Permutation groups and permutation patterns

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Permutations

We consider permutations of $\{1, \ldots, n\}$, for some $n \in \mathbb{N}_+$.

A **permutation** of rank n is a bijective map on $\{1, \ldots, n\}$.

We may consider a permutation $\pi \in S_n$ as a word of length n:

$$\pi = \pi_1 \pi_2 \ldots \pi_n,$$

where $\pi_i = \pi(i)$.

Permutation patterns

$$\sigma = \sigma_1 \dots \sigma_\ell \in S_\ell \qquad \tau = \tau_1 \dots \tau_\ell \in S_n \qquad (\ell \leq n)$$

σ is a **pattern** of τ (or τ **involves** σ), in symbols, $\sigma \leq \tau$, if there exists a scattered subword $\tau_{i_1} \dots \tau_{i_\ell}$ of τ ($i_1 < i_2 < \dots < i_\ell$) that is order-isomorphic to $\sigma_1 \dots \sigma_\ell$.

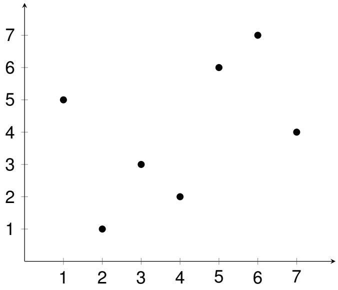
Example

5132674

5 26 4

3 14 2

$3142 \leq 5132674$



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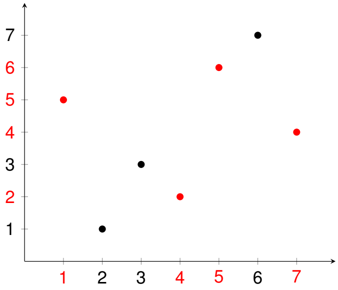

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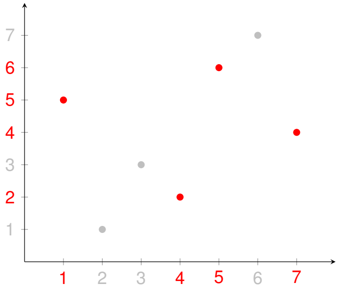

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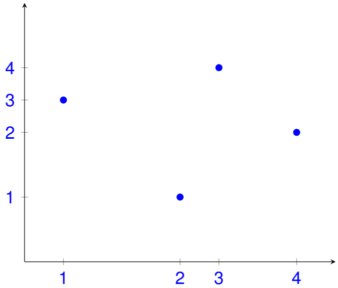

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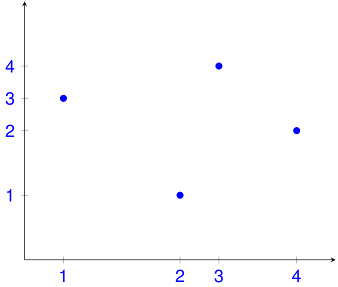

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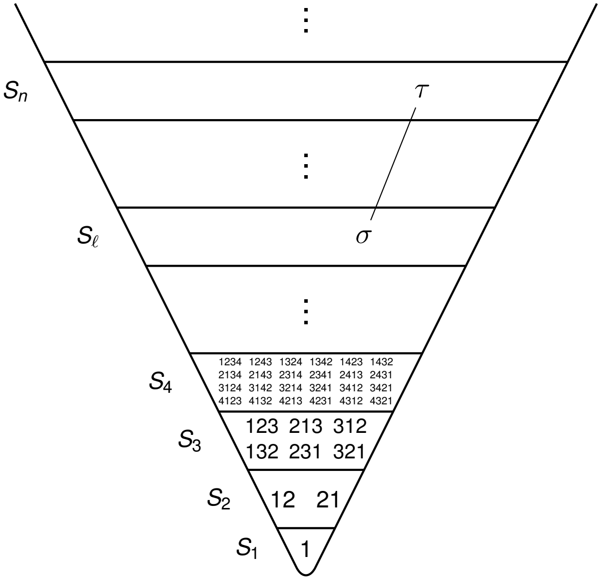
τ **avoids** σ if $\sigma \nleq \tau$.

The pattern involvement relation $\leq$ is a partial order on the set $\mathbb{P} := \bigcup_{n \geq 1} S_n$ of all finite permutations. Downward closed subsets of $\mathbb{P}$ under $\leq$ are called **permutation classes**.

