Identities of Vector Spaces and Nonassociative Linear Algebras

Kislitsin A.V., Isaev I.M.

Altai State Pedagogical University

19.11.2021

Basic Definitions

- Let F be a field, A be an linear associative F-algebra and E is a subspace in A (but E not necessary subalgebra of A) which generate A how linear F-algebra.
- In this case, we call E a *multiplicative vector space* (in short, an *L-space*) over the field F. The algebra A will be called *enveloping for the space* E, and the space E will be called *embedded in the algebra* A.
- The identity of an L-space E over a field F (embedded in an F-algebra A) is a associative polynomial $f(x_1, x_2, \ldots, x_n)$ which equal to zero in A if, instead of its variables $x_1, x_2, \ldots, x_n$ we substitute any elements from E.

- Let F be a field, A be an linear associative F-algebra and E is a subspace in A (but E not necessary subalgebra of A) which generate A how linear F-algebra.
- In this case, we call E a *multiplicative vector space* (in short, an *L-space*) over the field F. The algebra A will be called *enveloping for the space E*, and the space E will be called *embedded in the algebra A*.
- The identity of an L-space E over a field F (embedded in an F-algebra A) is a associative polynomial $f(x_1, x_2, \dots, x_n)$ which equal to zero in A if, instead of its variables $x_1, x_2, \dots, x_n$ we substitute any elements from E.

- Let F be a field, A be an linear associative F-algebra and E is a subspace in A (but E not necessary subalgebra of A) which generate A how linear F-algebra.
- In this case, we call E a *multiplicative vector space* (in short, an *L-space*) over the field F. The algebra A will be called *enveloping for the space E*, and the space E will be called *embedded in the algebra A*.
- The identity of an L-space E over a field F (embedded in an F-algebra A) is a associative polynomial $f(x_1, x_2, \ldots, x_n)$ which equal to zero in A if, instead of its variables $x_1, x_2, \ldots, x_n$ we substitute any elements from E.

- In this form, the concept of a multiplicative vector space and his identity was introduced in 2010 by I. M. Isaev and the speaker. However, (direct) analogs of this concept have been studied earlier.
- In 1978 I. V. L'vov considered algebras of the form $\overline{V} = V \oplus E$, where V is a vector space and $E \subseteq \mathrm{End}_F V$. Nonzero products of elements of this algebra are given by the rule: $v_i e_{ij} = v_j$. It's clear that $\overline{V} \in \mathrm{Var}\langle x(yz) = 0\rangle$. The nonassociative polynomial $zf(R_{x_1}, R_{x_2}, \ldots, R_{x_n})$ is an identity of the algebra $\overline{V}$ iff the associative polynomial $f(x_1, x_2, \ldots, x_n)$ is equal to zero when substituting instead of variables linear combinations of elements from E.
- In fact, this construction uses the concept of a multiplicative vector space.

- In this form, the concept of a multiplicative vector space and his identity was introduced in 2010 by I. M. Isaev and the speaker. However, (direct) analogs of this concept have been studied earlier.
- In 1978 I. V. L'vov considered algebras of the form $\overline{V} = V \oplus E$, where V is a vector space and $E \subseteq \mathrm{End}_F V$. Nonzero products of elements of this algebra are given by the rule: $v_i e_{ij} = v_j$. It's clear that $\overline{V} \in \mathrm{Var}\langle x(yz) = 0\rangle$. The nonassociative polynomial $zf(R_{x_1}, R_{x_2}, \ldots, R_{x_n})$ is an identity of the algebra $\overline{V}$ iff the associative polynomial $f(x_1, x_2, \ldots, x_n)$ is equal to zero when substituting instead of variables linear combinations of elements from E.
- In fact, this construction uses the concept of a multiplicative vector space.

- In this form, the concept of a multiplicative vector space and his identity was introduced in 2010 by I. M. Isaev and the speaker. However, (direct) analogs of this concept have been studied earlier.
- In 1978 I. V. L'vov considered algebras of the form $\overline{V} = V \oplus E$, where V is a vector space and $E \subseteq \mathrm{End}_F V$. Nonzero products of elements of this algebra are given by the rule: $v_i e_{ij} = v_j$. It's clear that $\overline{V} \in \mathrm{Var}\langle x(yz) = 0\rangle$. The nonassociative polynomial $zf(R_{x_1}, R_{x_2}, \ldots, R_{x_n})$ is an identity of the algebra $\overline{V}$ iff the associative polynomial $f(x_1, x_2, \ldots, x_n)$ is equal to zero when substituting instead of variables linear combinations of elements from E.
- In fact, this construction uses the concept of a multiplicative vector space.

- In 1973, Yu. P. Razmyslov introduced the concept of a weak identity of an associative Lie pair (A, L), where L is a Lie algebra and A is its associative enveloping.
- A *weak identity* of a pair (A, L) is an associative polynomial $f(x_1, x_2, \ldots, x_n)$ that is equal to zero in the algebra A when substituted instead of variable $x_1, x_2, \ldots, x_n$ elements of the algebra L.
- Following the above construction, the identity of the multiplicative vector space E (with the enveloping algebra A) can be considered (if necessary) as a weak identity of the pair (A, E). The pair (A, E) in this case will be called a *multiplicative vector pair*.

- In 1973, Yu. P. Razmyslov introduced the concept of a weak identity of an associative Lie pair (A, L), where L is a Lie algebra and A is its associative enveloping.
- A *weak identity* of a pair (A, L) is an associative polynomial $f(x_1, x_2, \ldots, x_n)$ that is equal to zero in the algebra A when substituted instead of variable $x_1, x_2, \ldots, x_n$ elements of the algebra L.
- Following the above construction, the identity of the multiplicative vector space E (with the enveloping algebra A) can be considered (if necessary) as a weak identity of the pair (A, E). The pair (A, E) in this case will be called a *multiplicative vector pair*.

- In 1973, Yu. P. Razmyslov introduced the concept of a weak identity of an associative Lie pair (A, L), where L is a Lie algebra and A is its associative enveloping.
- A *weak identity* of a pair (A, L) is an associative polynomial $f(x_1, x_2, \ldots, x_n)$ that is equal to zero in the algebra A when substituted instead of variable $x_1, x_2, \ldots, x_n$ elements of the algebra L.
- Following the above construction, the identity of the multiplicative vector space E (with the enveloping algebra A) can be considered (if necessary) as a weak identity of the pair (A, E). The pair (A, E) in this case will be called a *multiplicative vector pair*.

- Consider the set

$$E_0 = \left\{ \begin{pmatrix} \alpha & \alpha \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in F \right\}.$$

This set is a vector space over the field F, but it is not an F-algebra. The algebra $T_2(F)$ of upper triangular matrices is the enveloping algebra for E_0.

- Consider the polynomial $\mathrm{St}_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$.

Since $\dim_F E_0 = 2$, the polynomial $\mathrm{St}_3(x_1, x_2, x_3)$ is the identity of the F-space E_0.

- But this polynomial is not an identity of the algebra $T_2(F)$, because $\mathrm{St}_3(e_{11}, e_{12}, e_{22}) = e_{12} \neq 0$.

- Consider the set

$$E_0 = \left\{ \begin{pmatrix} \alpha & \alpha \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in F \right\}.$$

This set is a vector space over the field F, but it is not an F-algebra. The algebra $T_2(F)$ of upper triangular matrices is the enveloping algebra for E_0.

- Consider the polynomial $\mathrm{St}_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$.

Since $\dim_F E_0 = 2$, the polynomial $\mathrm{St}_3(x_1, x_2, x_3)$ is the identity of the F-space E_0.

- But this polynomial is not an identity of the algebra $T_2(F)$, because $\mathrm{St}_3(e_{11}, e_{12}, e_{22}) = e_{12} \neq 0$.

- Consider the set

$$E_0 = \left\{ \begin{pmatrix} \alpha & \alpha \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in F \right\}.$$

This set is a vector space over the field F, but it is not an F-algebra. The algebra $T_2(F)$ of upper triangular matrices is the enveloping algebra for E_0.

- Consider the polynomial $\mathrm{St}_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$.

Since $\dim_F E_0 = 2$, the polynomial $\mathrm{St}_3(x_1, x_2, x_3)$ is the identity of the F-space E_0.

- But this polynomial is not an identity of the algebra $T_2(F)$, because $\mathrm{St}_3(e_{11}, e_{12}, e_{22}) = e_{12} \neq 0$.

Identities of Vector Spaces

- Let $F\langle X\rangle$ be a free associative algebra and $\varnothing \neq G \subseteq F\langle X\rangle$. By $T(G)$ we denote the T-ideal of the algebra $F\langle X\rangle$ generated by the set G, and by $L(G)$ we denote the ideal of $F\langle X\rangle$ generated by the polynomials of the set G (as an ideal) and closed with respect to substitutions instead of variables of linear combinations of variables. These ideals will be called *L-ideals*. It's clear that $L(G) \subseteq T(G)$.
- The set of all identities of a vector space E is a L-ideal of $F\langle X\rangle$. We will denote such a L-ideal by $L(E)$.
- The converse is also true: every L-ideal of the algebra $F\langle X\rangle$ is the set of identities of some L-space.

- Let $F\langle X\rangle$ be a free associative algebra and $\varnothing \neq G \subseteq F\langle X\rangle$. By $T(G)$ we denote the T-ideal of the algebra $F\langle X\rangle$ generated by the set G, and by $L(G)$ we denote the ideal of $F\langle X\rangle$ generated by the polynomials of the set G (as an ideal) and closed with respect to substitutions instead of variables of linear combinations of variables. These ideals will be called *L-ideals*. It's clear that $L(G) \subseteq T(G)$.
- The set of all identities of a vector space E is a L-ideal of $F\langle X\rangle$. We will denote such a L-ideal by $L(E)$.
- The converse is also true: every L-ideal of the algebra $F\langle X\rangle$ is the set of identities of some L-space.

