On Krull dimension of noetherian super-rings

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Purely even case

Let R be a commutative ring. Recall that the Krull dimension of R is defined as

$$\sup\{n \mid \text{there is a chain } \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_n$$
$$\text{of pairwise different prime ideals in } R.\}$$

It is denoted by $\mathrm{Kdim}(R)$. Even if R is Noetherian, $\mathrm{Kdim}(R)$ is not necessary finite. Nevertheless, if R is a finitely generated K-algebra, then $\mathrm{Kdim}(R) < \infty$ and it coincides with the dimension of the underlying space of the affine scheme $\mathrm{Spec}(R)$.
Similarly, for any R-module M the Krull dimension of M is defined as $\mathrm{Kdim}(R/\mathrm{Ann}_R(M))$ and denoted by $\mathrm{Kdim}(M)$.

Definitions

Throughout these notes a super-ring R is a $\mathbb{Z}_2$-graded ring $R = R_0 \oplus R_1$ that is supposed to be super-commutative, i.e. each triple of homogeneous elements a, b and c satisfy $a^2 = 0$ and $bc = cb$ provided $a \in R_1, b \in R_0$. One can easily show that a prime (maximal) superideal of R has a form $\mathfrak{p} \oplus R_1$, where $\mathfrak{p}$ ia a prime (maximal) ideal of R_0. Let R be a Noetherian super-ring. This is equivalent to the condition that R is a Noetherian R_0-supermodule. We also assume that $\mathrm{Kdim}(R_0) < \infty$, unless stated otherwise. A collection of odd elements $y_1, \ldots, y_r$ is called a system of odd parameters of R if

$$\mathrm{Kdim}(R_0/\mathrm{Ann}_{R_0}(y_1 \ldots y_r)) = \mathrm{Kdim}(R_0).$$

This condition is equivalent to the existence of a longest prime chain $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_n$ in R such that $\mathrm{Ann}_{R_0}(y_1 \ldots y_r) \subseteq \mathfrak{p}_0$.

Then the Krull super-dimension of R can be defined as the couple of nonnegative integers $n \mid m$, where $n = \mathrm{Kdim}(R_0)$ and m is the length of a longest system of odd parameters. It is denoted by

$$\mathrm{Ksdim}(R) = \mathrm{Ksdim}_0(R) \mid \mathrm{Ksdim}_1(R),$$

where $n = \mathrm{Ksdim}_0(R) = \mathrm{Kdim}(R_0), m = \mathrm{Ksdim}_1(R)$.
Similarly to the purely even case one can define the Krull super-dimension of an R-supermodule M by

$$\mathrm{Ksdim}(M) = \mathrm{Ksdim}(R/\mathrm{Ann}_R(M)).$$

Elementary properties of Krull super-dimension

Proposition1

1. The Krull super-dimension of R coincides with its Krull super-dimension as a left R-supermodule.
2. If M is regarded as an R_0-module, then $\mathrm{Ksdim}_0(M) = \mathrm{Kdim}(M)$. Besides, $\mathrm{Ksdim}_1(M)$ is equal to the length of a longest system of odd parameters of M, i.e. the odd elements $y_1, \ldots, y_r$ which satisfy $\mathrm{Kdim}((y_1 \ldots y_r)M) = \mathrm{Kdim}(M)$.
3. There is also $\mathrm{Ksdim}_1(M) = \max\{t \mid \mathrm{Kdim}(R_1^t M) = \mathrm{Kdim}(M)\}$.

Noether normalization theorem

Theorem1

Let R be a finitely generated K-superalgebra with $\mathrm{Ksdim}_0(R) = n$. Fix a subalgebra B of R_0 that is isomorphic to the polynomial algebra in n variables, and such that R_0 is integral over B. For a set of odd elements $y_1, \ldots, y_r$ the following condition are equivalent to each other :

1. $y_1, \ldots, y_r$ is the system of odd parameters;
2. $\mathrm{Ann}_B(y_1 \ldots y_r) = 0$;
3. The map $z_1 \mapsto y_1, \ldots, z_r \mapsto y_r$ induces an embedding $\Lambda_B(Kz_1 + \ldots + Kz_r) \to R$. In other words, the subalgebra of R, generated by B and the elements $y_1, \ldots, y_r$, is isomorphic to the polynomial superalgebra in n even and r odd variables.

Regular super-rings

Let R be a local Noetherian super-ring with the maximal superideal $\mathfrak{M}$. Recall that an odd element $y \in R$ is called regular if $\mathrm{Ann}_R(Ry) = Ry$. A set of odd elements $y_1, \ldots, y_r$ is called an odd regular sequence if each y_i is regular modulo $Ry_1 + \ldots + Ry_{i-1}, 1 \leq i \leq r-1$. A set of even elements $r_1, \ldots, r_s$ is called an even regular sequence if it is a regular sequence modulo the nilpotent ideal R_1^2. A set of homogeneous elements is called regular if its even elements and odd elements form even and odd regular sequences respectively.

The super-ring R is said to be regular if any minimal system of generators of $\mathfrak{M}$ is a regular sequence. The regular local super-rings were first characterized in [2].