Permutation patterns



$\sigma \leq \tau$

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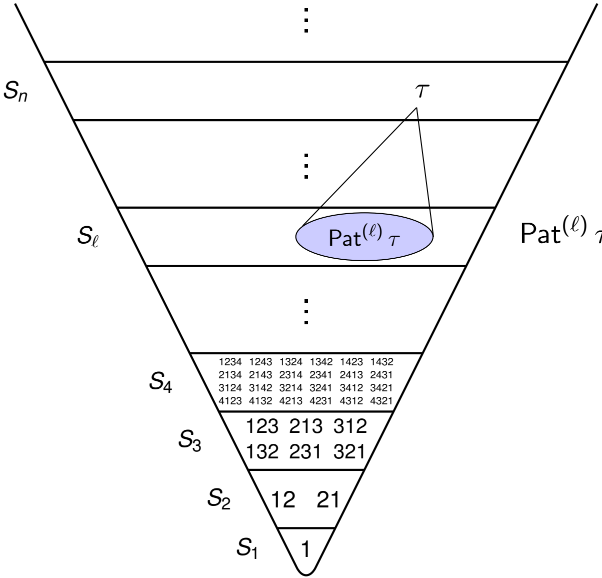
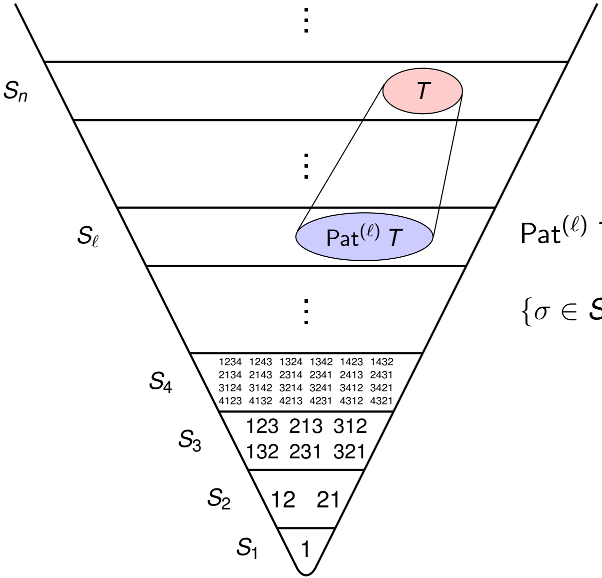


$$\mathrm{Pat}^{(\ell)}\,\tau = \{\sigma \in S_\ell \mid \sigma \leq \tau\}$$

Permutation patterns



$$\mathrm{Pat}^{(\ell)}\ T = \bigcup_{\tau \in T} \mathrm{Pat}^{(\ell)}\ \tau = \{\sigma \in S_\ell \mid \exists \tau \in T \colon \sigma \leq \tau\}$$

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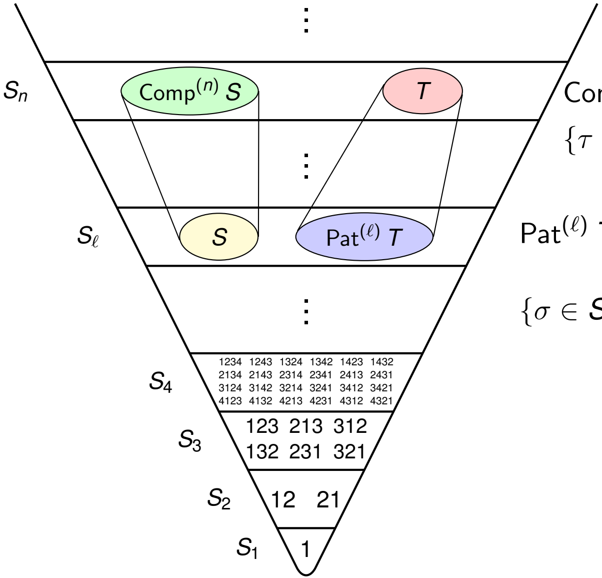

$$\mathrm{Comp}^{(n)}\, S = \{\tau \in S_n \mid \mathrm{Pat}^{(\ell)}\, \tau \subseteq S\}$$

$$\mathrm{Pat}^{(\ell)}\, T = \bigcup_{\tau \in T} \mathrm{Pat}^{(\ell)}\, \tau = \{\sigma \in S_\ell \mid \exists \tau \in T \colon \sigma \leq \tau\}$$

Galois connection

The operators $\mathrm{Pat}^{(\ell)}$ and $\mathrm{Comp}^{(n)}$ constitute a monotone Galois connection between $\mathcal{P}(S_\ell)$ and $\mathcal{P}(S_n)$.

This is in fact the monotone Galois connection induced by the pattern avoidance relation $\nleq$ between S_ℓ and S_n.

$S \subseteq S_\ell$, $T \subseteq S_n$ ($\ell \leq n$)

$$\mathrm{Comp}^{(n)}\, S := \{\tau \in S_n \mid \mathrm{Pat}^{(\ell)}\, \tau \subseteq S\} = \{\tau \in S_n \mid \forall \sigma \in S_\ell \setminus S \colon \sigma \nleq \tau\},$$
$$\mathrm{Pat}^{(\ell)}\, T := \bigcup_{\tau \in T} \mathrm{Pat}^{(\ell)}\, \tau = S_\ell \setminus \{\sigma \in S_\ell \mid \forall \tau \in T \colon \sigma \nleq \tau\}.$$

Closures and kernels:

$$\mathrm{Pat}^{(\ell)}\, \mathrm{Comp}^{(n)}\, S \subseteq S,$$
$$T \subseteq \mathrm{Comp}^{(n)}\, \mathrm{Pat}^{(\ell)}\, T,$$
$$\mathrm{Comp}^{(n)}\, S = \mathrm{Comp}^{(n)}\, \mathrm{Pat}^{(\ell)}\, \mathrm{Comp}^{(n)}\, S,$$
$$\mathrm{Pat}^{(\ell)}\, T = \mathrm{Pat}^{(\ell)}\, \mathrm{Comp}^{(n)}\, \mathrm{Pat}^{(\ell)}\, T.$$

Galois connection

The operators $\mathsf{Comp}^{(n)}$, $\mathsf{Pat}^{(\ell)}$ have a "transitive" property.