- Let $F\langle X\rangle$ be a free associative algebra and $\varnothing \neq G \subseteq F\langle X\rangle$. By $T(G)$ we denote the T-ideal of the algebra $F\langle X\rangle$ generated by the set G, and by $L(G)$ we denote the ideal of $F\langle X\rangle$ generated by the polynomials of the set G (as an ideal) and closed with respect to substitutions instead of variables of linear combinations of variables. These ideals will be called *L-ideals*. It's clear that $L(G) \subseteq T(G)$.
- The set of all identities of a vector space E is a L-ideal of $F\langle X\rangle$. We will denote such a L-ideal by $L(E)$.
- The converse is also true: every L-ideal of the algebra $F\langle X\rangle$ is the set of identities of some L-space.

- The identity f of the L-space E follows from the identities $f_1, f_2, \dots$ if $f \in L(f_1, f_2, \dots)$.
- Thus, for obtaining a corollary from the identity $f(x_1, x_2, \dots, x_n)$, instead of variables $x_1, x_2, \dots, x_n$, only linear combinations of variables can be substituted. If we substitute the product of variables instead of variables $x_1, x_2, \dots, x_n$ in $f(x_1, x_2, \dots, x_n)$, the resulting polynomial may not be an identity of the L-space.
- For example, the space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ satisfies the identity $\mathrm{St}_3(x, y, z) = 0$. However, E_0 does not satisfy the identity $\mathrm{St}_3(xt, y, z) = 0$. Indeed, if $x = e_{11} + e_{12}$, $t = e_{22}$, $y = e_{11} + e_{12}$, $z = e_{22}$ then $\mathrm{St}_3(xt, y, z) = \mathrm{St}_3(e_{12}, e_{11} + e_{12}, e_{22}) = -e_{12} \neq 0$.

- The identity f of the L-space E follows from the identities $f_1, f_2, \dots$ if $f \in L(f_1, f_2, \dots)$.
- Thus, for obtaining a corollary from the identity $f(x_1, x_2, \dots, x_n)$, instead of variables $x_1, x_2, \dots, x_n$, only linear combinations of variables can be substituted. If we substitute the product of variables instead of variables $x_1, x_2, \dots, x_n$ in $f(x_1, x_2, \dots, x_n)$, the resulting polynomial may not be an identity of the L-space.
- For example, the space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ satisfies the identity $\mathrm{St}_3(x, y, z) = 0$. However, E_0 does not satisfy the identity $\mathrm{St}_3(xt, y, z) = 0$. Indeed, if $x = e_{11} + e_{12}$, $t = e_{22}$, $y = e_{11} + e_{12}$, $z = e_{22}$ then $\mathrm{St}_3(xt, y, z) = \mathrm{St}_3(e_{12}, e_{11} + e_{12}, e_{22}) = -e_{12} \neq 0$.

- The identity f of the L-space E follows from the identities $f_1, f_2, \dots$ if $f \in L(f_1, f_2, \dots)$.
- Thus, for obtaining a corollary from the identity $f(x_1, x_2, \dots, x_n)$, instead of variables $x_1, x_2, \dots, x_n$, only linear combinations of variables can be substituted. If we substitute the product of variables instead of variables $x_1, x_2, \dots, x_n$ in $f(x_1, x_2, \dots, x_n)$, the resulting polynomial may not be an identity of the L-space.
- For example, the space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ satisfies the identity $\mathrm{St}_3(x, y, z) = 0$. However, E_0 does not satisfy the identity $\mathrm{St}_3(xt, y, z) = 0$. Indeed, if $x = e_{11} + e_{12}$, $t = e_{22}$, $y = e_{11} + e_{12}$, $z = e_{22}$ then $\mathrm{St}_3(xt, y, z) = \mathrm{St}_3(e_{12}, e_{11} + e_{12}, e_{22}) = -e_{12} \neq 0$.

- If there exists a finite set G of identities of a multiplicative vector space E, from which all the identities of this space follow, then the space E is called a *finitely based L-space* (*FB-space*) with a *basis of identities* G.
- If such a finite set does not exist for the L-space E, then we say that the L-space E is *infinitely based* or *not finitely based* (*NFB-space*).
- FB-algebras and NFB-algebras are similarly defined in other classes of algebras which may be related to Specht's problem [Specht, 1950].

- If there exists a finite set G of identities of a multiplicative vector space E, from which all the identities of this space follow, then the space E is called a *finitely based L-space* (*FB-space*) with a *basis of identities G*.
- If such a finite set does not exist for the L-space E, then we say that the L-space E is *infinitely based* or *not finitely based* (*NFB-space*).
- FB-algebras and NFB-algebras are similarly defined in other classes of algebras which may be related to Specht's problem [Specht, 1950].

- If there exists a finite set G of identities of a multiplicative vector space E, from which all the identities of this space follow, then the space E is called a *finitely based L-space* (*FB-space*) with a *basis of identities* G.
- If such a finite set does not exist for the L-space E, then we say that the L-space E is *infinitely based* or *not finitely based* (*NFB-space*).
- FB-algebras and NFB-algebras are similarly defined in other classes of algebras which may be related to Specht's problem [Specht, 1950].

- The construction of examples of NFB-algebras is an important direction in the study of varieties of algebras.
- There are known examples of NFB-algebras in various classes of algebraic systems:
  - in the class of groupoids [Lyndon, 1954];
  - in the class of semigroups [Perkins, 1969];
  - in the class of rings and linear algebras [Polin, 1976];
  - in the class of loops [Vaughan-Lee, 1979].
- Let us give examples of FB-spaces and NFB-spaces.

- The construction of examples of NFB-algebras is an important direction in the study of varieties of algebras.
- There are known examples of NFB-algebras in various classes of algebraic systems:
  - in the class of groupoids [Lyndon, 1954];
  - in the class of semigroups [Perkins, 1969];
  - in the class of rings and linear algebras [Polin, 1976];
  - in the class of loops [Vaughan-Lee, 1979].
- Let us give examples of FB-spaces and NFB-spaces.

- The construction of examples of NFB-algebras is an important direction in the study of varieties of algebras.
- There are known examples of NFB-algebras in various classes of algebraic systems:
    - in the class of groupoids [Lyndon, 1954];
    - in the class of semigroups [Perkins, 1969];
    - in the class of rings and linear algebras [Polin, 1976];
    - in the class of loops [Vaughan-Lee, 1979].
- Let us give examples of FB-spaces and NFB-spaces.

Proposition 1.1 [I., K., 2010].

Let F be an infinite field of arbitrary characteristic. The L-spaces $A_1 = \langle e_{11}, e_{12}\rangle_F$ and $A_2 = \langle e_{11}, e_{21}\rangle_F$ over the field F is an FB-spaces with a bases of identities $\{[x,y]z\}$ and $\{x[y,z]\}$ respectively.

Theorem 1.1 [I., K., 2010].

Let F be an infinite field of arbitrary characteristic. The vector space $A = A_1 \oplus A_2$ over the field F is an NFB-space with a basis of identities:

$$\{\mathrm{St}_3(x,y,z), x[y,u]v, [x,y][u,v], [x,y]z_1z_2\dots z_m[u,v] | m = 1,2,\dots\}.$$

- Note that the space $A = A_1 \oplus A_2$ is an NFB-space over an arbitrary infinite field. However, any linear associative algebra over a field of characteristic zero has a finite basis of its identities [Kemer, 1987].

Proposition 1.1 [I., K., 2010].

Let F be an infinite field of arbitrary characteristic. The L-spaces $A_1 = \langle e_{11}, e_{12}\rangle_F$ and $A_2 = \langle e_{11}, e_{21}\rangle_F$ over the field F is an FB-spaces with a bases of identities $\{[x,y]z\}$ and $\{x[y,z]\}$ respectively.

Theorem 1.1 [I., K., 2010].

Let F be an infinite field of arbitrary characteristic. The vector space $A = A_1 \oplus A_2$ over the field F is an NFB-space with a basis of identities:

$$\{\mathrm{St}_3(x,y,z), x[y,u]v, [x,y][u,v], [x,y]z_1z_2\ldots z_m[u,v] | m = 1, 2, \ldots\}.$$

- Note that the space $A = A_1 \oplus A_2$ is an NFB-space over an arbitrary infinite field. However, any linear associative algebra over a field of characteristic zero has a finite basis of its identities [Kemer, 1987].

Proposition 1.1 [I., K., 2010].

Let F be an infinite field of arbitrary characteristic. The L-spaces $A_1 = \langle e_{11}, e_{12}\rangle_F$ and $A_2 = \langle e_{11}, e_{21}\rangle_F$ over the field F is an FB-spaces with a bases of identities $\{[x,y]z\}$ and $\{x[y,z]\}$ respectively.

Theorem 1.1 [I., K., 2010].

Let F be an infinite field of arbitrary characteristic. The vector space $A = A_1 \oplus A_2$ over the field F is an NFB-space with a basis of identities:

$$\{\mathrm{St}_3(x,y,z), x[y,u]v, [x,y][u,v], [x,y]z_1z_2\ldots z_m[u,v] | m = 1,2,\ldots\}.$$

- Note that the space $A = A_1 \oplus A_2$ is an NFB-space over an arbitrary infinite field. However, any linear associative algebra over a field of characteristic zero has a finite basis of its identities [Kemer, 1987].

Proposition 1.2 [I., K., 2010].

Let $F = GF(q)$ be an finite field of q elements. The L-spaces $A_1 = \langle e_{11}, e_{12}\rangle_F$ and $A_2 = \langle e_{11}, e_{21}\rangle_F$ over the field F is an FB-spaces with a bases of identities $\{[x,y]z, (x^q - x)y\}$ and $\{x[y,z], x(y^q - y)\}$ respectively.

Theorem 1.2 [I., K., 2010].

