The mysterious Kronecker coefficients

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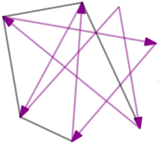
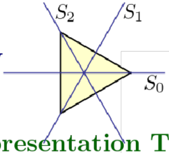
University of Southern California

IMI BAS Algebra and Logic Seminar

Complexity Theory

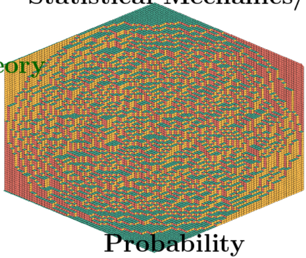
P vs NP





Representation Theory

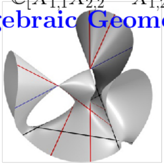
Statistical Mechanics/

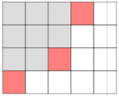


Probability

$\mathbb{C}[X_{1,1}X_{2,2} - X_{1,2}X_{2,1}]$

Algebraic Geometry



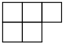


Algebraic Combinatorics

$$s_{(2,2)}(x_1,x_2,x_3) = \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}}{x_1^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}}{x_2^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}}{x_1^2x_2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}}{x_1x_2^2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}}{x_1x_2x_3^2}$$

Intro

Permutations $\pi = 42351$
$\pi : [1 \ldots n] \to [1 \ldots n]$

Integer Partitions
$\lambda = (3, 2)$ 

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Irreducible S_n modules
$\mathbb{S}_\lambda$, $\lambda \vdash n$

Standard **Y**oung **T**ableaux

1	3	4
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General Linear Group GL_N

Irreducible GL_N modules V_λ

Semi-**S**tandard **Y**oung **T**ableaux

1	1	3
2	3	

Multiplicities

GL_N–irreducible [polynomial] representations: the Weyl modules V_α for partitions $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_N)$ and $\alpha_i \in \mathbb{N}$.

Decomposition of tensor products into irreducibles:

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

Multiplicities $c^\nu_{\lambda\mu}$ – **Littlewood-Richardson coefficients**.

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The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

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(LR tableaux of shape $(6,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

Corollary: $\textsc{ComputeLR}$ is in $\#P$.

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$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

E.g.: $\mathbb{S}_{(2,1)} \otimes \mathbb{S}_{(2,1)} = \mathbb{S}_{(3)} \oplus \mathbb{S}_{(2,1)} \oplus \mathbb{S}_{(1,1,1)}$ and so $g((2,1),(2,1),\nu) = 1$ for $\nu = (3), (2,1), (1,1,1)$.

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Via the **irreducible characters** χ^λ of $\mathbb{S}_\lambda$:

$$g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w)\chi^\mu(w)\chi^\nu(w)$$

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Problem: the Kronecker coefficients

Problem (Murnaghan, 1938, then Stanley et al)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$. Alternatively, show that KRON is in $\#P$.

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Classical motivation:
Combinatorial understanding, analogous to
Littlewood–Richardson: for $c^{\nu}_{\lambda,\mu}$, $\mathcal{O}_{\lambda,\mu,\nu} = \{$ LR tableaux of shape ν/μ, type $\lambda\}$
LR – special case of Kroneckers:
[Murnaghan]: If $|\lambda| + |\mu| = |\nu|$ and $n > |\nu|$, then
$g((n + |\mu|, \lambda), (n + |\lambda|, \mu), (n, \nu)) = c^{\nu}_{\lambda\mu}$.

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Recent motivation: Computational Complexity.

1. A positive combinatorial formula
" $\Longleftrightarrow$ " Computing Kronecker coefficients is in #P.

2. Role in Geometric Complexity Theory [Strassen, Mulmuley, Sohoni, Landsberg, etc]

3. Lower bounds via Invariant Theory, moment polytopes [Bürgisser, Wigderson etc]

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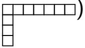
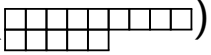
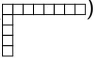
Combinatorial Results since then:
Combinatorial formulas for $g(\lambda,\mu,\nu)$, when:

- μ and ν are hooks (), [*Remmel, 1989*]
- $\nu = (n-k,k)$ () and $\lambda_1 \geq 2k-1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n-k,k)$, $\lambda = (n-r,r)$ [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2013]
- $\nu = (n-k,1^k)$ (), [*Blasiak, 2012*]
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova].

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Complexity results:

[Bürgisser-Ikenmeyer]: KRON is in GapP.
(Littlewood-Richardson, i.e. KRON's special case, is $\#$P-complete)

[Pak-P]: If ν is a hook, then KronPositivity is in P. If λ, μ, ν have fixed length there exists a linear time algorithm for deciding $g(\lambda, \mu, \nu) > 0$.

