КОМУТАТИВНИ ГРУПОВИ ПРЪСТЕНИ И АБЕЛЕВИ ГРУПИ
COMMUTATIVE GROUP RINGS AND ABELIAN GROUPS

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Представяне на встъпителна лекция за "**доцент**"

1. Комутативни Групови Пръстени - Commutative Group Rings

In 1988, in an important paper published in the Proc. Amer. Math. Soc., Warren Lee May asked of whether or not the commutaive group algebra RG over an unital ring R of prime characteristic p determines the isomorphism class of the torsion p-component of the group G whenever G_p is totally projective. We resolve this problem in the positive for direct sums of countable groups, which sums are of countable length. Specifically, the following holds:

Danchev - PAMS (1997)

Let R be a commutative unital ring of prime characteristic p and G an abelian group whose p-component G_p of torsion is totally projective of countable length. If H is an arbitrary group such that $RH \cong RG$ as R-algebras, then $H_p \cong G_p$.

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Let G be an abelian group with $\mathrm{supp}(G) = \{p \mid G_p \neq 1\}$, and let R be a commutative unitary ring with $\mathrm{inv}(R) = \{p \mid p \cdot 1 \in U(R)\}$. A classical question by Gregory Karpilovsky is when each normalized unit in the group ring RG is generated by the idempotents in R of augmentation 1 with basis from G. We settle this problem, provided that G contains at leat one p-component G_p which prime p inverts in R. Concretely, the following is true (see also [8]):

Danchev - CA (2010)

Suppose G ***is a non-trivial abelian group and*** R ***is a commutative ring with*** 1 ***such that*** $\mathrm{supp}(G) \cap \mathrm{inv}(R) \neq \emptyset$***. Then*** $V(RG) = Id(RG)$ ***if, and only if, either*** G ***has two elements and, for every*** $r \in R$***,*** $2r - 1 \in U(R)$ ***exactly when*** r ***is an idempotent; or*** G ***has three elements and, for every pair*** $(r, f) \in R \times R$***,*** $1 + 3r^2 + 3f^2 + 3rf - 3r - 3f \in U(R)$ ***exactly when*** r ***and*** f ***are orthogonal idempotents.***

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Let F be a perfect field of positive characteristic p and let G be a p-mixed abelian group. As usual, $V(FG)$ denotes the normalized unit group of the group ring FG with maximal divisible subgroup (i.e., a divisible part) $dV(FG)$ and with p-primary component $V_p(FG)$. The following formula gives an explicit form of the divisible part like this:

Danchev - RM (2004)

Suppose G is an abelian group whose torsion part is a p-group and F is a perfct field of non-zero characteristic p. Then the following equality holds:
$dV(FG) = (dG)(V_p(FG^d))$, where G^d is the maximal p-divisible subgroup of G.

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Very recently, in 2018, Simion Breaz along with Andrada Cimpean defined and studied in the Bull. Korean Math. Soc. the class of so-termed *weakly tripotent rings* as follows: A ring R is weakly tripoten if, for all $x \in R$, either $x^3 = x$ or $(1-x)^3 = 1-x$. These rings are generally non-commutative. However, they are properly contained in the class of so-called *strongly invo-clean rings*, defined by Danchev in the Commun. Korean Math. Soc. (2017), as for that purpose a special group ring R is exhibited which is strongly invo-clean but definitely *not* weakly tripotent. In fact, $R = BG$, where $B \ncong \mathbb{Z}_2$ is a Boolean ring and G is a bounded abelian group of exponent 2.

This construction leads us to the following criterion:

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Danchev - BASM (2020)

Suppose R is a commutative ring and G is a non-identity abelian group. Then the group ring RG is weakly tripotent if, and only if, $G^2 = 1$ and R is weakly tripotent having the decomposition $R \cong R_1 \times R_2$, where

- *$R_1 = 0$ or R_1 is a subdirect product of $L \times \mathbb{Z}_2$ for either $L = 0$ or $L/J(L) \cong \mathbb{Z}_2$ with $y^2 = 2y$ for any $y \in J(L)$*

and

- *$R_2 = 0$ or R_2 is a subdirect product of copies of $\mathbb{Z}_3$ or is a single copy of $\mathbb{Z}_3$,*

and precisely one of the next two points is valid:

(1) G has two elements and $2d = 2d^2$ for all $d \in L$ (thus $4 = 0$ in L)

or

(2) G has strictly more than two elements and $2 = 0$ in L.

A classical result of Roger Nunke, published in Math. Z. (1967), showed that if, for any ordinal α, both $p^\alpha G$ and $G/p^\alpha G$ are totally projective abelian p-groups, then so is the whole abelian p-group G, and vice versa. We shall prove now a similar result concerning the so-called *summable abelian p-groups*, which intersect the totally projective abelian p-groups in the class of direct sums of countable abelian p-groups.

Danchev - AMBP (2008)

Suppose G is an abelian p-group of length not exceeding the first uncountable ordinal Ω. If $p^\alpha G$ and $G/p^\alpha G$ are both summable groups for some ordinal α, then G is summable too.