For $\ell \leq m \leq n$, $S \subseteq S_\ell$, $T \subseteq S_n$:

$$\mathsf{Comp}^{(n)} \mathsf{Comp}^{(m)} S = \mathsf{Comp}^{(n)} S$$
$$\mathsf{Pat}^{(\ell)} \mathsf{Pat}^{(m)} T = \mathsf{Pat}^{(\ell)} T$$

Formalism

For any $I \in \mathcal{P}_\ell(n)$, let $h_I \colon [\ell] \to I$ be the unique order-isomorphism from $([\ell], \leq)$ to $(I, \leq)$.

For $\tau \in S_n$, define $\tau_I \colon [\ell] \to [\ell]$ as

$$\tau_I := h_{\tau(I)}^{-1} \circ \tau \circ h_I.$$

The patterns of τ are precisely the permutations of the form τ_I for some $\emptyset \neq I \subseteq [n]$.

Crucial lemma

Lemma

For any $\pi, \tau \in S_n$ *and* $\emptyset \neq I \subseteq [n]$, *we have* $(\pi\tau)_I = \pi_{\tau(I)} \circ \tau_I$.

Proof.

$$(\pi\tau)_I = h^{-1}_{(\pi\circ\tau)(I)} \circ \pi \circ \tau \circ h_I = h^{-1}_{\pi(\tau(I))} \circ \pi \circ h_{\tau(I)} \circ h^{-1}_{\tau(I)} \circ \tau \circ h_I = \pi_{\tau(I)} \circ \tau_I. \qquad \square$$

Groups and $\mathrm{Comp}^{(n)} S$

Proposition

If S is a subgroup of S_ℓ, then $\mathrm{Comp}^{(n)} S$ is a subgroup of S_n.

Sketch of a proof.

Assume that $S \leq S_\ell$. Let $\pi, \tau \in \mathrm{Comp}^{(n)} S$.
Thus $\mathrm{Pat}^{(\ell)} \pi, \mathrm{Pat}^{(\ell)} \tau \subseteq S$.
It holds that

$$\mathrm{Pat}^{(\ell)} \pi^{-1} = (\mathrm{Pat}^{(\ell)} \pi)^{-1} := \{\sigma^{-1} \mid \sigma \in \mathrm{Pat}^{(\ell)} \pi\},$$
$$\mathrm{Pat}^{(\ell)} \pi\tau \subseteq (\mathrm{Pat}^{(\ell)} \pi)(\mathrm{Pat}^{(\ell)} \tau) = \{\sigma\sigma' \mid \sigma \in \mathrm{Pat}^{(\ell)} \pi,\ \sigma' \in \mathrm{Pat}^{(\ell)} \tau\}.$$

Since S is a group, it contains the inverses and products of its members. Consequently, π^{-1} and $\pi\tau$ also belong to $\mathrm{Comp}^{(n)} S$. Thus $\mathrm{Comp}^{(n)} S$ is a group. □

Groups and $\mathrm{Comp}^{(n)} S$, continued

The converse of the Proposition does not hold.

There even exist subgroups $H \leq S_n$ which are of the form $\mathrm{Comp}^{(n)} S$ for some $S \subseteq S_\ell$ but there is no subgroup $G \leq S_\ell$ such that $H = \mathrm{Comp}^{(n)} G$.

However, for $\ell \leq 3$ and $n \geq \ell$, it holds that for every $S \subseteq S_\ell$, $\mathrm{Comp}^{(n)} S$ is a subgroup of S_n if and only if S is a subgroup of S_ℓ.

Groups and $\mathsf{Comp}^{(n)}\, S$, continued

Recall the Galois connection Inv–Aut between permutations of $\{1, \dots, n\}$ and relations on $\{1, \dots, n\}$:

$$\begin{aligned} \mathsf{Inv}\, T &:= \{\rho \in \mathrm{Rel}_n \mid \forall \pi \in T \colon \pi \triangleright \rho\} & \qquad T &\subseteq S_n, \\ \mathsf{Aut}\, R &:= \{\pi \in S_n \mid \forall \rho \in R \colon \pi \triangleright \rho\} & \qquad R &\subseteq \mathrm{Rel}_n\,. \end{aligned}$$

Groups and $\mathrm{Comp}^{(n)} S$, continued

Proposition

Let $H \leq S_n$ and assume that $H = \mathrm{Comp}^{(n)} S$ for some subset $S \subseteq S_\ell$. Then H is determined by its ℓ-ary invariant relations:

$$H = \mathrm{Aut}\,\mathrm{Inv}\,H = \mathrm{Aut}\,\mathrm{Inv}^{(\ell)}\,H.$$

In particular, the ℓ-orbits are enough to characterize the group:

$$H = \mathrm{Aut}\{(h_I)^H \mid I \in \mathcal{P}_\ell(n)\}.$$

$\mathbf{a}^H := \{\sigma(\mathbf{a}) \mid \sigma \in H\} = \{(\sigma(a_1), \ldots, \sigma(a_\ell)) \mid \sigma \in H\}$
The order-isomorphism $h_I \colon [\ell] \to I$ is an ℓ-tuple: $h_I \in [n]^\ell_{\neq}$.
$(h_I)^H$ is called an ℓ-orbit of H.