[Ikenmeyer-Mulmuley-Walter]: KronPositivity is NP-hard.

[Bürgisser-Christandl-Mulmuley-Walter]: membership in the moment polytope is NP and coNP.

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Bounds and positivity:

[Pak-Panova, JCTA'16]: $g(\lambda, \mu, \mu) \geq |\chi^\lambda(2\mu_1 - 1, 2\mu_2 - 3, \ldots)|$ when $\mu = \mu^T$.
Corollaries : $g(\lambda, \mu, \mu) > c\frac{2^{\sqrt{2k}}}{k^{9/4}}$ for $\lambda = (|\mu| - k, k)$, and $diag(\mu) \geq \sqrt{k}$.

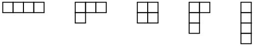
[Jan Saxl's conjecture]: For every $n > 9$ there exists a self-conjugate partition $\lambda \vdash n$, s.t. $g(\lambda, \lambda, \mu) > 0$ for all $\mu \vdash n$. When $n = \binom{m+1}{2}$, then $\lambda = (m, m-1, \ldots, 1)$.
[Partial results: Pak-P-Vallejo, Ikenmeyer, Luo–Sellke]

[Ikenmeyer-Panova, Adv.Math'16]:
$g((N - ab, a^b), (N - ab, a^b), (N - |\gamma|, \gamma)) > 0$ for almost all γ, a, b (with restrictions), related to [no occurrence-obstructions in] Geometric Complexity Theory.

[Pak-Panova-Yeliussizov, JCTA'19] The largest Kronecker coefficient is $g(\lambda, \mu, \nu) = \sqrt{n!}e^{-O(\sqrt{n})}$ and is attained at Plancherel shapes λ, μ, ν (i.e. such that $f^\lambda \geq \sqrt{n!}e^{-a\sqrt{n}}$).

Partitions

$$p_n := \#\{\lambda :\ \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots), \lambda_1 + \lambda_2 + \cdots = n\}$$



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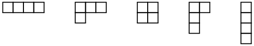

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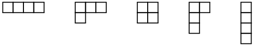

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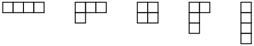

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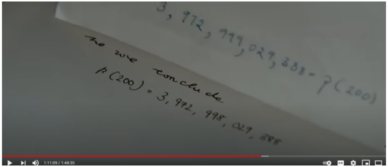


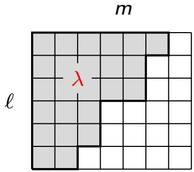
Hardy-Ramanujan:

$$p_n := \#\{\lambda \vdash n\} \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

Partitions inside a rectangle

$$p_n(\ell, m) := \#\{\lambda \vdash n;\ \lambda \subset (m^\ell)\}$$

$$\sum_{k\geq 0} p_n(\ell, m) q^n \ = \ \binom{m+\ell}{m}_q = \frac{\prod_{i=1}^{\ell+m}(1-q^i)}{\prod_{i=1}^{\ell}(1-q^i)\prod_{i=1}^{m}(1-q^i)}$$



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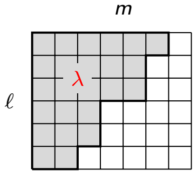

Theorem (Sylvester 1878, Cayley's conjecture 1856)

The sequence $p_0(\ell, m), \ldots, p_{\ell m}(\ell, m)$ is unimodal, i.e.

$$p_0(\ell, m) \leq \ p_1(\ell, m) \leq \ \ldots \leq p_{\lfloor \ell\, m/2 \rfloor}(\ell, m) = p_{\lceil \ell\, m/2 \rceil}(\ell, m) \geq \ \cdots \geq \ p_{\ell\, m}(\ell, m)$$

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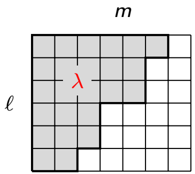

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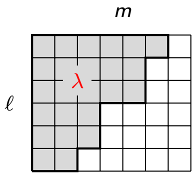

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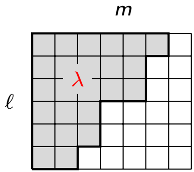

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Proofs:
Sylvester, 1878: *"by aid of a construction drawn from the resources of Imaginative Reason"* (Lie algebras, $\mathfrak{sl}_2$ representations)
Stanley, 1978: hard Lefschetz theorem (alg. geom.), gives Sperner property; 1982: Linear Algebra Paradigm.
Proctor, 1982: explicit linear operators.
O'Hara, 1990: constructive combinatorial proof.