Let $F = GF(q)$ be an finite field of q elements. The vector space $A = A_1 \oplus A_2$ over the field F is an NFB-space with a basis of identities:

$$\{\mathrm{St}_3(x,y,z), x[y,u]v, [x,y][u,v], x(y-y^q)z, (x-x^q)(y-y^q), [x,y](z-z^q), (x-x^q)[y,z], [x,y]z_1z_2\ldots z_m[u,v] | m = 1,2,\ldots\}.$$

Proposition 1.2 [I., K., 2010].

Let $F = GF(q)$ be an finite field of q elements. The L-spaces $A_1 = \langle e_{11}, e_{12} \rangle_F$ and $A_2 = \langle e_{11}, e_{21} \rangle_F$ over the field F is an FB-spaces with a bases of identities $\{[x, y]z, (x^q - x)y\}$ and $\{x[y, z], x(y^q - y)\}$ respectively.

Theorem 1.2 [I., K., 2010].

Let $F = GF(q)$ be an finite field of q elements. The vector space $A = A_1 \oplus A_2$ over the field F is an NFB-space with a basis of identities:

$$\{\mathrm{St}_3(x, y, z), x[y, u]v, [x, y][u, v], x(y - y^q)z, (x - x^q)(y - y^q), [x, y](z - z^q),$$
$$(x - x^q)[y, z], [x, y]z_1 z_2 \ldots z_m[u, v] | m = 1, 2, \ldots\}.$$

Theorem 1.3 [I., K., 2010].

Let F be an infinite field of arbitrary characteristic. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an NFB-space with a basis of identities:

$$\{[x,y][u,v],[x,y]z_1z_2\dots z_m[u,v]|m=1,2,\dots\}.$$

Theorem 1.4 [I., 1989].

Let $F=GF(q)$ be an finite field of q elements. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an NFB-space with a basis of identities:

$$G=\{[x,y][u,v],(x-x^q)(y-y^q),[x,y](z-z^q),\\ (x-x^q)[y,z],[x,y]z_1z_2\dots z_m[u,v]|m=1,2,\dots\}.$$

Theorem 1.3 [I., K., 2010].

Let F be an infinite field of arbitrary characteristic. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an NFB-space with a basis of identities:

$$\{[x,y][u,v],[x,y]z_1z_2\dots z_m[u,v]|m=1,2,\dots\}.$$

Theorem 1.4 [I., 1989].

Let $F=GF(q)$ be an finite field of q elements. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an NFB-space with a basis of identities:

$$G=\{[x,y][u,v],(x-x^q)(y-y^q),[x,y](z-z^q),\\ (x-x^q)[y,z],[x,y]z_1z_2\dots z_m[u,v]|m=1,2,\dots\}.$$

Theorem 1.5 [I., K., 2011].

Let F be a field of characteristic zero. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ is an NFB-space with a basis of identities:

$$\{\mathrm{St}_3(x,y,z), [x,y][u,v], [x,y]z_1z_2\dots z_m[u,v] | m = 1,2,\dots\}.$$

Theorem 1.6 [I., K., 2015].

Let $F = GF(q)$. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ over the field F is an NFB-space with a basis of identities:

$$\{x^{q^2-q+1} - x, \mathrm{St}_3(x,y,z), [x,y][u,v], (x-x^q)(y-y^q), [x,y](z-z^q), (x-x^q)[y,z], [x,y]z_1z_2\dots z_k[u,v] | k = 1,2,\dots\}.$$

- Note that the space E_0 is not an F-algebra.

Theorem 1.5 [I., K., 2011].

Let F be a field of characteristic zero. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ is an NFB-space with a basis of identities:

$$\{\mathrm{St}_3(x, y, z), [x, y][u, v], [x, y]z_1 z_2 \ldots z_m[u, v] | m = 1, 2, \ldots\}.$$

Theorem 1.6 [I., K., 2015].

Let $F = GF(q)$. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ over the field F is an NFB-space with a basis of identities:

$$\{x^{q^2-q+1} - x, \mathrm{St}_3(x, y, z), [x, y][u, v], (x - x^q)(y - y^q),$$
$$[x, y](z - z^q), (x - x^q)[y, z], [x, y]z_1 z_2 \ldots z_k[u, v] | k = 1, 2, \ldots\}.$$

- Note that the space E_0 is not an F-algebra.

Theorem 1.5 [I., K., 2011].

Let F be a field of characteristic zero. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ is an NFB-space with a basis of identities:

$$\{\mathrm{St}_3(x, y, z), [x, y][u, v], [x, y]z_1 z_2 \dots z_m[u, v] | m = 1, 2, \dots\}.$$

Theorem 1.6 [I., K., 2015].

Let $F = GF(q)$. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ over the field F is an NFB-space with a basis of identities:

$$\{x^{q^2-q+1} - x, \mathrm{St}_3(x, y, z), [x, y][u, v], (x - x^q)(y - y^q),$$
$$[x, y](z - z^q), (x - x^q)[y, z], [x, y]z_1 z_2 \dots z_k[u, v] | k = 1, 2, \dots\}.$$

- Note that the space E_0 is not an F-algebra.

- A finite NFB-algebra (of arbitrary signature) is called *inherently nonfinitely based* (*INFB-algebra*) if any locally finite variety containing this algebra does not have a finite basis of identities. Sometimes inherently nonfinitely based algebras are called *essentially nonfinitely based*.
- Any finite algebra containing an INFB-algebra as a subalgebra has no finite basis of identities. Thus, if we construct an example of INFB-algebra, then a series of examples of NFB-algebras is automatically constructed.

- A finite NFB-algebra (of arbitrary signature) is called *inherently nonfinitely based* (*INFB-algebra*) if any locally finite variety containing this algebra does not have a finite basis of identities. Sometimes inherently nonfinitely based algebras are called *essentially nonfinitely based*.
- Any finite algebra containing an INFB-algebra as a subalgebra has no finite basis of identities. Thus, if we construct an example of INFB-algebra, then a series of examples of NFB-algebras is automatically constructed.

- There are known examples of INFB-algebras in various classes of algebraic systems:
  - in the class of groupoids [Lyndon, 1954];
  - in the class of semigroups [Sapir, 1987];
  - in the class of rings and linear algebras [I., 1989].
- In 1987 M. V. Sapir gave a complete description of INFB-semigroups.
- Let us give examples of INFB-spaces.

- There are known examples of INFB-algebras in various classes of algebraic systems:
  - in the class of groupoids [Lyndon, 1954];
  - in the class of semigroups [Sapir, 1987];
  - in the class of rings and linear algebras [I., 1989].
- In 1987 M. V. Sapir gave a complete description of INFB-semigroups.
- Let us give examples of INFB-spaces.

- There are known examples of INFB-algebras in various classes of algebraic systems:
  - in the class of groupoids [Lyndon, 1954];
  - in the class of semigroups [Sapir, 1987];
  - in the class of rings and linear algebras [I., 1989].
- In 1987 M. V. Sapir gave a complete description of INFB-semigroups.
- Let us give examples of INFB-spaces.

Theorem 2.1 [I., K., 2013].

Let $F = GF(q)$ be an finite field of q elements. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an INFB-space.

- It follows from this theorem that the multiplicative vector space of matrices of any order over a finite field has no finite basis of identities.
- However, inherently nonfinitely based algebras and multiplicative vector spaces can be considered only for algebras and L-spaces over a finite field.

Theorem 2.1 [I., K., 2013].

Let $F = GF(q)$ be an finite field of q elements. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an INFB-space.

- It follows from this theorem that the multiplicative vector space of matrices of any order over a finite field has no finite basis of identities.
- However, inherently nonfinitely based algebras and multiplicative vector spaces can be considered only for algebras and L-spaces over a finite field.

Theorem 2.1 [I., K., 2013].

Let $F = GF(q)$ be an finite field of q elements. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an INFB-space.

- It follows from this theorem that the multiplicative vector space of matrices of any order over a finite field has no finite basis of identities.
- However, inherently nonfinitely based algebras and multiplicative vector spaces can be considered only for algebras and L-spaces over a finite field.

- Consider a monomial $w = w(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_k) \in F\langle X\rangle$ that is linear in each of the variables $x_1, x_2, \ldots, x_n$. Let $C_n^{(w)} = 0$ be the Capelli identity and $\mathrm{Cap}(n) = \mathrm{Var}\langle C_n^{(w)} = 0\rangle$ the variety of linear algebras satisfying all possible Capelli identities for a fixed n.
- A variety $\mathfrak{M}$ of linear algebras over a field F is called *strongly nonfinitely based* (*SNFB-variety*) if $\mathfrak{M} \subseteq \mathrm{Cap}(k)$ for some k and any variety of F-algebras containing $\mathfrak{M}$ and contained in $\mathrm{Cap}(n)$ for some n is NFB-variety.

- Consider a monomial $w = w(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_k) \in F\langle X\rangle$ that is linear in each of the variables $x_1, x_2, \ldots, x_n$. Let $C_n^{(w)} = 0$ be the Capelli identity and $\mathrm{Cap}(n) = \mathrm{Var}\langle C_n^{(w)} = 0\rangle$ the variety of linear algebras satisfying all possible Capelli identities for a fixed n.
- A variety $\mathfrak{M}$ of linear algebras over a field F is called *strongly nonfinitely based* (*SNFB-variety*) if $\mathfrak{M} \subseteq \mathrm{Cap}(k)$ for some k and any variety of F-algebras containing $\mathfrak{M}$ and contained in $\mathrm{Cap}(n)$ for some n is NFB-variety.

- The algebra generating the SNFB-variety of algebras will be called the *SNFB-algebra*.
- Any finite dimensional algebra containing an SNFB-algebra as a subalgebra has no finite basis of identities.
- Strongly nonfinitely based multiplicative vector spaces can be defined similarly.
- Let us give examples of SNFB-spaces.

- The algebra generating the SNFB-variety of algebras will be called the *SNFB-algebra*.
- Any finite dimensional algebra containing an SNFB-algebra as a subalgebra has no finite basis of identities.
- Strongly nonfinitely based multiplicative vector spaces can be defined similarly.
- Let us give examples of SNFB-spaces.