2. Абелеви Групи - Abelian Groups

A reduced abelian p-group G is called *quasi-complete* provided that, for all pure subgroups P of G, the quotient group $p^{\omega}(G/P)$ is always divisible. These groups are always separable, that is, $p^{\omega}G = 0$.
Likewise, a separable abelian p-group G is said to be *a Q-group* if, for all infinite subgroups A of G, the cardinality of $p^{\omega}(G/A)$ does not exceed the cardinality of A. Also, a separable abelian p-group G is said to be *weakly* $\aleph_1$*-separable* if, for all countable (finite or infinite) subgroups A of G, the cardinality of $p^{\omega}(G/A)$ is less than or equal to the cardinality of A. The following statement is a theorem in ZFC.

Danchev - VMJ (2008,2009)

Let G be a quasi-complete abelian p-group. Then the following two assertions hold:
(i) If G is a Q-group, then G is bounded.
(ii) If G is a weakly $\aleph_1$-separable group, then G is bounded.

In Comment. Math. Univ. St. Pauli (1986), the American mathematicians Irwin-Snabb-Cutler proved that if H is a pure and dense subgroup of the separable abelian p-group G such that G/H is at most countable, then G is $p^{\omega+1}$-projective if, and only if, H is $p^{\omega+1}$-projective. In what follows, we considerably extend this result by removing the condition on H to be dense in G and expanding the statement to arbitrary $p^{\omega+n}$-projectives for any natural number n.

Danchev - AMB (2006)

Given a separable abelian p-group G with a pure subgroup H such that G/H is either a finite or a countably infinite factor-group. Then G is $p^{\omega+n}$-projective if, and only if, H is $p^{\omega+n}$-projective for all $n \geq 1$.

Notice that the present theorem was noted in the recent monograph of Laszlo Fuchs *"Abelian Groups"* (2015) as a **"Lemma of Danchev"** and is also used intensively in the proofs of some other related results.

Mimicking Pierce (Homomorphisms of primary abelian groups, in Topics in Abelian Groups (Chicago, 1963), Scott, Foresman & Co. pp. 215–310), by a *large* subgroup of a given abelian p-group G we will mean a fully invaraint subgroup satisfying the equality $L + B = G$ for every basic subgroup of G. The following criterion is a connection between a group and their large subgroups having a specific property.

Danchev - PIAS (2004)

Let $n \in \mathbb{N} \cup \{0\}$. A reduced abelian p-group is $p^{\omega+n}$-projective if, and only if, it has a large subgroup that is $p^{\omega+n}$-projective.

Notice that the present theorem was also included in the recent monograph of Laszlo Fuchs *"Abelian Groups"* (2015).

A subgroup S of an abelian p-group G is known to be *projection-invariant* in G, provided $\pi(S) \leq S$ for all projections (i.e., idempotent endomorphisms) of G. We shall say that the group G is *projectively socle-regular* if, for each projection-invariant subgroup S of G, there exists an ordinal α depending on S such that $S[p] = (p^\alpha G)[p]$.

Danchev-Goldsmith (CONM, 2012)

Suppose G is an abelian p-group. Then the next two assertions are true:
(a) If G is projectively socle-regular, then so is the internal direct sum (= square) $G \oplus G$.
(b) A direct summand of a projectively socle-regular group need not be again projectively socle-regular, namely if the square $G \oplus G$ is projectively socle-regular, then G could not be such a group.

Авторски статии изградили представянето - Author's papers

I. Комутативни Групови Пръстени - Commutative Group Rings

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Перспективи за развитие - Further Perspectivities

В бъдеще авторът ще акцентира върху детайлно изследване в областта на представянето на настоящата лекция, засягащо следните две насоки:

• Да се опише структурата на нормираната мултипликативна група $V(RG)$ на груповия пръстен RG, когато R е съвършен пръстен (или съвършено поле) с проста характеристика p, а G е p-смесена Уорфилдова абелева група.
• To describe the structure of the normed unit group $V(RG)$ of the group ring RG, where R is a perfect ring (or a perfect field) of prime characteristic p, and G is a p-mixed Warfield group.

• Да се характеризират напълно много широки класове от абелеви групи, като например тези дефинирани и изследвани в следните три научни статии:
• To be completely characterized very large classes of Abelian groups as those defined and studied in the following three scientific articles:

Перспективи за развитие - Further Perspectivities

(1) A.R. Chekhlov, P.V. Danchev and B. Goldsmith, *On the socles of fully inert subgroups of Abelian p-groups,* Mediterranean J. Math. (3) **18** (2021).

(2) A.R. Chekhlov, P.V. Danchev and B. Goldsmith, *On the socles of characteristically inert subgroups of Abelian p-groups,* Forum Math. (4) **33** (2021).

(3) A.R. Chekhlov, P.V. Danchev and B. Goldsmith, *Weakly fully and characteristically inert socle-regular Abelian p-groups,* submitted to a scientific journal.

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