Characterization of the dissertation and evaluation of the main results and scientific contributions in it

The dissertation is in the volume of 112 pages. It is structured in 25 paragraphs, organized into 2 chapters, an introduction and a bibliography of 67 references. The dissertation abstract consists of 36 pages and reflects the main results achieved in the dissertation.

At the beginning of the first chapter, basic definitions and formulas for submanifolds relating to surfaces are briefly given. Further, the invariance of the two fundamental forms of a timelike surface M^2 in $\mathbb{R}^4_1$ under any change of the parameters on M^2, as well as the invariance of the second fundamental form under a change of the basis of the normal space N_pM^2, $p \in M^2$, are discussed in detail. Analogously to the surfaces in $\mathbb{R}^4$ and spacelike surfaces in $\mathbb{R}^4_1$, a linear map γ of Weingarten-type is introduced, as well as the functions $k = \det \gamma$, $\varkappa = -\frac{1}{2}\mathrm{tr}\, \gamma$, generated by γ. It is proved that $\varkappa$ coincides with the curvature of the normal connection of M^2.

It is known that for surfaces in $\mathbb{R}^4$ and spacelike surfaces in $\mathbb{R}^4_1$ the matrix of the Weingarten map can always be diagonalized, i.e. principal lines exist at every point on these surfaces, and they can be parameterized by principal parameters. In the present dissertation it is established that for timelike surfaces in $\mathbb{R}^4_1$ the diagonalization of the matrix of the Weingarten map depends on the sign of the function $\varkappa^2 - k$ at the points of the surface. Each of the cases $\varkappa^2 - k = 0$, $\varkappa^2 - k > 0$ and $\varkappa^2 - k < 0$ is considered separately and it is interpreted geometrically. For example, $\varkappa^2 - k = 0$ at a point on the surface is a necessary condition the point to be umbilical. It is proved that a timelike surface in $\mathbb{R}^4_1$ without inflection points is minimal if and only if it consists of umbilical points. Both types of surfaces without umbilical points have been studied in detail.

When $\varkappa^2 - k > 0$ at any point on the surface, the Weingarten map is diagonalizable. In this case, the surface is studied analogously to a spacelike surface in $\mathbb{R}^4_1$ - it is parameterized by principal parameters and a geometric frame field is determined by the principal directions and the mean curvature vector field. The derivative formulas of the Frenet-type with respect to the geometric frame field are derived, where 8 functions are involved. In terms of these functions, a geometric characterization of the considered surface is obtained. Necessary and sufficient conditions the surface to be flat, with a constant non-zero mean curvature, with a constant non-zero Gauss curvature, with a flat normal connection, with a constant non-zero normal curvature are given. A theorem like Bonnet's theorem in classical differential geometry is proved.

The Weingarten map is not diagonalizable for time-like surfaces in $\mathbb{R}^4_1$ for which at every point $\varkappa^2 - k < 0$. The doctoral student investigates such surfaces applying a different approach than in case $\varkappa^2 - k > 0$. It is based on the well-known fact that for any timelike surface M^2 in a pseudo-Euclidean space one can choose a local coordinate system $\{u, v\}$ with respect to which the metric tensor g on $M^2 : z = z(u, v)$ is defined so that the tangent vector fields z_u and z_v are isotropic (lightlike). The parameters (u, v) are called isotropic parameters of the surface, and the directions defined by z_u and z_v - isotropic directions. Further, the doctoral student studies the timelike surfaces in $\mathbb{R}^4_1$ with a non-diagonalizable Weingarten map in terms of isotropic parameters. By using the isotropic directions, a pseudo-orthonormal basis is constructed, with respect to which the derivative formulas are derived. With the help of the nine functions in these formulas, geometric characteristics of different classes of the considered surfaces are given. Depending on the values of some of the obtained geometric functions, three types of surfaces are distinguished, for each of which a fundamental theorem of existence and uniqueness is proved.

The last 4 paragraphs of the first chapter are devoted to the study of timelike surfaces in $\mathbb{R}^4_1$ with a parallel normalized mean curvature vector field, parameterized with respect to isotropic parameters. These surfaces are divided into two main classes. For the surfaces of both classes it is shown that they locally can be parameterized by special isotropic parameters, called canonical parameters. In this way, a reduction of the number of geometric functions and the partial differential equations of the integrability conditions of the considered surfaces is achieved. Fundamental existence and uniqueness theorems for both classes of surfaces are proved.

In the second chapter, two basic classes of timelike surfaces in $\mathbb{R}^4_1$ are constructed: the class of timelike general rotational surfaces of first and second types and the class of timelike meridian surfaces. Surfaces of the first class (resp. of the second class) are examples of timelike surfaces in $\mathbb{R}^4_1$ with diagonalizable (resp. nondiagonalizable) Weingarten map. For the study of the surfaces of each class, the doctoral student applies the relevant local theory, developed in the first chapter, i.e. she examines the surfaces of the first class (resp. of the second class), parameterized by principal parameters (resp. by isotropic parameters). Many types of surfaces with additional conditions (for the Gauss curvature, the mean curvature, the normal connection, the mean curvature vector field, the invariant k of the Weingarten map) are analytically characterized. The flat meridian surfaces of elliptic type and those with non-zero constant Gauss curvature are explicitly given.

4 Estimation of the publications on the dissertation

Victoria Bencheva-Petrova has presented 3 papers in which the main results of the dissertation were published. All papers are in English and they have been published in peer-reviewed and indexed Web of Science journals, of which 2 in Q2 and 1 in Q4. The presented publications significantly exceed the requirements of the Regulations for the terms and conditions for acquiring scientific degrees and for holding academic positions at BAS and the relevant Regulations at IMI-BAS.

5 Remarks and recommendations

The present dissertation is well formed in structural and technical terms. The exposition is distinguished by a logical sequence and a precise scientific style. I have the following remarks, which do not significantly affect the results in the dissertation: Statements (i) and (ii) from Theorem 2.1.4. should be omitted because they contradict the conditions for the functions f and g from the definitions of general rotational surfaces of the first and second type; It is good to clarify what is meant by motion in $\mathbb{R}^4_1$. My recommendation to the doctoral student is to illustrate and support the proven assertions with more explicit examples in future research.

6 Conclusion

The results in the dissertation work are new, meaningful and significant. The presented dissertation work and the scientific publications attached to it fully meet the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria, the Regulations for its Implementation, the Regulations for the Terms and Procedures for Acquiring Scientific Degrees and for Holding Academic Positions at the BAS and the relevant Regulations in IMI-BAS.

Due to the above, I confidently give my **positive** assessment of the research, the achieved results and contributions and **propose** to the honorable Scientific Jury to award the educational and scientific degree **"doctor"** to **Victoria Gerasimova Bencheva-Petrova** in the area of higher education 4. Natural sciences, Mathematics and Informatics, professional direction 4.5. Mathematics, doctoral program "Geometry and Topology".

26. 08. 2024 Member of the Scientific Jury:

V. Tarnovo (Prof. Galya Nakova, PhD)