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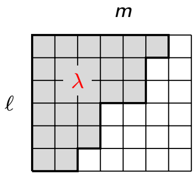

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$$p_0(\ell,m) \leq p_1(\ell,m) \leq \ldots \leq p_{\lfloor \ell m/2\rfloor}(\ell,m) = p_{\lceil \ell m/2\rceil}(\ell,m) \geq \cdots \geq p_{\ell m}(\ell,m)$$

Proofs:
Sylvester, 1878: *"by aid of a construction drawn from the resources of Imaginative Reason"* (Lie algebras, $\mathfrak{sl}_2$ representations)
Stanley, 1978: hard Lefschetz theorem (alg. geom.), gives Sperner property; 1982: Linear Algebra Paradigm.
Proctor, 1982: explicit linear operators.
O'Hara, 1990: constructive combinatorial proof.
Pak-Panova, 2014: proof and asymptotics via Kronecker coefficients.
Melczer-Panova-Pemantle, 2018: tight asymptotics, difference, via random variables.

Lower bounds

Theorem[Pak-P (2014)] Let $\mu = \mu'$ be a self-conjugate partition and let $\widehat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, \ldots) \vdash n$. Let $\chi^\lambda[\pi]$ be the character of $\mathbb{S}^\lambda$ on the permutations of cycle type π. Then:

$$g(\lambda, \mu, \mu) \geq \left|\chi^\lambda[\widehat{\mu}]\right|, \quad \text{for every} \quad \lambda \vdash n,$$

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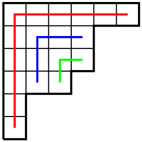
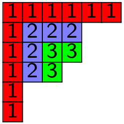
Example:

$\mu =$ 

$\lambda = \mu =$ 

$|\chi^\mu[(\widehat{\mu})]| = 1$

Corollary:[Bessenrodt–Behns] For every μ, s.t. $\mu = \mu'$ we have $\langle \chi^\mu \otimes \chi^\mu, \chi^\mu \rangle \neq 0$.

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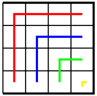
$\mu = (n^n)$:

$$\lambda = (n^2 - k, k)$$

$\mu =$ 

$$|\chi^\lambda[(1, 3, 5, \ldots)]| = |b_k(n) - b_{k-1}(n)|$$

$$\prod_{i=1}^{n} \left(1 + q^{2i-1}\right) = \sum_{k=0}^{n^2} b_k(n)\, q^k.$$

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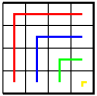
$\mu =$ 

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Lemma[Pak-P, Vallejo, etc]:

$$g((m^\ell), (m^\ell), (m\ell - k, k)) = p_k(m, \ell) - p_{k-1}(m, \ell)$$

Corollary[Stanley, 1982; Pak-P, 2016] The following polynomial in q is symmetric and unimodal

$$\binom{2n}{n}_q - \prod_{i=1}^{n} \left(1 + q^{2i-1}\right) .$$

Effective bounds

$$\prod_{i=1}^{n} \left(1+q^{2i-1}\right) \;=\; \sum_{k=0}^{n^2} b_k(n)\, q^k$$

1. **Theorem**[Pak-P, cor to Almkvist]

$$b_k(n) \,-\, b_{k-1}(n) \,\geq\, C\, 2^{\sqrt{2\,k}}\, \frac{1}{(2\,k)^{9/4}}$$

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$\Longrightarrow$ **Theorem**[Pak-P (2016)]

$g(m^\ell, m^\ell, (m\ell-k,k)) = p_k(\ell,m) \;-\; p_{k-1}(\ell,m) \;>\; 0.004\, \dfrac{2^{\sqrt{s}}}{s^{9/4}}$, where $s = \min\{2k, \ell^2\}$,

The difference $p_{n+1}(m,\ell) - p_n(m,\ell)$

Theorem (Melczer-P-Pemantle)

Given m,ℓ and n, let $A := \ell/m$ and $B := n/m^2$ and define c,d as solutions of a given system of integral equations. Suppose m,ℓ and n go to infinity so that (A,B) remains in a compact subset of $\{(x,y) : x > 2y > 0\}$. Then

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Corollaries:
Sylvester's unimodality, [Pak-P] bound on the Kronecker to exact asymtptotics etc.

