Schur's exponent conjecture

Michael Vaughan-Lee

Oxford

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The conjecture

Let G be a finite group and write $G = F/R$ where F is a free group. Then the Schur multiplier $M(G) = (R \cap F')/[R, F]$.

Conjecture

Let G be a finite group with Schur multiplier $M(G)$. Then the exponent of $M(G)$ divides the exponent of G.

It is easy to show that the conjecture holds true for groups of exponent 2 or 3, but it has been known since 1973 that the conjecture fails for exponent 4. Bayes, Kautsky and Wamsley found an example of a class 4 group G of order 2^{68} with exponent 4, where the multiplier $M(G)$ has exponent 8.

The conjecture has remained open for groups of odd exponent up till now.

I have found an example of a finite group with exponent 5 with Schur multiplier of exponent 25, and an example of a finite group with exponent 9 with Schur multiplier of exponent 27.

Wamsley's contribution

Wamsley was the first person to get the p-quotient algorithm for computing finite p-groups to work. The paper mentioned above was the first significant calculation using the p-quotient algorithm, and in that paper they also computed $B(3,4)$.

A PCP for a group of order p^n has generators $a_1, a_2, \ldots, a_n$ and relations

$$a_i a_j = a_j a_i a_{i+1}^* \ldots a_n^* \ (i > j)$$

$$a_i^p = a_{i+1}^* \ldots a_n^*$$

Using the relations you can write any product of the generators a_i as a collected word $a_1^* a_2^* \ldots a_n^*$ in normal form

The collection process

If we have a product of the generators a_i which is not in normal form then it must have a subword of the form $a_i a_j$ $(i > j)$ or a subword a_i^p, and in the collection process we replace one of these subwords by the right hand side of the corresponding relation.
There are various strategies for deciding which non-normal subword of this form to collect.

Collection from the right: collect the rightmost non-normal subword.

Collection from the left: collect the leftmost non-normal subword.

Hall collection: look for a non-normal subword involving a_1. If there are none of these look for one involving a_2, then a_3, and so on.

Hall collection is really beautiful theoretically, especially in free groups, but turns out to be hopeless in p-groups.

An example

Collect $(ab)^5$ in $R(2,5;9)$

	iterations	max length
left	118,005	826
right	2,660,395	124
Hall	112,420,884,852	712,090

Collect $(ab)^5$ in $R(2,5)$

	iterations	max length
left	371,958	1802
right	40,687,832	144
Hall	$>>$112,000,000,000	$>$7,000,000

The groups $R(d, 5)$

Viji Thomas wrote to me last year saying he was investigating the Schur multipliers of the groups $R(d, 5)$ $(d = 2, 3, 4)$. $(R(d, 5)$ is the largest d-generator group of exponent 5.)

He asked me if I knew what the nilpotency class of $R(4, 5)$ is. It is known that it has class at most 24, but he said that if in fact the class is less than 24 then he could prove that its Schur multiplier has exponent 5.
It seems likely that the class is less than 24, but as far as I know the actual class has not been determined.

This was the first I had heard about Schur's exponent conjecture, and out of interest I thought I would see if I could compute the Schur multiplier of $R(2, 5)$.

The Schur multiplier of $R(2,5)$

$R(2,5)$ has order 5^{34} and class 12. As a class 12 group $R(2,5)$ has a presentation on generators a, b with 31 5^{th} power relations

$$U = \{a^5, b^5, (ab)^5, \ldots, (a^2ba^2bab^2)^5\}.$$

So if we let F be the free group of rank 2 generated by a, b and we let N be the normal closure of U in F, then $R(2,5) = F/R$ where $R = N\gamma_{13}(F)$.

So $F/[F,R]$ is the class 13 quotient of $\langle a, b \mid \{[u,a],[u,b] : u \in U\}\rangle$.

The Schur multiplier of $R(2,5)$

You can use the nilpotent quotient algorithm to compute $F/[F,R]$. The Schur multiplier of $R(2,5)$ is elementary abelian of order 5^{31}.

