Dessins d'enfants and moduli spaces of curves

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Dessins d'enfants theory, initiated by Alexander Grothendieck in 1970's, establishes an equivalence between the category of certain graphs on topological surfaces and some arithmetic-geometric category (of *Belyi pairs*, i.e. $(\mathbf{X}, \beta)$'s, where $\mathbf{X}$ is a curve and $\beta : \mathbf{X} \to \mathbf{P_1}(\mathbb{C})$ is a covering with 3 branch points). We are going to discuss two relations of this equivalence with the moduli spaces of curves.

(1) It turned out (Mumford-Penner-Kontsevich-...) that the *decorated moduli spaces of curves* $\mathcal{M}_{g,N}(\mathbb{C}) \times \mathbb{R}^N_{>0}$ admit the orbifold cell decomposition in which the cells are parametrized by certain dessins d'enfants. (In 1992 Kontsevich has applied this construction to the proof of the famous Witten conjecture). The relation of this decomposition with the Grothendieck-Belyi construction will be explained.

(2) For any triple of natural numbers (b, d, g) and for any algebraically closed ground field $\Bbbk$ we consider the *critical filtration* of the moduli space $\mathcal{M}_g(\Bbbk)$ by the subvarieties

$$\mathsf{Cr}_{g;d,b}(\Bbbk) := \{\mathbf{X} \in \mathcal{M}_g(\Bbbk) \mid \exists f \in \Bbbk(\mathbf{X}), \deg f = d, \#\mathrm{CritVal}(f) \leq b\}$$

(the set of curves of genus g carrying rational functions of degree d with no more than b critical values – or, alternatively, admitting a degree-d covering of the projective line with no more than b branch points).

According to Grothendieck-Belyi, the zero-dimensional stratum $\mathsf{Cr}_{g;d,3}(\mathbb{C}) = \mathsf{Cr}_{g;d,3}(\overline{\mathbb{Q}})$ coresponds to dessins d'enfants. The combinatorial, algebro-geometrical and arithmetical problems, related to the higher-dimensional strata of the critical filtration, will be discussed.