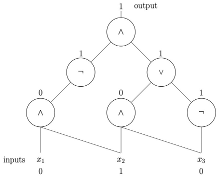
Corollary

The Kronecker coefficient of $S_{m\ell}$ for the (rectangle, rectangle, two-row) case is asymptotically given by

$$g((m^\ell),(m^\ell),(m\ell-n-1,n+1)) = p_{n+1}(\ell,m) - p_n(\ell,m) \sim \frac{d}{m} p_n(\ell,m)$$
$$\sim \frac{de^{m\left[cA+2dB-\log(1-e^{-c-d})\right]}}{2\pi m^3\sqrt{D}}.$$

Computational Complexity

`Input:` I, size$(I) = n$ (bits)



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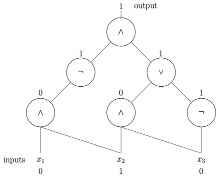

Decision problems: is there...

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Is $C(I) \neq \emptyset$?

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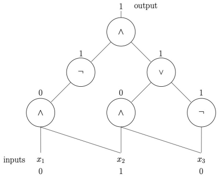

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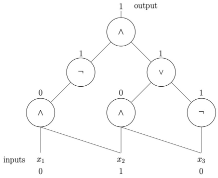

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Given X, is $X \in C(I)$? Answer in $O(n^d)$.

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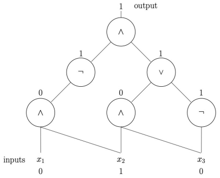

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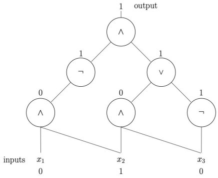

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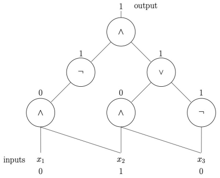

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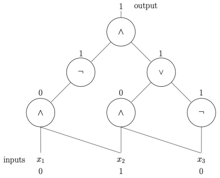

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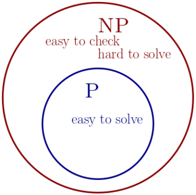

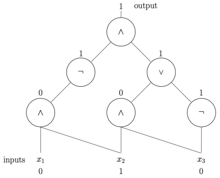

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The P vs NP Millennium Problem:
Is P = NP ?

Complexity of Computing Multiplicities

Dimension of irreducible representations:

$$\dim \mathbb{S}_\lambda = f^\lambda = \#SYT(\lambda) = \frac{n!}{\prod_{(i,j)\in\lambda}(\lambda_i - i + \lambda'_j - j + 1)}.$$

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Kostka numbers: $K_{\lambda\mu} = \#SSYT(\text{shape} = \lambda,\ \text{content} = \mu)$,
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$K_{(4,2,1)(3,2,2)} = 2$ $\quad\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}$

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$$\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}$$

KostkaPos: Input λ, μ, Output: Is $K_{\lambda\mu} > 0$.
Answer: Iff $\lambda \succ \mu$ $(\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$ for every $i)$.

KostkaPos $\in$ P

ComputeKostka: Input λ, μ. Output: the integer $K_{\lambda\mu}$.

Complexity of Computing Multiplicities

Dimension of irreducible representations: $\in$ P

$$\dim \mathbb{S}_\lambda = f^\lambda = \#SYT(\lambda) = \frac{n!}{\prod_{(i,j)\in\lambda}(\lambda_i - i + \lambda'_j - j + 1)}.$$

$f^{(3,2)} = \frac{5!}{4.3.2.1.1} = 5$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$$

Kostka numbers: $K_{\lambda\mu} = \#SSYT(\text{shape} = \lambda,\ \text{content} = \mu)$,
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Theorem (Narayanan'06)

ComputeKostka *is $\#P$-complete. (input – binary)*

Conjecture (Pak-Panova'20+)

ComputeKostka *is strongly $\#P$-complete. (input – unary)*

Complexity of Computing LR

Littlewood-Richardson coefficients: $c^{\lambda}_{\mu\nu} = \mathrm{mult}_{\lambda} V_{\mu} \otimes V_{\nu} = \#LR - tableaux$

$c^{(6,4,3)}_{(4,3,2)(3,1)} = 2$:

$$\begin{array}{cccccc} & & & \boxed{1} & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \boxed{2} & & \\ \boxed{2} & \boxed{3} & \boxed{3} & & & \end{array} \qquad \begin{array}{cccccc} & & & \boxed{1} & \boxed{1} & \boxed{1} \\ & \boxed{2} & \boxed{2} & \boxed{2} & & \\ \boxed{1} & \boxed{3} & \boxed{3} & & & \end{array}$$

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Conjecture (Pak-Panova'20+)

ComputeLR *is strongly #P-hard.*

Complexity of Computing ...

$\chi^\lambda(\alpha) =$ **character** of $\mathbb{S}_\lambda$ at permutation of cycle type α.
CharPos:
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ComputeKron *is in* GapP *($= F - G$, where $F, G \in \#P$).*

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Conjecture

ComputeKron *is in* #P *.*

Thank you!