$[b,a]^5 \in \gamma_{10}(F/[F,R])$. This implies that if G is any group of exponent 5 with class less than 9 then $M(G)$ has exponent 5.
In fact $[b,a]^5$ is a product of commutators in $F/[F,R]$ of multiweight (r,s) in the generators a,b where $r,s \geq 5$.

Similarly, in any Schur cover of $R(2,7)$, $[b,a]^7$ is a product of commutators of multiweight (r,s) where $r,s \geq 7$, and this implies that $M(R(2,7))$ is elementary abelian.

Look for a class 9 counterexample of exponent 5

Let G be a class 9 group of exponent 5, write $G = F/R$, and let $H = F/[F,R]$. Then H' has exponent 25 and $\gamma_3(H)$ has exponent 5.

So if $M(G)$ has exponent 25 then G must satisfy a relation $r = 1$ where $r \in F'$, but $r \notin (F')^5\gamma_3(F)$.

You need at least 4 generators for this work. After trying various possibilities I came up with

$$G = \langle a, b, c, d \mid [b,a] = [d,c], \text{ exponent 5, class 9} \rangle.$$

G has order 5^{4122} and

$$M(G) = C_{25} \times C_5^{9170}.$$

Embarrassment?

I emailed George Havas about this example, and he replied:

You won't know this, since you never read anyone else's papers. But your example is exactly the same as the Bayes, Kautsky, Wamsley example in exponent 4.

Actually, they tried your exponent 5 example as well, but only up to class 5.

An exponent 9 example

I also found an exponent 9 example:

$$A = \langle a, b, c, d \mid a^3,\ b^3,\ c^3\ d^3,\ [b, a] = [d, c],\ \text{exponent 9, class 9}\rangle.$$

A has order 3^{11983} and

$$M(A) = C_{27} \times C_3^{25184}.$$

Tricky computations

The p-covering group of the class 8 quotient of our exponent 5 group G has order 5^{7044}, so we need 2921 fifth powers, together with the relation $[b, a] = [d, c]$ to define G as a class 9 group.

Let

$$S = \{a^5, b^5, c^5, d^5, (ab)^5, \ldots, (ab^4cbcd)^5\}$$

be a suitable set of 2921 fifth powers, and let $T = S \cup \{[b, a][c, d]\}$ so that G is the class 9 quotient of $\langle a, b, c, d \mid T \rangle$.

Then the group we want is the class 10 quotient of

$$\langle a, b, c, d \mid \{[x, y] \,:\, x \in T,\ y \in \{a, b, c, d\}\}\rangle.$$

Tricky computations

This is an infinite group, but the subgroup $\langle a^5 \rangle$ is central and intersects the derived group trivially. So we can add in the relation $a^5 = 1$ without impacting the Schur multiplier. Similarly we can add in the relations $b^5 = c^5 = d^5 = 1$.

This means that we now have a finite group, and we can use the p-quotient algorithm to compute it. The computation takes about 12 hours of CPU time. I used the nilpotent quotient algorithm to compute the Schur multiplier of $R(2,5)$, but the nilpotent quotient algorithm would have taken months of CPU time for this calculation.

I tried the same approach for calculating the Schur multiplier of my group A of exponent 9, but MAGMA crashed! However I was able to do the calculation using my own fortran programs. Then Eamonn O'Brien showed me how to use *p*QuotientProcess in MAGMA to do the calculation without crashing.

A two generator example

Let

$$L = \langle a, b \mid [a, b]^2 = 1, \text{ exponent } 8, \text{ class } 12 \rangle.$$

Then

$$M(L) = C_2^{251} \times C_4^4 \times C_{16}.$$

Some more examples

Let

$$H(q, c) = \langle a, b, c, d \mid [b, a] = [d, c], \text{ exponent } q, \text{ class } c \rangle.$$

Then the Schur multipliers of $H(8, 5)$, $H(16, 6)$, $H(32, 7)$, $H(64, 8)$ have exponents 16, 32, 64, 128 (respectively).