Primary decompositions of unital locally matrix algebras and Steinitz numbers

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We assign a Steinitz number st(A) to an arbitrary unital locally matrix algebra A.

Steinitz numbers

Let $\mathbb{P}$ be the set of all primes. A *Steinitz number* (E.Steinitz) is an infinite formal product of the form

$$\prod_{p\in\mathbb{P}} p^{k_p}$$

where $k_p \in \mathbb{N} \cup \{0, \infty\}$. Define the product of two Steinitz numbers as follows:

$$\prod_{p\in\mathbb{P}} p^{r_p} \cdot \prod_{p\in\mathbb{P}} p^{k_p} = \prod_{p\in\mathbb{P}} p^{r_p+k_p}, \qquad r_p, k_p \in \mathbb{N} \cup \{0, \infty\},$$

where

$$r_p + k_p := \begin{cases} r_p + k_p & \text{if } r_p < \infty \text{ and } k_p < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Steinitz numbers

Denote by $\mathbb{SN}$ the set of all Steinitz numbers. Note, that the set of all positive integers $\mathbb{N}$ is subset of $\mathbb{SN}$. The numbers $\mathbb{SN} \setminus \mathbb{N}$ are called *infinite* Steinitz numbers.

An infinite Steinitz number $\prod_{p\in\mathbb{P}} p^{r_p}$ is called *locally finite* if $r_p \neq \infty$ for any $p \in \mathbb{P}$.

Algebras of countable dimension

J. G. Glimm [1] parametrised countable dimensional locally matrix algebras (under the name uniformly hyperfinite algebras) with Steinitz number.
O. Bezushchak, B. Oliynyk and V. Sushchanskii [5] extended their parametrisation to regular relation structures. The idea of diagonal embeddings was introduced by A. E. Zalesskii in [4].
In a series of papers A. A. Baranov and A. G. Zhilinskii used Steinitz numbers to classify diagonal locally simple Lie algebras of countable dimension [1], [2].

Algebras of countable dimension

Theorem

[See [1] [2], [5]] If A and B are unital locally matrix algebras of countable dimension then A and B are isomorphic if and only if $\mathrm{st}(A) = \mathrm{st}(B)$.

Steinitz number st(A) of the algebra A

Definition [O.Bezushchak, B.O.]

Let A be an infinite dimensional unital locally matrix algebra over a field F and let $D(A)$ be the set of all positive integers n, such that there is a subalgebra A', $A' \subseteq A$, $A' \cong M_n(F)$ and $1_A \in A'$, where 1_A is the identity of A. Then the least common multiple of the set $D(A)$ is called the *Steinitz number* $\mathrm{st}(A)$ of the algebra A.

Let A and B be unital locally matrix algebras. Then the algebra $A \otimes_F B$ is a unital locally matrix and

$$\mathrm{st}(A) \cdot \mathrm{st}(B) = \mathrm{st}(A \otimes_F B).$$

Let A be an algebraic system (see [1]). The universal elementary theory $UTh(A)$ consists of universal closed formulas (see [1]) that are valid on A. The systems A and B of the same signature are universally equivalent if $UTh(A) = UTh(B)$.

Theorem 1 [O. Bezushchak, B.O., 2020]

Let A and B be unital locally matrix algebras. Then A and B are universally equivalent if and only if their Steinitz numbers st(A) and st(B) are equal, i.e.

$$\text{st}(A) = \text{st}(B).$$

Clifford algebra

Let V be a vector space over a field F, $charF \neq 2$ and $f: \ V \to F$ be nondegenerate quadratic form. The Clifford algebra $Cl(V,f)$ is a unital algebra generated by the vector space V and 1 and defined by relations

$$v^2 = f(v) \cdot 1, \text{for all } v \in V.$$

Let $\{v_i\}_{i \in I}$ be a basis of the vector space V. Assume, that the set of indexes I is ordered. Then all possible ordered products $v_{i_1} v_{i_2} \dots v_{i_k}$, $i_1 < i_2 < \dots < i_k$, and 1 (that can be defined as the empty product) is a basis of the Clifford algebra $Cl(V,f)$.

Clifford algebra

Theorem 2 [O. Bezushchak, B.O., 2020]

Let V be an infinite dimensional vector space. Then the Clifford algebra $Cl(V, f)$ is locally matrix and $\mathrm{st}(Cl(V, f)) = 2^{\infty}$.

A generalization of Clifford algebras

Let $l > 1$ be an integer. If char $F > 0$ then we assume that l is coprime with char F. Let $\xi \in F$ be an l-th primitive root of 1. Let I be an ordered set. The generalized Clifford algebra $Clg(l, I)$ is presented by generators x_i, $i \in I$, and relations:

$$x_i^l = 1\,, \quad x_i^{-1} x_j x_i = \xi x_j \ \text{ for } i < j\,,$$

$$x_i^{-1} x_j x_i = \xi^{-1} x_j \ \text{ for } i > j\,, \ i, j \in I\,.$$

$Clg(l, I)$ is a unital locally matrix algebra. Ordered monomials

$$x_{i_1}^{k_1} \cdots x_{i_r}^{k_r}, \ i_1 < \ldots < i_r, \ 1 \leq k_j \leq l-1, \ 1 \leq j \leq r,$$

form a basis of $Clg(l, I)$.