Groups and $\mathrm{Comp}^{(n)} S$, continued

Theorem

Let $H \leq S_n$, and consider the ℓ-orbits $\rho_I := (h_I)^H$ for all $I \in \mathcal{P}_\ell(n)$. Then H is of the form $H = \mathrm{Comp}^{(n)} S$ for some $S \subseteq S_\ell$ if and only if

1. $H = \mathrm{Aut}\{\rho_I \mid I \in \mathcal{P}_\ell(n)\}$,
2. *the ρ_I satisfy the following property: for every $x \in [n]^n_{\neq}$ we have*

$$(\forall I \in \mathcal{P}_\ell(n)\, \exists J \in \mathcal{P}_\ell(n) \colon\ \mathrm{red}(x[I]) \in \mathrm{red}(\rho_J)) \implies \forall I \in \mathcal{P}_\ell(n) \colon\ x[I] \in \rho_I.$$

Groups and $\mathrm{Comp}^{(n)} S$, continued

Theorem

Let $H \leq S_n$. Then H is of the form $H = \mathrm{Comp}^{(n)} G$ for some $G \leq S_\ell$ if and only if $H = \mathrm{Aut}\,\rho$ for some k-ary ($k \leq \ell$) irreflexive relation ρ satisfying $\rho = \rho^{\vee\wedge}$.

$$\rho^{\vee} := \{h_I^{-1}(\mathbf{r}) \mid \mathbf{r} \in \rho,\ \mathrm{Im}\,\mathbf{r} \subseteq I \in \mathcal{P}_\ell(n)\},$$
$$\sigma^{\wedge} := \{h_J(\mathbf{s}) \mid \mathbf{s} \in \sigma,\ J \in \mathcal{P}_\ell(n)\}.$$

Groups and $\mathsf{Comp}^{(n)}\, S$, continued

Further details on the Galois connection $\mathsf{Comp}^{(n)}$–$\mathsf{Pat}^{(\ell)}$ and the related Galois connection between the subgroup lattices of S_ℓ and S_n in

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Group classes

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M. D. ATKINSON, R. BEALS,
Permutation involvement and groups,
Q. J. Math. **52** (2001) 415–421.

Group classes

Theorem (Atkinson, Beals)

*If C is a permutation class in which every level $C^{(n)}$ is a permutation group, then the level sequence $C^{(1)}, C^{(2)}, \ldots$ eventually coincides with one of the following **stable** families of groups:*

1. *the groups $S_n^{a,b}$ for some fixed $a, b \in \mathbb{N}_+$,*
2. *the natural cyclic groups Z_n,*
3. *the full symmetric groups S_n,*
4. *the groups $\langle G_n, \delta_n \rangle$, where $(G_n)_{n \in \mathbb{N}}$ is one of the above families (with $a = b$ in (1)).*

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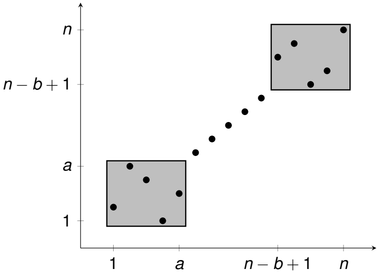
$S_n^{a,b}$



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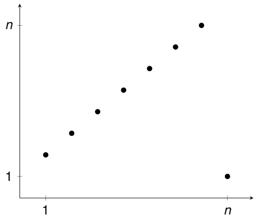
$$\zeta_n = (1\ 2\ \cdots n)$$



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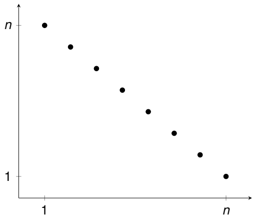
$$\delta_n = n(n-1)\dots 1$$



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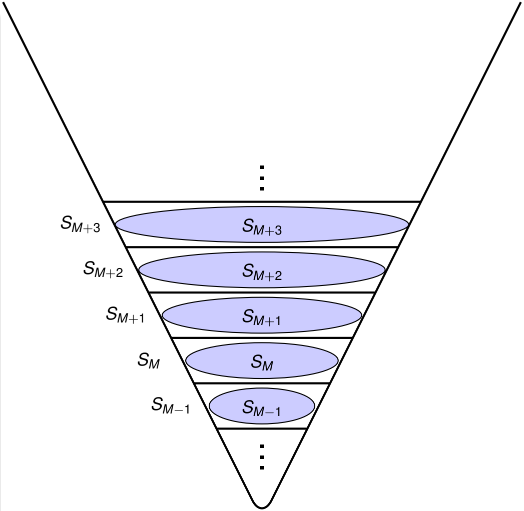
Theorem (Atkinson, Beals)

Let C be a permutation class in which every level $C^{(n)}$ is a transitive group. Then, with the exception of at most two levels, one of the following holds.

