Enumeration Reducibility: 60 years of investigations

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Computability

Different models of computation:

- Turing Machines;
- Recursive functions;
- λ-calculus;
- Random Access Machines;
- ...

All leading to one and the same thing: computable functions.

Coding

- The (effective) coding is a procedure definig an injective (bijective) mapping from a collection A to the set of natural numbers. Given any element in A we should be able to determine its code using finite amount of time and memory
- *Example 1.* We can code every word (string) built from the letters a and b in the following way: the code of the word $x_1 x_2 \dots x_k$ is the natural number $p_1^{SN(x_1)} p_2^{SN(x_2)} \dots p_k^{SN(x_k)}$, where $p_1 < p_2 < \cdots$ are the prime numbers, and $SN(a) = 1$, $SN(b) = 2$.
- *Example 2.* The code of the tuple $(m, n) \in \mathbb{N} \times \mathbb{N}$ is the number $\langle m, n \rangle = 2^n(2m + 1) - 1$.
- *Example 3.* Given a finite set D of natural numbers its cannonical code is the number $u = \sum_{n \in D} 2^n$.

A modern point of view

- A computer with unbounded binary memory;
- Every executable file (programme, algorithm) has the form of a binary string, i.e. a natural number, $\{e\}$.
- Every natural number e can be loaded in the memory of the computer as an executable file (in most cases this programme will cause the computer to crash).
- The programme takes some input and after the execution produces an output. The input and the output are once again stored in the memory, so that they are binary strings and hence natural numbers.
- Thus $\{e\}$ is a function with a domain and range consisting of natural numbers.

Computably enumerable (c.e.) sets

- Usually $dom\{e\} \neq \mathbb{N}$. We use the notation $W_e = dom\{e\}$. W_e is called the c.e. set with index e.
- For every c.e. set W_e there is an effective procedure, such that for every natural n it says "YES" if $n \in W$, and it does never give an answer if $n \notin W$ (just compute $\{e\}(n)$). C.e. sets are aslo called semi-decidable.
- There is a computable surjection $f : \mathbb{N} \to W_e$. Compute simultaneously $\{e\}(0)$, $\{e\}(1)$, ..., $\{e\}(k)$,.... Then $f(0)$ is the first natural number for which $\{e\}$ gives an output, $f(1)$ is the second one, etc.
- The set $K = \{e \mid e \in W_e\}$ is c.e., but $\overline{K}$ is not c.e. (Halting problem).

The problem with the natural numbers

- The axiomatic approach to natural numbers.
- Peano Arithmetic
- The set of theorems of PA is c.e., but it is undecidable (not computable).
- Every consistent extension of PA is undecidable

Relative computing

- (Turing) Computing with oracle: we allow the computer to use an auxiliary function f when executing a programme e. We denote the resulting function by $\{e\}^f$.
- Turing reducibility: $A \leq_T B$ *iff* $\chi_A = \{e\}^{\chi_B}$ for some programme e.
- $\leq_T$ is a reflexive and transitive relation, so that the relation $\equiv_T$ defined by $A \equiv_T B \iff A \leq_T B \wedge B \leq_T A$ is an equivalence relation.
- The equivalence classes are called Turing degrees. $\leq_T$ is a partial ordering on the Turing degrees.
- $\mathbf{0} = \{A \mid A \text{ is decidable}\}$ is the least element.
- $\mathbf{d}(A \oplus B)$ is the l.u.b. of $\mathbf{d}(A)$ and $\mathbf{d}(B)$, where $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$.

- $W \leq_T K$ for every c.e. set W.
- (Post) Is there a c.e. set X, such that $\emptyset <_T X <_T K$?
- No natural examples.
- (Friedberg; Muchnick) There are incomparable c.e. sets X and Y, such that $\emptyset <_T X, Y <_T K$.

Enumeration reducibility

- An enumeration of A is any surjective mapping of $\mathbb{N}$ onto A.
- (Friedberg, Rodgers) $A \leq_e B$ iff there is an algorithm e such that for every enumeration f of B, $\{e\}^f$ is an enumeration of A.
- $A \leq_e B \iff A = W_i(B) = \{n \mid \exists u(\langle n, u\rangle \in W_i \ \&\ D_u \subseteq B)\}$ for some i.
- $\leq_e$ is a reflexive and transitive relation, which generates an equivalence relation $\equiv_e$.
- The equivalence classes are called enumearion degrees. $\leq_e$ becomes a partial ordering. $\mathbf{d}_e(A \oplus B) = \mathbf{d}_e(A) \vee \mathbf{d}_e(B)$.
- $\mathbf{0}_e = \{W_i \mid i \in \mathbb{N}\}$ is the least element.