- The algebra generating the SNFB-variety of algebras will be called the *SNFB-algebra*.
- Any finite dimensional algebra containing an SNFB-algebra as a subalgebra has no finite basis of identities.
- Strongly nonfinitely based multiplicative vector spaces can be defined similarly.
- Let us give examples of SNFB-spaces.

- The algebra generating the SNFB-variety of algebras will be called the *SNFB-algebra*.
- Any finite dimensional algebra containing an SNFB-algebra as a subalgebra has no finite basis of identities.
- Strongly nonfinitely based multiplicative vector spaces can be defined similarly.
- Let us give examples of SNFB-spaces.

Theorem 2.2 [I., K., 2013].

Let F be an arbitrary field. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an SNFB-space.

Theorem 2.3 [I., K., 2015].

Let F be an field of characteristic zero. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ over the field F is an SNFB-space.

Theorem 2.4 [I., K., 2013].

Let F be an arbitrary field. The multiplicative vector space $A = A_1 \oplus A_2$ over the field F is an NFB space, but it is not an SNFB-space.

Theorem 2.2 [I., K., 2013].

Let F be an arbitrary field. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an SNFB-space.

Theorem 2.3 [I., K., 2015].

Let F be an field of characteristic zero. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ over the field F is an SNFB-space.

Theorem 2.4 [I., K., 2013].

Let F be an arbitrary field. The multiplicative vector space $A = A_1 \oplus A_2$ over the field F is an NFB space, but it is not an SNFB-space.

Theorem 2.2 [I., K., 2013].

Let F be an arbitrary field. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an SNFB-space.

Theorem 2.3 [I., K., 2015].

Let F be an field of characteristic zero. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ over the field F is an SNFB-space.

Theorem 2.4 [I., K., 2013].

Let F be an arbitrary field. The multiplicative vector space $A = A_1 \oplus A_2$ over the field F is an NFB space, but it is not an SNFB-space.

- Let $G \subseteq F\langle X \rangle$. The class of all multiplicative vector pairs of the form (A, E) satisfying all the identities of the set G is called an *L-variety defined by the set of identities* G and is denoted by $\mathrm{Var}_L\langle g = 0 \,|\, g \in G \rangle$.
- If G is a basis of identities in the space E, then the L-variety $\mathrm{Var}_L\langle g = 0 \,|\, g \in G \rangle$ is denoted by $\mathrm{Var}_L E$ and called the *L-varieties generated by the space E*.
- If we need information about the enveloping algebra A of the space E, then $\mathrm{Var}_L\langle g = 0 \,|\, g \in G \rangle$ is denoted as $\mathrm{Var}_L(A, E)$ and called the *L-varieties generated by the pair* (A, E).
- Using the concept of an L-variety, it was possible to partially describe the FB-space and SNFB-spaces.

- Let $G \subseteq F\langle X \rangle$. The class of all multiplicative vector pairs of the form (A, E) satisfying all the identities of the set G is called an *L-variety defined by the set of identities* G and is denoted by $\mathrm{Var}_L\langle g = 0 \,|\, g \in G \rangle$.
- If G is a basis of identities in the space E, then the L-variety $\mathrm{Var}_L\langle g = 0 \,|\, g \in G \rangle$ is denoted by $\mathrm{Var}_L E$ and called the *L-varieties generated by the space E.*
- If we need information about the enveloping algebra A of the space E, then $\mathrm{Var}_L\langle g = 0 \,|\, g \in G \rangle$ is denoted as $\mathrm{Var}_L(A, E)$ and called the *L-varieties generated by the pair* (A, E).
- Using the concept of an L-variety, it was possible to partially describe the FB-space and SNFB-spaces.

- Let $G \subseteq F\langle X\rangle$. The class of all multiplicative vector pairs of the form (A, E) satisfying all the identities of the set G is called an *L-variety defined by the set of identities* G and is denoted by $\mathrm{Var}_L\langle g = 0 \mid g \in G\rangle$.
- If G is a basis of identities in the space E, then the L-variety $\mathrm{Var}_L\langle g = 0 \mid g \in G\rangle$ is denoted by $\mathrm{Var}_L E$ and called the *L-varieties generated by the space E.*
- If we need information about the enveloping algebra A of the space E, then $\mathrm{Var}_L\langle g = 0 \mid g \in G\rangle$ is denoted as $\mathrm{Var}_L(A, E)$ and called the *L-varieties generated by the pair* (A, E).
- Using the concept of an L-variety, it was possible to partially describe the FB-space and SNFB-spaces.

- Let $G \subseteq F\langle X\rangle$. The class of all multiplicative vector pairs of the form (A, E) satisfying all the identities of the set G is called an *L-variety defined by the set of identities* G and is denoted by $\mathrm{Var}_L\langle g = 0 \,|\, g \in G\rangle$.
- If G is a basis of identities in the space E, then the L-variety $\mathrm{Var}_L\langle g = 0 \,|\, g \in G\rangle$ is denoted by $\mathrm{Var}_L E$ and called the *L-varieties generated by the space E.*
- If we need information about the enveloping algebra A of the space E, then $\mathrm{Var}_L\langle g = 0 \,|\, g \in G\rangle$ is denoted as $\mathrm{Var}_L(A, E)$ and called the *L-varieties generated by the pair* (A, E).
- Using the concept of an L-variety, it was possible to partially describe the FB-space and SNFB-spaces.

Theorem 3.1 [K., 2015].

Let F be a field of characteristic zero, A a finite dimensional L-space over the field F, which is also an F-algebra with a unity element. An L-space A has a finite basis of identities iff $T_2(F) \notin \mathrm{Var}_L A$.

Theorem 3.2 [K., 2015].

Let F be a field of characteristic zero, A a finite dimensional L-space over the field F, which is also an F-algebra with a unity element. An L-space A is strongly infinitely based iff $T_2(F) \in \mathrm{Var}_L A$.

Theorem 3.1 [K., 2015].

Let F be a field of characteristic zero, A a finite dimensional L-space over the field F, which is also an F-algebra with a unity element. An L-space A has a finite basis of identities iff $T_2(F) \notin \mathrm{Var}_L A$.

Theorem 3.2 [K., 2015].

Let F be a field of characteristic zero, A a finite dimensional L-space over the field F, which is also an F-algebra with a unity element. An L-space A is strongly infinitely based iff $T_2(F) \in \mathrm{Var}_L A$.

- It can be shown that any class of multiplicative vector pairs is an L-variety iff it is closed with respect to taking subpairs, homomorphic images of pairs, and direct products of pairs, that is, for L-varieties an analogue of Birkhof's theorem holds.
- In the study of L-varieties we can pose the same problems as for varieties of linear algebras.

- It can be shown that any class of multiplicative vector pairs is an L-variety iff it is closed with respect to taking subpairs, homomorphic images of pairs, and direct products of pairs, that is, for L-varieties an analogue of Birkhof's theorem holds.
- In the study of L-varieties we can pose the same problems as for varieties of linear algebras.

- An L-variety $\mathcal{M}$ is called *Specht* if any pair $(A, E) \in \mathcal{M}$ has a finite basis of identities.
- The *union* of the L-varieties $\mathcal{M}_1$ and $\mathcal{M}_2$ is the smallest L-variety containing $\mathcal{M}_1$ and $\mathcal{M}_2$.
- Let F is an arbitrary field, $A_1 = \langle e_{11}, e_{21} \rangle_F$, $A_2 = \langle e_{11}, e_{12} \rangle_F$ is the vector spaces over field F, $\mathcal{M}_1 = \mathrm{Var}_L A_1$, $\mathcal{M}_2 = \mathrm{Var}_L A_2$, $\mathcal{M} = \mathrm{Var}_L (A_1 \oplus A_2)$. It's obvious that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$.

- An L-variety $\mathcal{M}$ is called *Specht* if any pair $(A, E) \in \mathcal{M}$ has a finite basis of identities.
- The *union* of the L-varieties $\mathcal{M}_1$ and $\mathcal{M}_2$ is the smallest L-variety containing $\mathcal{M}_1$ and $\mathcal{M}_2$.
- Let F is an arbitrary field, $A_1 = \langle e_{11}, e_{21} \rangle_F$, $A_2 = \langle e_{11}, e_{12} \rangle_F$ is the vector spaces over field F, $\mathcal{M}_1 = \mathrm{Var}_L A_1$, $\mathcal{M}_2 = \mathrm{Var}_L A_2$, $\mathcal{M} = \mathrm{Var}_L(A_1 \oplus A_2)$. It's obvious that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$.

- An L-variety $\mathcal{M}$ is called *Specht* if any pair $(A, E) \in \mathcal{M}$ has a finite basis of identities.
- The *union* of the L-varieties $\mathcal{M}_1$ and $\mathcal{M}_2$ is the smallest L-variety containing $\mathcal{M}_1$ and $\mathcal{M}_2$.
- Let F is an arbitrary field, $A_1 = \langle e_{11}, e_{21} \rangle_F$, $A_2 = \langle e_{11}, e_{12} \rangle_F$ is the vector spaces over field F, $\mathcal{M}_1 = \mathrm{Var}_L A_1$, $\mathcal{M}_2 = \mathrm{Var}_L A_2$, $\mathcal{M} = \mathrm{Var}_L(A_1 \oplus A_2)$. It's obvious that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$.

Theorem 3.3 [K., 2018].

Let F be an arbitrary field. An infinitely based L-variety $\mathcal{M}$ is the union of the Specht L-varieties $\mathcal{M}_1$ and $\mathcal{M}_2$.

Corollary 3.1.

Let F be an infinite field. An arbitrary L-space over the field F satisfying either the identity $[x, y]z = 0$ or the identity $x[y, z] = 0$ has a finite basis of identities.

Corollary 3.2.