1. $C^{(n)} = S_n$ for all $n \in \mathbb{N}_+$.
2. For some $M \in \mathbb{N}$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, and $C^{(n)} = D_n$ for $n > M$.
3. For some $M, N \in \mathbb{N}$ with $M \leq N$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, $C^{(n)} = D_n$ for $M + 1 \leq n \leq N$, and $C^{(n)} = Z_n$ for $n > N$.

The exceptions, if any, may occur in the second and third cases and are of the following two possible types:

- (i) $C^{(M+1)} = A_{M+1}$ and $C^{(M+2)}$ is an anomalous group that is neither D_{M+2} nor Z_{M+2}, or
- (ii) $C^{(M+1)}$ is a proper overgroup of Z_{M+1} but is not D_{M+1}.



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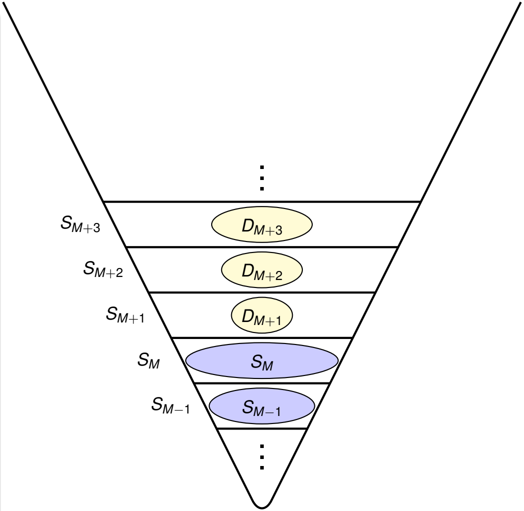

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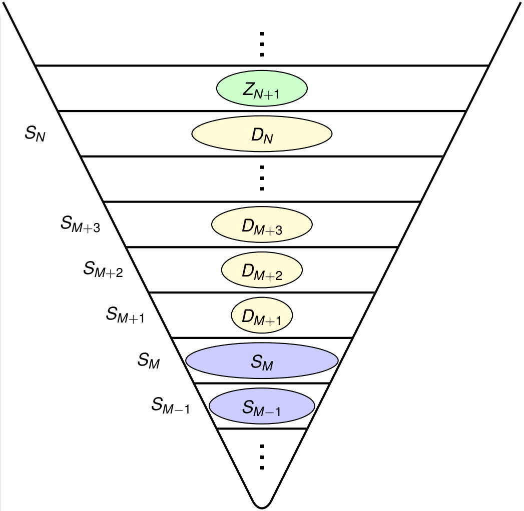

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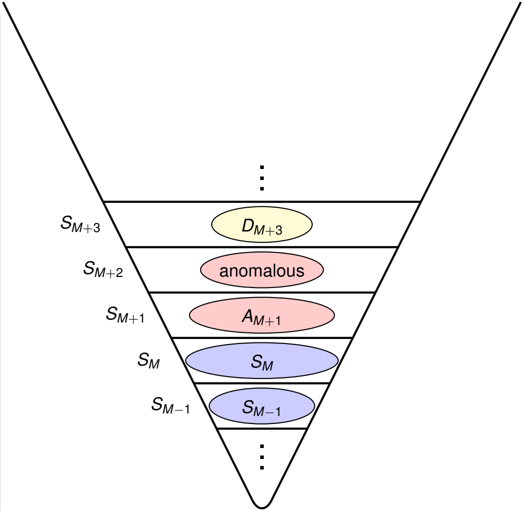

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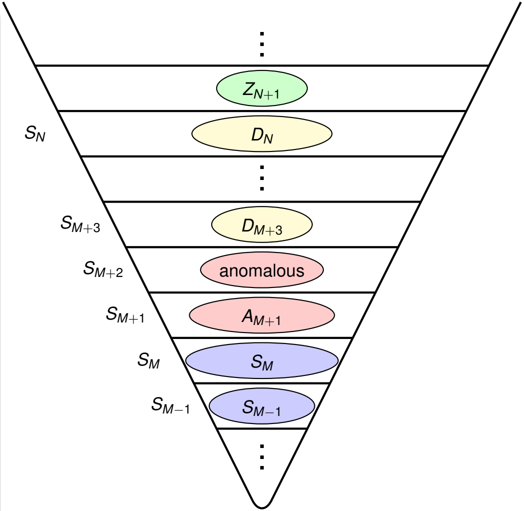