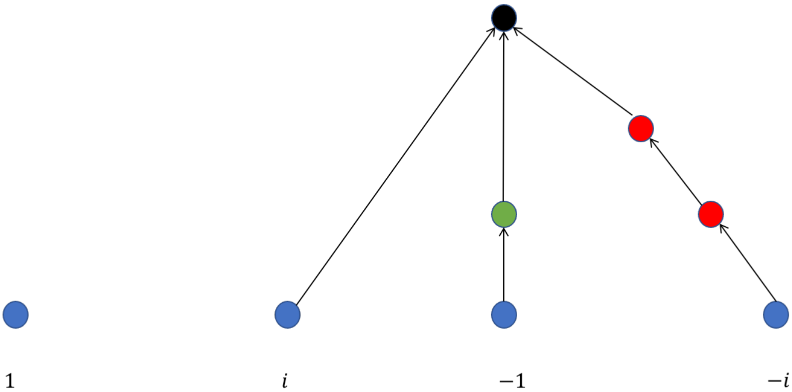
Connection between Turing and enumeartion reducibility

- $A \leq_T B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$.
- A is c.e. in B iff $A \leq_e B \oplus \overline{B}$.
- $\iota : \mathcal{D}_T \to \mathcal{D}_e$ defined by $\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A})$ is the natural embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$.
- The images of Turing degrees under ι are called total degrees.
- An enumeration degree is total iff it contains a total set, i.e. a set A, such that $\overline{A} \leq_e A$.

Initial segments

- There are minimal (non-zero) Turing degrees.
- (Gutteridge; Cooper) The enumeration degrees are downwards dense.
- (Lachlan, Lebeuf) Every countable upper semi-lattice is an initial segment of Turing degrees.
- (Slaman, Sorbi) Every partial order can be embedded in every initial segment of enumeration degrees.

$\mathbb{C}_4$



The complexity of $\mathcal{D}_T$ and $\mathcal{D}_e$

- (Simpson; Slaman, Woodin) The first order theory both $\mathcal{D}_T$ and $\mathcal{D}_e$ are computably isomorphic to second order arithmetic.
- Biinterpretability conjecture: For every degree **a** the respective degree structure can correctly guess a generator of **a**.
- The biinterpretability conjecture is equivalent to the rigidity (the lack of nontrivial automorphisms) of the structure.
- (Slaman, Woodin; M. Soskova) There is a finite automorphism base for both degree structures.

A canonical way for obtaining more complex sets

- $K_A = \{e \mid e \in W_e^A\}$.
- $A <_T K_A$ and $W_e^A \leq_e K_A$.
- $\mathbf{d}_T(K_A) = \mathbf{d}_T(A)'$ is called the jump of $\mathbf{d}_T(A)$.
- $E_A = \{e \mid e \in W_e(A)\}$
- $A <_e E_A \oplus \overline{E}_A$ and $W_e(A) \oplus \overline{W}_e(A) \leq_e E_A \oplus \overline{E}_A$.
- $\mathbf{d}_e(E_A \oplus \overline{E}_A) = \mathbf{d}_e(A)'$ is called the jump of $\mathbf{d}_e(A)$.
- The jump operation agrees with the natural embedding ι, i.e. $\iota(\mathbf{a}') = \iota(\mathbf{a})'$.

- (Shore) The jump is definable in $\mathcal{D}_T$.

Kalimullin pairs

- The pair of enumeration degrees $\{\mathbf{x}, \mathbf{y}\}$ is called a $\mathcal{K}$-pair iff

$$\mathbf{a} = (\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{a} \vee \mathbf{y})$$

  for every enumeration degree $\mathbf{a}$.
- (Kalimullin) The jump is definable in $\mathcal{D}_e$, namely, $\mathbf{a}'$ is the biggest l.u.b of a $\mathcal{K}$-pair $\{\mathbf{x}, \mathbf{y}\}$ such that $\mathbf{x} \leq \mathbf{a}$.

- (Cai, G., Lempp, Miller, M. Soskova) The total degrees (the images of the Turing degrees under the embedding ι) are definable in $\mathcal{D}_e$, namely **a** is total iff **a** is the l.u.b. of a maximal $\mathcal{K}$-pair.
- The relation c.e. in for total degrees is definable in $\mathcal{D}_e$, namely, the total degree **a** is c.e. in the total degrees **b** iff **a** is the l.u.b. of a $\mathcal{K}$-pair $\{\mathbf{x}, \mathbf{y}\}$ such that $\mathbf{x} \leq \mathbf{b}$.
- (Slaman, Soskova) $\mathcal{D}_e$ has a finite automorphism base consisting of total degrees below $\mathbf{0}'$.

Thank you!