Let $F = GF(q)$. An arbitrary L-space over the field F satisfying either the identities $[x, y]z = 0$ and $(x^q - x)y = 0$ or the identities $x[y, z] = 0$ and $x(y^q - y) = 0$ has a finite basis of identities.

Theorem 3.3 [K., 2018].

Let F be an arbitrary field. An infinitely based L-variety $\mathcal{M}$ is the union of the Specht L-varieties $\mathcal{M}_1$ and $\mathcal{M}_2$.

Corollary 3.1.

Let F be an infinite field. An arbitrary L-space over the field F satisfying either the identity $[x,y]z = 0$ or the identity $x[y,z] = 0$ has a finite basis of identities.

Corollary 3.2.

Let $F = GF(q)$. An arbitrary L-space over the field F satisfying either the identities $[x,y]z = 0$ and $(x^q - x)y = 0$ or the identities $x[y,z] = 0$ and $x(y^q - y) = 0$ has a finite basis of identities.

Theorem 3.3 [K., 2018].

Let F be an arbitrary field. An infinitely based L-variety $\mathcal{M}$ is the union of the Specht L-varieties $\mathcal{M}_1$ and $\mathcal{M}_2$.

Corollary 3.1.

Let F be an infinite field. An arbitrary L-space over the field F satisfying either the identity $[x, y]z = 0$ or the identity $x[y, z] = 0$ has a finite basis of identities.

Corollary 3.2.

Let $F = GF(q)$. An arbitrary L-space over the field F satisfying either the identities $[x, y]z = 0$ and $(x^q - x)y = 0$ or the identities $x[y, z] = 0$ and $x(y^q - y) = 0$ has a finite basis of identities.

- An L-variety $\mathcal{M}$ is called *locally finite* if for any pair $(A, E) \in \mathcal{M}$, where E is a finite dimensional vector space, the algebra A is finite.
- A pair (A, E) will be called *critical* if the algebra A is finite and the pair (A, E) does not belong to the L-variety generated by its proper factors.
- Any locally finite L-variety is generated by its critical pairs.
- Critical rings and algebras are well studied. They are an effective means for the study of varieties of rings and linear algebras.

- An L-variety $\mathcal{M}$ is called *locally finite* if for any pair $(A, E) \in \mathcal{M}$, where E is a finite dimensional vector space, the algebra A is finite.
- A pair (A, E) will be called *critical* if the algebra A is finite and the pair (A, E) does not belong to the L-variety generated by its proper factors.
- Any locally finite L-variety is generated by its critical pairs.
- Critical rings and algebras are well studied. They are an effective means for the study of varieties of rings and linear algebras.

- An L-variety $\mathcal{M}$ is called *locally finite* if for any pair $(A, E) \in \mathcal{M}$, where E is a finite dimensional vector space, the algebra A is finite.
- A pair (A, E) will be called *critical* if the algebra A is finite and the pair (A, E) does not belong to the L-variety generated by its proper factors.
- Any locally finite L-variety is generated by its critical pairs.
- Critical rings and algebras are well studied. They are an effective means for the study of varieties of rings and linear algebras.

- An L-variety $\mathcal{M}$ is called *locally finite* if for any pair $(A, E) \in \mathcal{M}$, where E is a finite dimensional vector space, the algebra A is finite.
- A pair (A, E) will be called *critical* if the algebra A is finite and the pair (A, E) does not belong to the L-variety generated by its proper factors.
- Any locally finite L-variety is generated by its critical pairs.
- Critical rings and algebras are well studied. They are an effective means for the study of varieties of rings and linear algebras.

Theorem 3.4 [I., Unpublished].

Let $M_2(F)$ be the algebra of second-order matrices over a finite field $F = GF(q)$. Pair $(M_2(F), M_2(F))$ is critical.

Theorem 3.5 [I., K., Unpublished].

Let $T_2(F)$ be the algebra of upper triangular matrices of second-order over a finite field $F = GF(q)$. Pair $(T_2(F), T_2(F))$ is critical.

Theorem 3.6 [I., K, Unpublished].

Let $F = GF(q)$, $A_1 = \langle e_{11}, e_{12} \rangle_F$, $A_2 = \langle e_{11}, e_{21} \rangle_F$. Pairs (A_1, A_1) and (A_2, A_2) are critical.

Theorem 3.4 [I., Unpublished].

Let $M_2(F)$ be the algebra of second-order matrices over a finite field $F = GF(q)$. Pair $(M_2(F), M_2(F))$ is critical.

Theorem 3.5 [I., K., Unpublished].

Let $T_2(F)$ be the algebra of upper triangular matrices of second-order over a finite field $F = GF(q)$. Pair $(T_2(F), T_2(F))$ is critical.

Theorem 3.6 [I., K, Unpublished].

Let $F = GF(q)$, $A_1 = \langle e_{11}, e_{12}\rangle_F$, $A_2 = \langle e_{11}, e_{21}\rangle_F$. Pairs (A_1, A_1) and (A_2, A_2) are critical.

Theorem 3.4 [I., Unpublished].

Let $M_2(F)$ be the algebra of second-order matrices over a finite field $F = GF(q)$. Pair $(M_2(F), M_2(F))$ is critical.

Theorem 3.5 [I., K., Unpublished].

Let $T_2(F)$ be the algebra of upper triangular matrices of second-order over a finite field $F = GF(q)$. Pair $(T_2(F), T_2(F))$ is critical.

Theorem 3.6 [I., K, Unpublished].

Let $F = GF(q)$, $A_1 = \langle e_{11}, e_{12}\rangle_F$, $A_2 = \langle e_{11}, e_{21}\rangle_F$. Pairs (A_1, A_1) and (A_2, A_2) are critical.

- A variety $\mathfrak{M}$ of linear algebras is called a *almost θ-variety* if $\mathfrak{M}$ does not satisfy the θ property, but any proper subvariety of the variety $\mathfrak{M}$ satisfies this property.
- Almost θ-varieties of algebras play an important role in the study of varieties of linear algebras.
- In the theory of associative rings and linear algebras, almost θ-varieties are related to the indicator characterization of varieties.

- A variety $\mathfrak{M}$ of linear algebras is called a *almost θ-variety* if $\mathfrak{M}$ does not satisfy the θ property, but any proper subvariety of the variety $\mathfrak{M}$ satisfies this property.
- Almost θ-varieties of algebras play an important role in the study of varieties of linear algebras.
- In the theory of associative rings and linear algebras, almost θ-varieties are related to the indicator characterization of varieties.

- A variety $\mathfrak{M}$ of linear algebras is called a *almost θ-variety* if $\mathfrak{M}$ does not satisfy the θ property, but any proper subvariety of the variety $\mathfrak{M}$ satisfies this property.
- Almost θ-varieties of algebras play an important role in the study of varieties of linear algebras.
- In the theory of associative rings and linear algebras, almost θ-varieties are related to the indicator characterization of varieties.

- If the property θ is a concrete identity, then in the class of associative rings and linear algebras there are descriptions of almost θ-varieties:
  - almost commutative varieties of rings [Maltsev, 1976];
  - almost commutative varieties of Φ-algebras, where Φ is a Noetherian commutative Jacobson ring with unity element [Maltsev, 1996];
  - almost Engel varieties of the linear algebras [Finogenova, 2004];
  - almost permutative varieties of algebras over an infinite field [Finogenova, 2013].
- Description problems can be formulated for almost θ-varieties of L-spaces.

- If the property θ is a concrete identity, then in the class of associative rings and linear algebras there are descriptions of almost θ-varieties:
  - almost commutative varieties of rings [Maltsev, 1976];
  - almost commutative varieties of Φ-algebras, where Φ is a Noetherian commutative Jacobson ring with unity element [Maltsev, 1996];
  - almost Engel varieties of the linear algebras [Finogenova, 2004];
  - almost permutative varieties of algebras over an infinite field [Finogenova, 2013].
- Description problems can be formulated for almost θ-varieties of L-spaces.

Theorem 3.7 [K., 2018].

Let F be a finite field and $\mathcal{M}$ a nonnilpotent L-variety generated by an F-algebra considered as a vector space. Then an L-variety $\mathcal{M}$ is almost commutative if and only if it is generated by one of the following spaces:

$$A_1 = \langle e_{11}, e_{12} \rangle_F, \qquad A_2 = \langle e_{11}, e_{21} \rangle_F$$

Corollary 3.3.

Let

$$A_3 = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \middle| a, b \in P \right\},$$

where $\sigma \in \mathrm{Aut} P$, $\sigma \neq 1$ and the field of invariants P^σ is the only maximal subfield of P containing F. L-variety $\mathcal{M}_3 = \mathrm{Var}_L A_3$ contains a noncommutative proper L-subvariety.

Theorem 3.7 [K., 2018].

Let F be a finite field and $\mathcal{M}$ a nonnilpotent L-variety generated by an F-algebra considered as a vector space. Then an L-variety $\mathcal{M}$ is almost commutative if and only if it is generated by one of the following spaces:

$$A_1 = \langle e_{11}, e_{12} \rangle_F, \qquad A_2 = \langle e_{11}, e_{21} \rangle_F$$

Corollary 3.3.

Let

$$A_3 = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \middle| a, b \in P \right\},$$

where $\sigma \in \mathrm{Aut} P$, $\sigma \neq 1$ and the field of invariants P^σ is the only maximal subfield of P containing F. L-variety $\mathcal{M}_3 = \mathrm{Var}_L A_3$ contains a noncommutative proper L-subvariety.

Theorem 3.8 [K., Unpublished].

Let F be a arbitrary field and $\mathcal{M}$ a L-variety generated by an F-algebra considered as a vector space. Then an L-variety $\mathcal{M}$ is almost Engel if and only if it is generated by one of the following spaces:

$$A_1 = \langle e_{11}, e_{12} \rangle_F, \qquad A_2 = \langle e_{11}, e_{21} \rangle_F$$

Corollary 3.4.