Set $I_R = RR_1, \overline{R} = R/I_R \simeq R_0/R_1^2$.

Theorem (T. Schmitt)

A local super-ring R is regular iff the following conditions hold :

1. $\overline{R}$ is regular;
2. $V = R_1/R_1^3$ is a free $\overline{R}$-module;
3. the natural super-ring morphism $\Lambda_{\overline{R}}(V) \to \mathrm{gr}(R) = \oplus_{i\geq 0} I_R^i/I_R^{i+1}$ is an isomorphism.

Theorem 1(MZ)

A local super-ring R is regular iff $\mathrm{sdim}_K(\mathfrak{M}/\mathfrak{M}^2) = \mathrm{Ksdim}(R)$, where $K = R/\mathfrak{M}$. Moreover, R is regular iff its completion $\widehat{R}$ is. In the latter case, and if R contains a field also, then

$$\widehat{R} \simeq K[[x_1, \ldots, x_m, y_1, \ldots, y_n]] = K[[x_1, \ldots, x_n]][y_1, \ldots, y_m],$$

where $m = \mathrm{Ksdim}_0(R)$ and $n = \mathrm{Ksdim}_1(R)$.

If R is not local, then R is said to be regular whenever $R_{\mathfrak{P}}$ is regular for any prime superideal $\mathfrak{P}$ of R. The above two theorems imply that R is regular iff the following conditions hold :

1. $\overline{R}$ is regular;
2. $V = R_1/R_1^3$ is a projective $\overline{R}$-module;
3. $\Lambda_{\overline{R}}(V) \to \mathrm{gr}(R) = \oplus_{i\geq 0} I_R^i/I_R^{i+1}$ is an isomorphism.

Completion and super-dimension

If I is a superideal of R, then let $\widehat{R}$ denote its completion in the I-adic topology. The following theorem generalizes the well known theorem from the commutative ring theory.

Theorem 3 (MZ)

The Krull super-dimension of $\widehat{R}$ is equal to $\sup_{I\subseteq\mathfrak{P}} \mathrm{Ksdim}(R_{\mathfrak{P}})$, where supremum is taken with respect to the (left) lexicographical order on the couples of nonnegative integers.

In particular, if R is local and $I = \mathfrak{M}$, then $\mathrm{Ksdim}(R) = \mathrm{Ksdim}(\widehat{R})$.

Super-dimension of a certain superscheme

Let X be an irreducible superscheme of finite type.

Lemma-Definition

If U and V are nonempty open affine subsuperschemes of X, then $\mathrm{Ksdim}(\mathcal{O}(U)) = \mathrm{Ksdim}(\mathcal{O}(V))$.

Thus one can define the super-dimension $\mathrm{sdim}(X)$ of X as the Krull super-dimension of the super-ring of global sections of any nonempty open affine subsuperscheme of X.

Recall that X is said to be nonsingular at a point x of the underlying topological space of X if the local super-ring $\mathcal{O}_x$ is regular, and X is just nonsingular if it is nonsingular at any its point.

Proposition 1 (MZ)

The superscheme X is nonsingular iff the sheaf of Kahler superdifferentials Ω_X is a locally free $\mathcal{O}_X$-supermodule of the super-rank equal to $\mathrm{sdim}(X)$.

A superscheme X is said to be generically nonsingular if there is a nonempty open nonsingular supersubscheme Y of X.

Theorem 2 (MZ)

Let X be an integral generically nonsingular superscheme and Y be its integral generically nonsingular closed supersubscheme, then $\mathrm{sdim}(X) = \mathrm{sdim}(Y)$ implies $X = Y$.

Grading and super-dimension

Let I be a superideal of R and M be an R-supermodule. We define graded and bigraded super-rings

$$\mathsf{gr}_I(R) = \oplus_{k\geq 0} I^k/I^{k+1}, I^0 = R,$$

and

$$\mathsf{bgr}_I(R) = \oplus_{k,l\geq 0} I_0^k I_1^l R/(I_0^{k+1} I_1^l R + I_0^k I_1^{l+1} R),$$

as well as graded $\mathsf{gr}_I(R)$-supermodule and bigraded $\mathsf{bgr}_I(R)$-supermodule

$$\mathsf{gr}_I(M) = \oplus_{k\geq 0} I^k M/I^{k+1} M,$$

and

$$\mathsf{bgr}_I(M) = \oplus_{k,l\geq 0} I_0^k I_1^l M/(I_0^{k+1} I_1^l M + I_0^k I_1^{l+1} M)$$

respectively.

Theorem 4

Assume that I is contained in the radical of R. Then

$$\mathrm{Ksdim}_0(M) = \mathrm{Ksdim}_0(\mathsf{gr}_I(M)),\ \mathrm{Ksdim}_1(M) \geq \mathrm{Ksdim}_1(\mathsf{gr}_I(M))$$

and $\mathrm{Ksdim}(M) = \mathrm{Ksdim}(\mathsf{gr}_I(M))$ provided $I = I_R = RR_1$.