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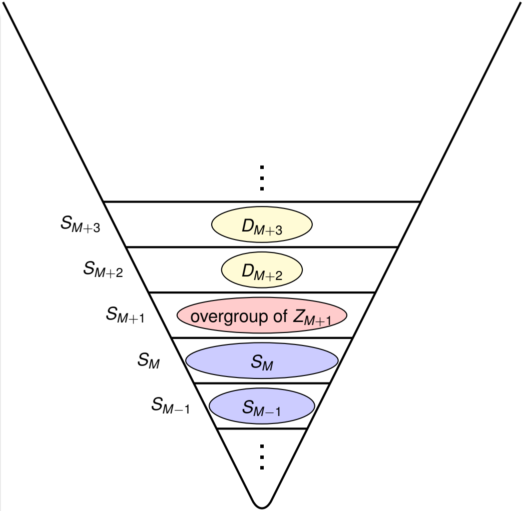
3. For some $M, N \in \mathbb{N}$ with $M \leq N$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, $C^{(n)} = D_n$ for $M + 1 \leq n \leq N$, and $C^{(n)} = Z_n$ for $n > N$.

The exceptions, if any, may occur in the second and third cases and are of the following two possible types:

- (i) $C^{(M+1)} = A_{M+1}$ and $C^{(M+2)}$ is an anomalous group that is neither D_{M+2} nor Z_{M+2}, or
- (ii) $C^{(M+1)}$ is a proper overgroup of Z_{M+1} but is not D_{M+1}.



Group classes

Theorem (Atkinson, Beals)

Let C be a permutation class in which every level $C^{(n)}$ is a transitive group. Then, with the exception of at most two levels, one of the following holds.

1. $C^{(n)} = S_n$ for all $n \in \mathbb{N}_+$.
2. For some $M \in \mathbb{N}$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, and $C^{(n)} = D_n$ for $n > M$.
3. For some $M, N \in \mathbb{N}$ with $M \leq N$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, $C^{(n)} = D_n$ for $M + 1 \leq n \leq N$, and $C^{(n)} = Z_n$ for $n > N$.

The exceptions, if any, may occur in the second and third cases and are of the following two possible types:

(i) $C^{(M+1)} = A_{M+1}$ and $C^{(M+2)}$ is an anomalous group that is neither D_{M+2} nor Z_{M+2}, or

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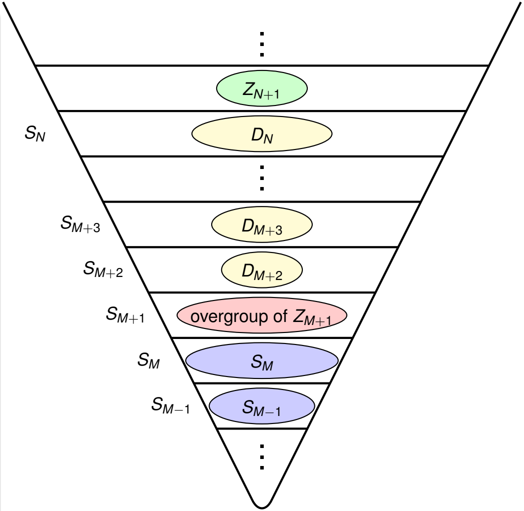
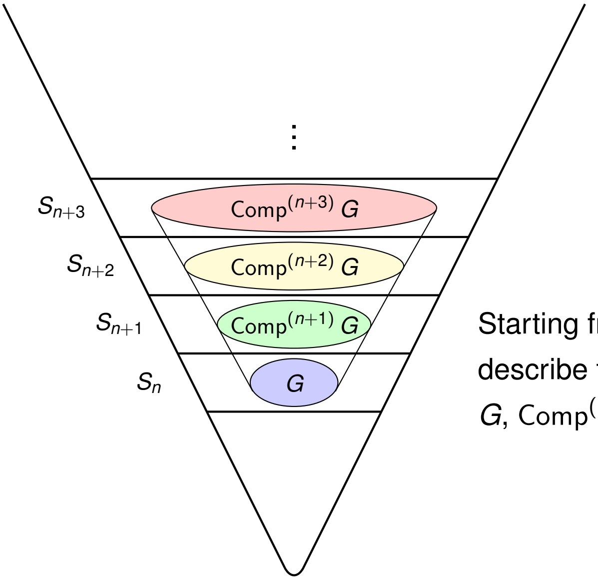


Permutation groups arising from pattern avoidance

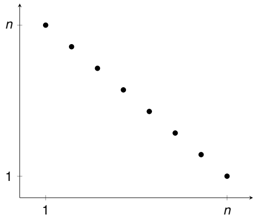


Starting from $G \leq S_n$,
describe the sequence
$G, \mathrm{Comp}^{(n+1)}\,G, \mathrm{Comp}^{(n+2)}\,G, \ldots$

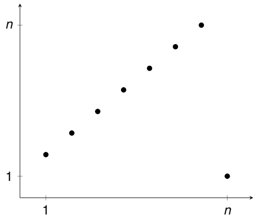
Roadmap

- S_n, $\langle \delta_n \rangle$, trivial
- A_n
- $\zeta_n \in G$ and $A_n \not\leq G$
- $\zeta_n \notin G$:
  - intransitive
  - transitive:
    - imprimitive
    - primitive

$\delta_n = n(n-1)\dots 1$



$\zeta_n = (1\ 2\ \cdots n)$



Simple observations

Lemma

Let $n, m \in \mathbb{N}_+$ with $n \leq m$. Let $G \leq S_n$. Then $\delta_m \in \operatorname{Comp}^{(m)} G$ if and only if $\delta_n \in G$.