Let

$$A_3 = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \middle| a, b \in P \right\},$$

where $\sigma \in \mathrm{Aut}P$, $\sigma \neq 1$ and the field of invariants P^σ is the only maximal subfield of P containing F. L-variety $\mathcal{M}_3 = \mathrm{Var}_L A_3$ contains a non-Engel proper L-subvariety.

Theorem 3.8 [K., Unpublished].

Let F be a arbitrary field and $\mathcal{M}$ a L-variety generated by an F-algebra considered as a vector space. Then an L-variety $\mathcal{M}$ is almost Engel if and only if it is generated by one of the following spaces:

$$A_1 = \langle e_{11}, e_{12} \rangle_F, \qquad A_2 = \langle e_{11}, e_{21} \rangle_F$$

Corollary 3.4.

Let

$$A_3 = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \middle| \, a, b \in P \right\},$$

where $\sigma \in \mathrm{Aut}P$, $\sigma \neq 1$ and the field of invariants P^σ is the only maximal subfield of P containing F. L-variety $\mathcal{M}_3 = \mathrm{Var}_L A_3$ contains a non-Engel proper L-subvariety.

- An L-variety $\mathcal{M}$ is called a *minimal nonzero L-variety* (with respect to the inclusion) or an *atom* if for any L-variety $\mathcal{N}$ it follows from inclusion $\mathcal{N} \subseteq \mathcal{M}$ that either $\mathcal{M} = \mathcal{N}$ or $\mathcal{N}$ is the zero L-variety.
- In 1956 A. Tarski showed that atoms in the class of rings are generated either by a simple field $GF(p)$ or by a ring with zero multiplication.
- The problem of describing atoms in the class of L-spaces is of interest for study.

- An L-variety $\mathcal{M}$ is called a *minimal nonzero L-variety* (with respect to the inclusion) or an *atom* if for any L-variety $\mathcal{N}$ it follows from inclusion $\mathcal{N} \subseteq \mathcal{M}$ that either $\mathcal{M} = \mathcal{N}$ or $\mathcal{N}$ is the zero L-variety.
- In 1956 A. Tarski showed that atoms in the class of rings are generated either by a simple field $GF(p)$ or by a ring with zero multiplication.
- The problem of describing atoms in the class of L-spaces is of interest for study.

- An L-variety $\mathcal{M}$ is called a *minimal nonzero L-variety* (with respect to the inclusion) or an *atom* if for any L-variety $\mathcal{N}$ it follows from inclusion $\mathcal{N} \subseteq \mathcal{M}$ that either $\mathcal{M} = \mathcal{N}$ or $\mathcal{N}$ is the zero L-variety.
- In 1956 A. Tarski showed that atoms in the class of rings are generated either by a simple field $GF(p)$ or by a ring with zero multiplication.
- The problem of describing atoms in the class of L-spaces is of interest for study.

Theorem 3.9 [K., Unpublished].

An L-variety of multiplicative vector spaces over a field $GF(2)$ is an atom iff it coincides with either $\mathcal{M}_0$, or $\mathcal{M}_1$, or $\mathcal{M}_{p(x)}$, where

$$\mathcal{M}_0 = \mathrm{Var}_L\langle xy = 0\rangle,$$
$$\mathcal{M}_1 = \mathrm{Var}_L\langle [x, y] = 0, x + x^2 = 0\rangle,$$
$$\mathcal{M}_{p(x)} = \mathrm{Var}_L\langle [x, y] = 0, x^2 y = xy^2, x \cdot p(x) = 0\rangle,$$

$p(x)$ is an irreducible polynomial over the field $GF(2)$.

- Note that the identities of the L-variety $\mathcal{M}_{p(x)}$ do not define an atom in the class of linear algebras over the field $GF(q)$.

Theorem 3.9 [K., Unpublished].

An L-variety of multiplicative vector spaces over a field $GF(2)$ is an atom iff it coincides with either $\mathcal{M}_0$, or $\mathcal{M}_1$, or $\mathcal{M}_{p(x)}$, where

$$\mathcal{M}_0 = \mathrm{Var}_L\langle xy = 0\rangle,$$
$$\mathcal{M}_1 = \mathrm{Var}_L\langle [x, y] = 0, x + x^2 = 0\rangle,$$
$$\mathcal{M}_{p(x)} = \mathrm{Var}_L\langle [x, y] = 0, x^2y = xy^2, x \cdot p(x) = 0\rangle,$$

$p(x)$ is an irreducible polynomial over the field $GF(2)$.

- Note that the identities of the L-variety $\mathcal{M}_{p(x)}$ do not define an atom in the class of linear algebras over the field $GF(q)$.

Identities of Nonassociative Linear Algebras

- Let $\mathfrak{P} = \mathrm{Var}\langle x(yz) = 0\rangle$ be the variety of left-nilpotent algebras of index 3. This variety of linear algebras was first considered in 1976 by S. V. Polin.
- Let V be a vector space and E is the (sub)space of linear transformations of the space V. Consider an algebra $\overline{V} = V \oplus E$, nonzero products of basis elements of of this algebra are given by the rule: $v_i e_{ij} = v_j$ for $v_i \in V$, $e_{ij} \in E$. It is easy to see that $\overline{V} \in \mathfrak{P}$.
- As we said earlier, the identities of the algebra $\overline{V} = V \oplus E$ and L-space E are very close related.

- Let $\mathfrak{P} = \mathrm{Var}\langle x(yz) = 0\rangle$ be the variety of left-nilpotent algebras of index 3. This variety of linear algebras was first considered in 1976 by S. V. Polin.
- Let V be a vector space and E is the (sub)space of linear transformations of the space V. Consider an algebra $\overline{V} = V \oplus E$, nonzero products of basis elements of of this algebra are given by the rule: $v_i e_{ij} = v_j$ for $v_i \in V$, $e_{ij} \in E$. It is easy to see that $\overline{V} \in \mathfrak{P}$.
- As we said earlier, the identities of the algebra $\overline{V} = V \oplus E$ and L-space E are very close related.

- Let $\mathfrak{P} = \mathrm{Var}\langle x(yz) = 0\rangle$ be the variety of left-nilpotent algebras of index 3. This variety of linear algebras was first considered in 1976 by S. V. Polin.
- Let V be a vector space and E is the (sub)space of linear transformations of the space V. Consider an algebra $\overline{V} = V \oplus E$, nonzero products of basis elements of of this algebra are given by the rule: $v_i e_{ij} = v_j$ for $v_i \in V$, $e_{ij} \in E$. It is easy to see that $\overline{V} \in \mathfrak{P}$.
- As we said earlier, the identities of the algebra $\overline{V} = V \oplus E$ and L-space E are very close related.

Theorem 4.1 [L'vov, 1978].

The nonassociative polynomial $zf(R_{x_1}, R_{x_2}, \dots, R_{x_n})$ is an identity of the algebra $\overline{V} = V \oplus E$ iff the associative polynomial $f(x_1, x_2, \dots, x_n)$ is an identity of the L-space E.

Corollary 4.1.

Let $G = \{f_i(x_{i_1}, x_{i_2}, \dots, x_{i_k}) | i \in I\} \subseteq F\langle X \rangle$,

$zG = \{zf_i(R_{x_{i_1}}, R_{x_{i_2}}, \dots, R_{x_{i_k}}) | i \in I\}$.

The set zG is a basis of an identities for the algebra $\overline{V} = V \oplus E$ iff the set G is a basis of an identities for the L-space E.

Theorem 4.1 [L'vov, 1978].

The nonassociative polynomial $zf(R_{x_1}, R_{x_2}, \dots, R_{x_n})$ is an identity of the algebra $\overline{V} = V \oplus E$ iff the associative polynomial $f(x_1, x_2, \dots, x_n)$ is an identity of the L-space E.

Corollary 4.1.

Let $G = \{f_i(x_{i_1}, x_{i_2}, \dots, x_{i_k}) | i \in I\} \subseteq F\langle X \rangle$,
$zG = \{zf_i(R_{x_{i_1}}, R_{x_{i_2}}, \dots, R_{x_{i_k}}) | i \in I\}$.
The set zG is a basis of an identities for the algebra $\overline{V} = V \oplus E$ iff the set G is a basis of an identities for the L-space E.

- Using Corollary 4.1 and the results obtained earlier for multiplicative vector spaces and their identities, we can obtain a number of consequences for nonassociative algebras of the form $\overline{V} = V \oplus E$.
- We formulate some corollaries of theorems proved for L-spaces.
- We assume that modulo $x(yz) = 0$ the brackets in arbitrary word $x_1x_2\dots x_k$ are placed according to rule: $x_1x_2\dots x_k = ((\dots((x_1x_2)x_3)\dots)x_{k-1})x_k$ and the writing $zf(x_1, x_2, \dots, x_n)$ is a short form of the writing of $zf(R_{x_1}, R_{x_2}, \dots, R_{x_n})$.

- Using Corollary 4.1 and the results obtained earlier for multiplicative vector spaces and their identities, we can obtain a number of consequences for nonassociative algebras of the form $\overline{V} = V \oplus E$.
- We formulate some corollaries of theorems proved for L-spaces.
- We assume that modulo $x(yz) = 0$ the brackets in arbitrary word $x_1x_2\dots x_k$ are placed according to rule: $x_1x_2\dots x_k = ((\dots((x_1x_2)x_3)\dots)x_{k-1})x_k$ and the writing $zf(x_1, x_2, \dots, x_n)$ is a short form of the writing of $zf(R_{x_1}, R_{x_2}, \dots, R_{x_n})$.

- Using Corollary 4.1 and the results obtained earlier for multiplicative vector spaces and their identities, we can obtain a number of consequences for nonassociative algebras of the form $\overline{V} = V \oplus E$.
- We formulate some corollaries of theorems proved for L-spaces.
- We assume that modulo $x(yz) = 0$ the brackets in arbitrary word $x_1 x_2 \dots x_k$ are placed according to rule: $x_1 x_2 \dots x_k = ((\dots((x_1 x_2)x_3)\dots)x_{k-1})x_k$ and the writing $zf(x_1, x_2, \dots, x_n)$ is a short form of the writing of $zf(R_{x_1}, R_{x_2}, \dots, R_{x_n})$.