The proof of this theorem also infers that if the cosets of odd elements $y_1, \ldots, y_m$ modulo R_1^3 form a longest system of odd parameters in $\mathsf{gr}_{I_R}(M)$, then $y_1, \ldots, y_m$ form a longest system of odd parameters of M.

Let R be a local Noetherian super-ring with the maximal superideal $\mathfrak{M} = \mathfrak{m} \oplus R_1$. Let $\mathfrak{N}$ be a (not necessary prime) $\mathfrak{M}$-*primary* superideal, that is $\mathfrak{N}_0 = \mathfrak{n}$ is a $\mathfrak{m}$-primary ideal. Assume also that R contans a field and $\mathfrak{n}$ is generated by m elements, which form a system of even parameters modulo R_1^2. In particular, $\mathrm{Kdim}(R_0) = m$.

Theorem 5

We have $\mathrm{Ksdim}_0(R) = \mathrm{Ksdim}_0(\mathsf{bgr}_{\mathfrak{N}}(R))$ and $\mathrm{Ksdim}_1(R) \geq \mathrm{Ksdim}_1(\mathsf{bgr}_{\mathfrak{N}}(R))$. Moreover, if $\mathfrak{N}_1 = R_1$, then $\mathrm{Ksdim}(R) = \mathrm{Ksdim}(\mathsf{bgr}_{\mathfrak{N}}(R))$.

We can associate with the bigraded super-ring $B = \mathsf{bgr}_{\mathfrak{N}}(R)$ a polynomial in two variables that seems to be a superversion of Hilbert polynomial. Let $B(k, l)$ denote a bihomogeneous component of B of bidegree (k, l). Set

$$g_l(s) = \sum_{0 \leq k \leq s} \dim B(k, l), s \geq 0.$$

Then for all sufficiently large s the function $g_l(s)$ is a polynomial of degree at most m.

Since $B(k,l) = 0$ for all sufficiently large l, we have a polynomial $g(x,y) = \sum_{l \geq 0} g_l(x) y^l$ that is called a Hilbert polynomial of R (associated with a $\mathfrak{M}$-primary superideal $\mathfrak{N}$). Then

$$\mathrm{Ksdim}_1(B) = \max\{l \mid \text{ the degree of } g_l(x) \text{ is equal to } m\}.$$

Combining with Theorem 5 we obtain that

$$\mathrm{Ksdim}_1(R) = \max\{l \mid \text{ the degree of } g_l(x) \text{ is equal to } m\},$$

provided $\mathfrak{N}_1 = R_1$.

Odd regular elements and some cohomology theory

Let A be a (not necessary supercommutative) super-ring and M be an R-superbimodule. We define a cochain complex $\tilde{C}^n(A, M) = \mathrm{Hom}_{\mathbb{Z}}(A^{\otimes(n+1)}, M), n \geq 0$, with coboundary maps

$$\delta_n(f)(a_0, \ldots, a_{n+1}) = \sum_{0 \leq i \leq n} (-1)^i f(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1})$$

$$-(-1)^{|f||a_0|} a_0 f(a_1, \ldots, a_{n+1}) + (-1)^{n+1} f(a_0, \ldots, a_n) a_{n+1},$$

$$f \in \tilde{C}^n(A, M), a_0, \ldots, a_{n+1} \in A.$$

Observe that each δ_n is a parity preserving ($\mathbb{Z}_2$-graded) map.

Assume that A is supercommutative and M is a left A-supermodule regarded as a right supermodule via

$$m \cdot a = (-1)^{|a||m|} am, a \in A, m \in M.$$

The above complex contains a subcomplex $C^n(A, M)$ consisting of all $f \in \tilde{C}^n(A, M)$ such that for any $a_0, \ldots, a_n \in A$ the following conditions hold :

1. $f(1, a_1, \ldots, a_n) = 0$;
2. $f(a_n, a_{n-1}, \ldots, a_0) = (-1)^{\frac{n(n-1)}{2} + \sum_{0 \leq i < j \leq n} |a_i||a_j|} f(a_0, \ldots, a_n)$.

Let R be a K-superalgebra and let y be an odd regular element of R. Let A denote R/Ry. We have an exact sequence

$$0 \to Ry \to R \to A \to 0,$$

where Ry is isomorphic to a free A-supermodule of super-rank $0 \mid 1$.

Theorem 6

The couple (R, y) is uniquely defined by an odd cocyle $\pi \in C^1(A, A)$. Moreover, let (R', y') be another couple defined by an odd cocyle $\pi' \in C^1(A, A)$. Then there is a superalgebra isomorphism $R \to R'$ that sends y to y' iff the cosets of π and π' in the first cohomology group $H^1(A, A)$ are the same.

Nonstandard properties of Krull super-dimension

Fact1

One can construct a finitely generated K-superalgebras A and a superideal I of A such that $\mathrm{Ksdim}_1(A) < \mathrm{Ksdim}_1(A/I)$!

Fact2

Any odd regular sequence is a system of odd parameters, but there are super-rings in which any odd regular sequence can not be included in any longest system of odd parameters.

References

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2. T. Schmitt, *Regular sequences in $\mathbb{Z}_2$-graded commutative algebra*, J.Algebra, 124 (1989), 60–118.