Lemma

Let $G \leq S_n$.

- (a) *The following statements are equivalent.*
  - (i) $Z_n \leq G$.
  - (ii) $Z_{n+1} \leq \operatorname{Comp}^{(n+1)} G$.
  - (iii) $\operatorname{Comp}^{(n+1)} G$ *contains a permutation* $\pi \in Z_{n+1} \setminus \{\iota_{n+1}\}$.
- (b) *The following statements are equivalent.*
  - (i) $D_n \leq G$.
  - (ii) $D_{n+1} \leq \operatorname{Comp}^{(n+1)} G$.
  - (iii) $\operatorname{Comp}^{(n+1)} G$ *contains a permutation* $\pi \in D_{n+1} \setminus (Z_{n+1} \cup \{\delta_{n+1}\})$.

Symmetric, trivial, ...

Theorem

The following statements hold for all $n \in \mathbb{N}_+$.

(a) $\mathrm{Comp}^{(n+1)} S_n = S_{n+1}$.

(b) *If $n \geq 2$, then $\mathrm{Comp}^{(n+1)} \{\iota_n\} = \{\iota_{n+1}\}$.*

(c) *If $n \geq 3$, then $\mathrm{Comp}^{(n+1)} \langle \delta_n \rangle = \langle \delta_{n+1} \rangle$.*

Notation

Let Π be a partition of $[n]$.

$$S_\Pi := \{\pi \in S_n \mid \forall B \in \Pi \colon \pi(B) = B\}$$

Alternating groups

$Œ_n$ – partition of $[n]$ into odd and even numbers
$S_{Œ_n}$ – permutations preserving blocks of $Œ_n$
$W_{Œ_n}$ – permutations interchanging blocks of $Œ_n$
A_n – even permutations
O_n – odd permutations
$\Xi_n := (S_{Œ_n} \cap A_n) \cup (W_{Œ_n} \cap O_n)$

Theorem

$$\operatorname{Comp}^{(n+1)} A_n = \Xi_{n+1}$$

$$\operatorname{Comp}^{(n+2)} A_n = \begin{cases} \langle \delta_{n+2} \rangle, & \text{if } n \equiv 0 \pmod 4, \\ D_{n+2}, & \text{if } n \equiv 1 \pmod 4, \\ \{\iota_{n+2}\}, & \text{if } n \equiv 2 \pmod 4, \\ Z_{n+2}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Groups containing the natural cycle

Theorem

Let $G \leq S_n$, and assume that G contains the natural cycle ζ_n.

1. (i) *If $D_n \leq G$ and $G \notin \{S_n, A_n\}$, then $\operatorname{Comp}^{(n+1)} G = D_{n+1}$.*
2. (ii) *If $D_n \nleq G$, then $\operatorname{Comp}^{(n+1)} G = Z_{n+1}$.*

Intransitive groups

Let $G \leq S_n$ be an intransitive group.

Let $\operatorname{Orb} G$ be the set of orbits of G.

Then $G \leq S_{\operatorname{Orb} G}$.

Moreover, $\operatorname{Orb} G$ is the finest partition Π such that $G \leq S_\Pi$.

Intransitive groups

Let Π be a partition of $[n]$.

Define the partition Π' of $[n+1]$ as follows.

Let I_Π be the coarsest interval partition that refines Π.

$$I_\Pi^1 := \{B + 1 \mid B \in I_\Pi, 1 \notin B\} \cup \{((1/I_\Pi) + 1) \cup \{1\}\}$$

$$I_\Pi^{n+1} := \{B \mid B \in I_\Pi, n \notin B\} \cup \{n/I_\Pi \cup \{n+1\}\}$$

$$\Pi' := I_\Pi^1 \wedge I_\Pi^{n+1}$$

Example

$$\Pi = \{\{1,2,3,7,8,9,10\}, \{4,5,6,12,13,14\}, \{11\}\}$$

$$I_\Pi = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9,10\}, \{11\}, \{12,13,14\}\}$$

$$\Pi' = \{\{1,2,3\}, \{4\}, \{5,6\}, \{7\}, \{8,9,10\}, \{11\}, \{12\}, \{13,14,15\}\}$$

Intransitive groups

Theorem

Let Π *be a partition of* $[n]$.

1. *If* $\delta_n \notin S_\Pi$, *then* $\mathrm{Comp}^{(n+1)} S_\Pi = S_{\Pi'}$.
2. *If* $\delta_n \in S_\Pi$, *then* $\mathrm{Comp}^{(n+1)} S_\Pi = S_{\Pi'} \cup \delta_{n+1} S_{\Pi'} = \langle S_{\Pi'}, \delta_{n+1} \rangle$; *moreover,* $\Pi' = \delta_{n+1}(\Pi')$.