A generalization of Clifford algebras

- The algebra $Clg(I, m)$ does not contain proper ideals that are invariant under Aut $Clg(I, m)$.
- The algebra $Clg(I, m)$ is semisimple.

A generalization of Clifford algebras

Theorem 3 [O. Bezushchak, B.O., 2020]

The Steinitz number of a unital locally matrix algebra $Clg(I, I)$, where the set I is infinite, is I^{∞}.

Theorem 4 [O. Bezushchak, B.O., 2020]

If $\tau = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots$ is a Steinitz number, such that $k_j = \infty$ for some positive integer j, then for any infinite dimension α there is a unital locally matrix algebra A, such that $\dim_F A = \alpha$ and $\mathrm{st}(A) = \tau$.

Non isomorphic algebras with equal Steinitz numbers

Consider the vector space:

$$V = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{C}, \sum_{i=1}^{\infty} |a_i|^2 < \infty\}.$$

Let f be the quadratic form:

$$f((a_1, a_2, \ldots)) = \sum_{i=1}^{\infty} a_i^2 \in \mathbb{C}.$$

The vector space V has uncountable dimension.

Non isomorphic algebras with equal Steinitz numbers

Assume that I is a set of indexes, whose cardinality $\mathfrak{Card} I$ is equal to the dimension of the vector space V. Let now W be a complex vector space with basis $w_i, i \in I$. Assume that g is the quadratic form on W determined for arbitrary $w = \alpha_1 w_{i_1} + \alpha_2 w_{i_2} + \ldots + \alpha_n w_{i_n}$, $\alpha_i \in \mathbb{C}$ by the rule:

$$g(w) = \sum_{i \in 1}^{n} \alpha_i^2.$$

Non isomorphic algebras with equal Steinitz numbers

Define Clifford algebras $A = Cl(V, f)$ and $B = Cl(W, g)$. As follows from their constructions, algebras A and B are unital locally matrix and

$$\dim_F A = \dim_F B = \mathfrak{Card}\, I.$$

In addition

$$\mathrm{st}(A) = \mathrm{st}(B) = 2^{\infty}.$$

Theorem 5 [O. Bezushchak, B.O., 2020]

Clifford algebras A and B are not isomorphic.

Theorem 6 [O. Bezushchak, B.O., 2020]

For an arbitrary infinite locally finite Steinitz number s there exists a unital locally matrix algebra A of uncountable dimension with Steinitz number s.

The crucial role in the proof will be played by the theorem of A. G. Kurosh ([5], Theorem 10) which is reformulated as follows:

let A be a countable dimensional locally matrix algebra with a unit 1_A. Then A contains a proper subalgebra $1_A \in B \subset A$ such that $A \cong B$.

A primary decomposition

A unital locally matrix algebra A over a field F is called primary if $\operatorname{st}(A) = p^s$, where p is a prime number and $s \in \mathbb{N}$ or $s = \infty$.

We say that the decomposition

$$A = \bigotimes_{p \in \mathbb{P}} A_p$$

of a unital locally matrix algebra A over F is a primary decomposition if each algebra A_p is primary for all $p \in \mathbb{P}$.

Locally standard Hamming space

In [3] G. Koethe proved that every countable dimensional unital locally matrix algebra admits a decomposition into an (infinite) tensor product of finite dimensional matrix algebras and admits a primary decomposition.

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A. G. Kurosh [5] and V. M. Kurochkin [4] further studied existence and uniqueness of such decompositions of unital locally matrix algebras of arbitrary dimensions. In particular V. M. Kurochkin [4] formulated the question:

does every locally matrix algebra have a primary decomposition?

It gives negative answers to this question

Theorem 7 [O. Bezushchak, B.O., 2020]

There exists a unital locally matrix algebra of uncountable dimension that has no primary decomposition.

Proof

From the Theorem 6 follows the existence of a unital locally matrix algebra A such that $\mathrm{st}(A) = s$ and $\dim_F A > \aleph_0$. If the algebra A admits a primary decomposition then

$$A \cong \bigotimes_{p \in \mathbb{P}} A_p, \quad \mathrm{st}(A_p) = p^{r_p} \text{ and } r_p < \infty \text{ for all } p \in \mathbb{P}.$$

Hence $A_p \cong M_{p^{r_p}}(F)$ and therefore $\dim_F A \leq \aleph_0$.

A.G. Kurosh in [5] constructed an example of a unital locally matrix algebra of uncountable dimension that does not admit a decomposition into an infinite tensor product of finite dimensional matrix algebras. This example has Steinitz number 2^{∞}.

Theorem 8 [O. Bezushchak, B.O., 2020]

If I is an odd number then $Clg(I, \mathbb{R})$ is not isomorphic to a tensor product of finite dimensional matrix algebras.

Theorem 9 [O. Bezushchak, B.O., 2020]

For an arbitrary infinite Steinitz number s there exists a unital locally matrix algebra A such that $\mathrm{st}(A) = s$ and A does not admit a decomposition into a tensor product of finite dimensional matrix algebras.

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Thank you for your attention!