Theorem 4.2 [I., K., 2013].

Let F be an infinite field of arbitrary characteristic. The nonassociative algebra $A = \langle v_1, v_2, v_3, v_4\rangle_F \oplus \langle e_{11}, e_{12}, e_{33}, e_{43}\rangle_F$ is an NFB-algebra with a basis of identities:

$$\{x(yz), z\mathrm{St}_3(x,y,t), zx[y,u]v, z[x,y][u,v],\\ z[x,y]z_1z_2\dots z_m[u,v] | m = 1,2,\dots\}.$$

Theorem 4.3 [I., K., 2013].

Let $F = GF(q)$ be an finite field. The nonassociative algebra $A = \langle v_1, v_2, v_3, v_4\rangle_F \oplus \langle e_{11}, e_{12}, e_{33}, e_{43}\rangle_F$ is an NFB-algebra with a basis of identities:

$$\{x(yz), z\mathrm{St}_3(x,y,t), zx[y,u]v, z[x,y][u,v], zx(y-y^q)t, z(x-x^q)(y-y^q),\\ z[x,y](t-t^q), z(x-x^q)[y,t], z[x,y]z_1z_2\dots z_m[u,v] | m = 1,2,\dots\}.$$

Theorem 4.2 [I., K., 2013].

Let F be an infinite field of arbitrary characteristic. The nonassociative algebra $A = \langle v_1, v_2, v_3, v_4\rangle_F \oplus \langle e_{11}, e_{12}, e_{33}, e_{43}\rangle_F$ is an NFB-algebra with a basis of identities:

$$\{x(yz), z\mathrm{St}_3(x,y,t), zx[y,u]v, z[x,y][u,v], z[x,y]z_1z_2\dots z_m[u,v] | m = 1,2,\dots\}.$$

Theorem 4.3 [I., K., 2013].

Let $F = GF(q)$ be an finite field. The nonassociative algebra $A = \langle v_1, v_2, v_3, v_4\rangle_F \oplus \langle e_{11}, e_{12}, e_{33}, e_{43}\rangle_F$ is an NFB-algebra with a basis of identities:

$$\{x(yz), z\mathrm{St}_3(x,y,t), zx[y,u]v, z[x,y][u,v], zx(y-y^q)t, z(x-x^q)(y-y^q), z[x,y](t-t^q), z(x-x^q)[y,t], z[x,y]z_1z_2\dots z_m[u,v] | m = 1,2,\dots\}.$$

Theorem 4.4 [I., K., 2015].

Let F be an infinite field of characteristic zero. The nonassociative algebra $A = \langle v_1, v_2, e_{11} + e_{12}, e_{22} \rangle_F$ is an NFB-algebra with a basis of identities:

$$\{x(yz), z\mathrm{St}_3(x, y, t), z[x, y][u, v], z[x, y]z_1 z_2 \dots z_m[u, v] | m = 1, 2, \dots \}.$$

Theorem 4.5 [I., K., 2015].

Let $F = GF(q)$ be an finite field. The nonassociative algebra $A = \langle v_1, v_2, e_{11} + e_{12}, e_{22} \rangle_F$ is an NFB-algebra with a basis of identities:

$$\{x(yz), z(x^{q^2-q+1} - x), z\mathrm{St}_3(x, y, t), z[x, y][u, v], z(x - x^q)(y - y^q), z[x, y](t - t^q), z(x - x^q)[y, t], z[x, y]z_1 z_2 \dots z_k[u, v] | k = 1, 2, \dots \}.$$

Theorem 4.4 [I., K., 2015].

Let F be an infinite field of characteristic zero. The nonassociative algebra $A = \langle v_1, v_2, e_{11} + e_{12}, e_{22} \rangle_F$ is an NFB-algebra with a basis of identities:

$$\{x(yz), z\mathrm{St}_3(x, y, t), z[x, y][u, v], z[x, y]z_1 z_2 \dots z_m[u, v] | m = 1, 2, \dots\}.$$

Theorem 4.5 [I., K., 2015].

Let $F = GF(q)$ be an finite field. The nonassociative algebra $A = \langle v_1, v_2, e_{11} + e_{12}, e_{22} \rangle_F$ is an NFB-algebra with a basis of identities:

$$\{x(yz), z(x^{q^2-q+1} - x), z\mathrm{St}_3(x, y, t), z[x, y][u, v], z(x - x^q)(y - y^q), z[x, y](t - t^q), z(x - x^q)[y, t], z[x, y]z_1 z_2 \dots z_k[u, v] | k = 1, 2, \dots\}.$$

- The nonassociative algebras constructed in Theorems 4.4 and 4.5 are examples of four-dimensional NFB-algebras. Examples of five-dimensional NFB-algebras were previously known [Maltsev, Parfenov, 1977; L'vov, 1978].
- If we put $F = GF(2)$ in Theorem 4.5, then we obtain an example of an NFB-ring containing 16 elements.
- Also, having examples of INFB-spaces and SNFB-spaces, we can construct examples of non-associative INFB-algebras and SNFB-algebras, respectively.

- The nonassociative algebras constructed in Theorems 4.4 and 4.5 are examples of four-dimensional NFB-algebras. Examples of five-dimensional NFB-algebras were previously known [Maltsev, Parfenov, 1977; L'vov, 1978].
- If we put $F = GF(2)$ in Theorem 4.5, then we obtain an example of an NFB-ring containing 16 elements.
- Also, having examples of INFB-spaces and SNFB-spaces, we can construct examples of non-associative INFB-algebras and SNFB-algebras, respectively.

- The nonassociative algebras constructed in Theorems 4.4 and 4.5 are examples of four-dimensional NFB-algebras. Examples of five-dimensional NFB-algebras were previously known [Maltsev, Parfenov, 1977; L'vov, 1978].
- If we put $F = GF(2)$ in Theorem 4.5, then we obtain an example of an NFB-ring containing 16 elements.
- Also, having examples of INFB-spaces and SNFB-spaces, we can construct examples of non-associative INFB-algebras and SNFB-algebras, respectively.

Theorem 4.6 [I., K., 2013].

Let F be an arbitrary field. The algebra $A = V \oplus T_2(F)$ is an SNFB-algebra.

Theorem 4.7 [I., K., 2015].

Let F be an field of characteristic zero, $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$. The algebra $A = V \oplus E_0$ over the field F is an SNFB-algebra.

Corollary 4.2.

Any finite dimensional F-algebra (over the corresponding field F) containing the algebra A as a subalgebra has no finite basis of identities.

Theorem 4.6 [I., K., 2013].

Let F be an arbitrary field. The algebra $A = V \oplus T_2(F)$ is an SNFB-algebra.

Theorem 4.7 [I., K., 2015].

Let F be an field of characteristic zero, $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$. The algebra $A = V \oplus E_0$ over the field F is an SNFB-algebra.

Corollary 4.2.

Any finite dimensional F-algebra (over the corresponding field F) containing the algebra A as a subalgebra has no finite basis of identities.

Theorem 4.6 [I., K., 2013].

Let F be an arbitrary field. The algebra $A = V \oplus T_2(F)$ is an SNFB-algebra.

Theorem 4.7 [I., K., 2015].

Let F be an field of characteristic zero, $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$. The algebra $A = V \oplus E_0$ over the field F is an SNFB-algebra.

Corollary 4.2.

Any finite dimensional F-algebra (over the corresponding field F) containing the algebra A as a subalgebra has no finite basis of identities.

- Above, we gave an example of an L-variety that does not have a finite basis of identities and is an union of two Specht varieties.
- A similar construction can be constructed for algebras from the variety $\mathfrak{P} = \langle x(yz) = 0 \rangle$.
- Let F be a field of characteristic zero, $B_1 = \langle v_1, v_2, e_{11}, e_{12} \rangle_F$, $B_2 = \langle v_1, v_2 + e_{11}, e_{21} \rangle_F$ be F-algebras from the variety $\mathfrak{P}$.

- Above, we gave an example of an L-variety that does not have a finite basis of identities and is an union of two Specht varieties.
- A similar construction can be constructed for algebras from the variety $\mathfrak{P} = \langle x(yz) = 0 \rangle$.
- Let F be a field of characteristic zero, $B_1 = \langle v_1, v_2, e_{11}, e_{12} \rangle_F$, $B_2 = \langle v_1, v_2 + e_{11}, e_{21} \rangle_F$ be F-algebras from the variety $\mathfrak{P}$.

- Above, we gave an example of an L-variety that does not have a finite basis of identities and is an union of two Specht varieties.
- A similar construction can be constructed for algebras from the variety $\mathfrak{P} = \langle x(yz) = 0 \rangle$.
- Let F be a field of characteristic zero, $B_1 = \langle v_1, v_2, e_{11}, e_{12} \rangle_F$, $B_2 = \langle v_1, v_2 + e_{11}, e_{21} \rangle_F$ be F-algebras from the variety $\mathfrak{P}$.

Theorem 4.8 [I., 2018].

Let F be an field of characteristic zero. Varieties $\mathfrak{M}_1 = \operatorname{Var} B_1$ and $\mathfrak{M}_2 = \operatorname{Var} B_2$ are Specht.

Theorem 4.9 [I., 2018].

Let F be an field of characteristic zero. The variety $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2 = \operatorname{Var} B_1 \oplus B_2$ is an NFB-variety with a basis of identities

$$\{x(yz), z\mathrm{St}_3(x, y, t), zx[y, u]v, x[y, u]v - v[y, u]x,$$
$$z[x, y]z_1 z_2 \ldots z_m[u, v] | m = 1, 2, \ldots\}.$$

Theorem 4.8 [I., 2018].