Theorem

Let Π *be a partition of* $[n]$.

1. Π' *is an interval partition with no consecutive non-trivial blocks.*
2. *If* Π *is an interval partition with no consecutive non-trivial blocks and* $\Pi = \delta_n(\Pi)$, *then* $\mathrm{Comp}^{(n+1)} \langle S_\Pi, \delta_n \rangle = \langle S_{\Pi'}, \delta_{n+1} \rangle$; *moreover,* $\Pi' = \delta_{n+1}(\Pi')$.

Theorem

Let $G \leq S_n$ be an intransitive group, and let $\Pi := \operatorname{Orb} G$. Let a and b be the largest numbers α and β, respectively, such that $S_n^{\alpha,\beta} \leq G$. Then for all $\ell \geq M_{a,b}(\Pi)$, it holds that $\operatorname{Comp}^{(n+\ell)} G = S_{n+\ell}^{a,b}$ or $\operatorname{Comp}^{(n+\ell)} G = \langle S_{n+\ell}^{a,b}, \delta_{n+\ell} \rangle$.

$$M(\Pi) := \max(\{|B| : B \in I_\Pi^-\} \cup \{1\})$$
$$M_{a,b}(\Pi) := \max(M(\Pi), |1/I_\Pi| - a + 1, |n/I_\Pi| - b + 1)$$

Notation

Let Π be a partition of $[n]$.

$$\operatorname{Aut} \Pi := \{\pi \in S_n \mid \forall B \in \Pi \colon \pi(B) \in \Pi\}$$

Theorem

Let Π be a partition of $[n]$ with no trivial blocks. Then

$$\operatorname{Comp}^{(n+1)} \operatorname{Aut} \Pi = \begin{cases} \langle S_{\Pi'}, E_\Pi \rangle, & \text{if } \delta_n \notin \operatorname{Aut} \Pi, \\ \langle S_{\Pi'}, E_\Pi, \delta_{n+1} \rangle, & \text{if } \delta_n \in \operatorname{Aut} \Pi, \end{cases}$$

where E_Π is the set of permutations satisfying the following conditions:

- *If $[1, \ell] \propto \Pi$ for some ℓ with $1 < \ell < n$, then $\nu_\ell^{(n+1)} \in E_\Pi$.*
- *If $[m, n] \propto \Pi$ for some m with $1 < m < n$, then $\lambda_{n-m+1}^{(n+1)} \in E_\Pi$.*
- *If $[1, n] \propto \Pi$, then $\zeta_{n+1} \in E_\Pi$.*
- *E_Π does not contain any other elements than the ones implied by the previous conditions.*

Primitive groups

Theorem

Assume that $G \leq S_n$ is a primitive group such that $\zeta_n \notin G$ and $A_n \nleq G$.

(i) (a) $n = 6$

G	$\mathrm{Comp}^{(n+1)} G$
$\langle (1\ 2\ 3\ 4), (3\ 4\ 5\ 6) \rangle$	$\{1234567, 2154376, 6734512, 7654321\}$
$\langle (1\ 2\ 3\ 4), (2\ 3\ 4\ 5\ 6) \rangle$	$\{1234567, 1276543, 1543276, 1567234\}$
$\langle (1\ 2\ 3\ 4\ 5), (3\ 4\ 5\ 6) \rangle$	$\{1234567, 2165437, 4561237, 5432167\}$
$\langle (1\ 2\ 3\ 4\ 5), (1\ 3\ 4)(2\ 5\ 6) \rangle$	$\langle \nu_5^{(7)} \rangle$
$\langle (2\ 3\ 4\ 5\ 6), (1\ 2\ 5)(3\ 4\ 6) \rangle$	$\langle \lambda_5^{(7)} \rangle$

(b) $n \neq 6$

G	$\mathrm{Comp}^{(n+1)} G$	G	$\mathrm{Comp}^{(n+1)} G$
$D_{[1,n-1]} \leq G$	$\langle \nu_{n-1}^{(n+1)} \rangle$	$D_{[2,n]} \leq G$	$\langle \lambda_{n-1}^{(n+1)} \rangle$
$D_{[1,n-2]} \leq G$	$\langle \nu_{n-2}^{(n+1)} \rangle$	$D_{[3,n]} \leq G$	$\langle \lambda_{n-2}^{(n+1)} \rangle$

(c) *Otherwise* $\mathrm{Comp}^{(n+1)} G \leq \langle \delta_{n+1} \rangle$.

(ii) $\mathrm{Comp}^{(n+2)} G \leq \langle \delta_{n+2} \rangle$.

In a nutshell

Corollary

Let $G \leq S_n$ and let m be the smallest number i such that $\operatorname{Comp}^{(n+i)} G$ belongs to one of the stable families of groups.

1. *If G is intransitive, then $m \leq n - 1$.*
2. *If G is imprimitive and $\zeta_n \notin G$, then $m \leq p$, where p is the largest proper divisor of n.*
3. *Otherwise $m \leq 2$.*

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The end

Thank you for your attention.