Let F be an field of characteristic zero. Varieties $\mathfrak{M}_1 = \mathrm{Var}\, B_1$ and $\mathfrak{M}_2 = \mathrm{Var}\, B_2$ are Specht.

Theorem 4.9 [I., 2018].

Let F be an field of characteristic zero. The variety $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2 = \mathrm{Var}\, B_1 \oplus B_2$ is an NFB-variety with a basis of identities

$$\{x(yz), z\mathrm{St}_3(x,y,t), zx[y,u]v, x[y,u]v - v[y,u]x, \\ z[x,y]z_1z_2 \dots z_m[u,v] | m = 1,2,\dots\}.$$

- In 1993, I.P. Shestakov formulated in Dniester notebook a question: do there exist finite dimensional central simple algebras over a field of characteristic zero that do not have a finite basis of identities?
- Note that any finite dimensional simple algebra over an algebraically closed field is uniquely determined by their identities up to isomorphism [Shestakov, Zaycev, 2011].
- Using the found SNFB-algebras and INFB-algebras, the required example is constructed for an arbitrary field.

- In 1993, I.P. Shestakov formulated in Dniester notebook a question: do there exist finite dimensional central simple algebras over a field of characteristic zero that do not have a finite basis of identities?
- Note that any finite dimensional simple algebra over an algebraically closed field is uniquely determined by their identities up to isomorphism [Shestakov, Zaycev, 2011].
- Using the found SNFB-algebras and INFB-algebras, the required example is constructed for an arbitrary field.

- In 1993, I.P. Shestakov formulated in Dniester notebook a question: do there exist finite dimensional central simple algebras over a field of characteristic zero that do not have a finite basis of identities?
- Note that any finite dimensional simple algebra over an algebraically closed field is uniquely determined by their identities up to isomorphism [Shestakov, Zaycev, 2011].
- Using the found SNFB-algebras and INFB-algebras, the required example is constructed for an arbitrary field.

Theorem 4.10 [I., K., 2012].

Let $A = \langle 1, v_1, v_2, e_{11}, e_{12}, e_{22}, p \rangle_F$ be an algebra over arbitrary field F, where 1 is a unity element of A, and nonzero products of basis elements which is not equal to unity element are defined by rules: $v_i e_{ij} = v_j$, $v_2 p = 1$. Then the algebra A is a central simple F-algebra and A has no finite basis of identities.

- After giving this example, the question was posed about the existence of a finite dimensional simple commutative or anticommutative NFB-algebra.
- In the case of characteristic zero, an positive answer to this question was obtained.

- After giving this example, the question was posed about the existence of a finite dimensional simple commutative or anticommutative NFB-algebra.
- In the case of characteristic zero, an positive answer to this question was obtained.

Theorem 4.11 [K., 2015].

Let $A = \langle 1, v_1, v_2, e_{11}, e_{12}, e_{22}, p \rangle_F$ be an algebra over field F of characteristic zero, where 1 is a unity element of A, and nonzero products of basis elements which is not equal to unity element are defined by rules: $v_i e_{ij} = e_{ij} v_i = v_j$, $v_2 p = p v_2 = 1$. Then the algebra A is a central simple commutative F-algebra and A has no finite basis of identities.

Theorem 4.12 [K., 2017].

Let $A = \langle e, v_1, v_2, e_{11}, e_{12}, e_{22}, p \rangle_F$ be an algebra over field F of characteristic zero, where nonzero products of basis elements are defined by rules: $v_i e_{ij} = -e_{ij} v_i = v_j$, $v_2 p = -p v_2 = e$, $v_i e = -e v_i = v_i$, $e_{ij} e = -e e_{ij} = e_{ij}$, $pe = -ep = p$. Then the algebra A is a simple anticommutative F-algebra and A has no finite basis of identities.

Theorem 4.11 [K., 2015].

Let $A = \langle 1, v_1, v_2, e_{11}, e_{12}, e_{22}, p\rangle_F$ be an algebra over field F of characteristic zero, where 1 is a unity element of A, and nonzero products of basis elements which is not equal to unity element are defined by rules: $v_i e_{ij} = e_{ij} v_i = v_j$, $v_2 p = p v_2 = 1$. Then the algebra A is a central simple commutative F-algebra and A has no finite basis of identities.

Theorem 4.12 [K., 2017].

Let $A = \langle e, v_1, v_2, e_{11}, e_{12}, e_{22}, p\rangle_F$ be an algebra over field F of characteristic zero, where nonzero products of basis elements are defined by rules: $v_i e_{ij} = -e_{ij} v_i = v_j$, $v_2 p = -p v_2 = e$, $v_i e = -e v_i = v_i$, $e_{ij} e = -e e_{ij} = e_{ij}$, $pe = -ep = p$. Then the algebra A is a simple anticommutative F-algebra and A has no finite basis of identities.

- Theorems 4.10–4.12 give examples of nonfinitely based simple algebras of dimension seven.
- In the case of a field of characteristic zero, the dimension of the algebra from Theorem 4.10 can be decrease to six.

- Theorems 4.10–4.12 give examples of nonfinitely based simple algebras of dimension seven.
- In the case of a field of characteristic zero, the dimension of the algebra from Theorem 4.10 can be decrease to six.

Theorem 4.13 [K., Unpublished].

Let $A = \langle 1, v_1, v_2, e_{11} + e_{12}, e_{22}, p\rangle_F$ be an algebra over the field F of characteristic zero, where 1 is a unity element of A, and nonzero products of basis elements which is not equal to unity element (taking into account the law of distributive) are defined by rules: $v_i e_{ij} = v_j$, $v_2 p = 1$. Then the algebra A is a central simple F-algebra and A has no finite basis of identities.

Unsolved problems

- Description of NFB-spaces (SNFB-spaces, INFB-spaces).
- Description of minimal nonzero L-varieties of vector spaces.
- Study the structure of the lattice of L-subvarieties.
- Description of almost commutative (almost nilpotent, almost Engel, etc) L-varieties of vector spaces.
- When is $T(G) = L(G)$ for the set G of associative polynomials?
- Construct an example of a not finitely based L-variety, any proper L-subvariety of which is given by a finite number of identities.
- Is there a three dimensional nonassociative NFB-algebra?
- etc.

- Description of NFB-spaces (SNFB-spaces, INFB-spaces).
- Description of minimal nonzero L-varieties of vector spaces.
- Study the structure of the lattice of L-subvarieties.
- Description of almost commutative (almost nilpotent, almost Engel, etc) L-varieties of vector spaces.
- When is $T(G) = L(G)$ for the set G of associative polynomials?
- Construct an example of a not finitely based L-variety, any proper L-subvariety of which is given by a finite number of identities.
- Is there a three dimensional nonassociative NFB-algebra?
- etc.

- Description of NFB-spaces (SNFB-spaces, INFB-spaces).
- Description of minimal nonzero L-varieties of vector spaces.
- Study the structure of the lattice of L-subvarieties.
- Description of almost commutative (almost nilpotent, almost Engel, etc) L-varieties of vector spaces.
- When is $T(G) = L(G)$ for the set G of associative polynomials?
- Construct an example of a not finitely based L-variety, any proper L-subvariety of which is given by a finite number of identities.
- Is there a three dimensional nonassociative NFB-algebra?
- etc.

- Description of NFB-spaces (SNFB-spaces, INFB-spaces).
- Description of minimal nonzero L-varieties of vector spaces.
- Study the structure of the lattice of L-subvarieties.
- Description of almost commutative (almost nilpotent, almost Engel, etc) L-varieties of vector spaces.
- When is $T(G) = L(G)$ for the set G of associative polynomials?
- Construct an example of a not finitely based L-variety, any proper L-subvariety of which is given by a finite number of identities.
- Is there a three dimensional nonassociative NFB-algebra?
- etc.

- Description of NFB-spaces (SNFB-spaces, INFB-spaces).
- Description of minimal nonzero L-varieties of vector spaces.
- Study the structure of the lattice of L-subvarieties.
- Description of almost commutative (almost nilpotent, almost Engel, etc) L-varieties of vector spaces.
- When is $T(G) = L(G)$ for the set G of associative polynomials?
- Construct an example of a not finitely based L-variety, any proper L-subvariety of which is given by a finite number of identities.
- Is there a three dimensional nonassociative NFB-algebra?
- etc.

- Description of NFB-spaces (SNFB-spaces, INFB-spaces).
- Description of minimal nonzero L-varieties of vector spaces.
- Study the structure of the lattice of L-subvarieties.
- Description of almost commutative (almost nilpotent, almost Engel, etc) L-varieties of vector spaces.
- When is $T(G) = L(G)$ for the set G of associative polynomials?
- Construct an example of a not finitely based L-variety, any proper L-subvariety of which is given by a finite number of identities.
- Is there a three dimensional nonassociative NFB-algebra?
- etc.

- Description of NFB-spaces (SNFB-spaces, INFB-spaces).
- Description of minimal nonzero L-varieties of vector spaces.
- Study the structure of the lattice of L-subvarieties.
- Description of almost commutative (almost nilpotent, almost Engel, etc) L-varieties of vector spaces.
- When is $T(G) = L(G)$ for the set G of associative polynomials?
- Construct an example of a not finitely based L-variety, any proper L-subvariety of which is given by a finite number of identities.
- Is there a three dimensional nonassociative NFB-algebra?
- etc.

- Description of NFB-spaces (SNFB-spaces, INFB-spaces).
- Description of minimal nonzero L-varieties of vector spaces.
- Study the structure of the lattice of L-subvarieties.
- Description of almost commutative (almost nilpotent, almost Engel, etc) L-varieties of vector spaces.
- When is $T(G) = L(G)$ for the set G of associative polynomials?
- Construct an example of a not finitely based L-variety, any proper L-subvariety of which is given by a finite number of identities.
- Is there a three dimensional nonassociative NFB-algebra?
- etc.

Thank you for your attention!